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New Fixed Point Theorems in Dislocated Quasi-b-Metric Space

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Abstract: In this article, fixed point theorems for rational type contraction and generalized type of contraction are established in the the frame work of complete dislocated quasi-b-metric space. Our established theorems extend and generalize some well-known results in the literature.

Keywords: Complete dislocated quasi-b-metric space, Cauchy sequence, self-mapping, fixed point.

1 Introduction

In 1906, Frechet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways. An incomplete list of the results of such an attempt is the following: quasi-symmetric space [14], A-metric space [15], S-metric space [16] and so on.

In the field of metric fixed point theory the first significant result was proved by Banach in complete metric space which may be stated as following:
"Every contraction mapping of a complete metric space X into itself has a unique fixed point".

Most of the work done in the field of metric fixed point theory after Banach contraction principle involve the continuity of self-mappings for different type of contractions. Therefore generally a natural question arises wether the condition of continuity of mappings is essential for the existence of fixed point. This question has been affirmatively answered by Kannan [4]. In [4], Kannan established the following result in which the continuity of mapping is not necessary at each point.
"If a mapping $\theta : B_1 \to B_1$ where (B_1, ς) is a complete metric space and the following condition holds

$$\varsigma(\theta b_1, \theta b_2) \leq \mu \cdot [\varsigma(b_1, \theta b_1) + \varsigma(b_2, \theta b_2)]$$

$\forall\ b_1, b_2 \in B_1$with $0 \leq \mu < \frac{1}{2}$. Then θ has a unique fixed point". The mapping satisfying the above axiom is known is Kannan type of mapping. Kannan [4] provide a new direction for the researchers to work in the area of metric fixed point theory. Almost similar type of contraction condition has been studied by Chetterjea [5] whose result may be stated as following:
"Suppose (A_1, ς) is a complete metric space a mapping $T_1 : A_1 \to A_1$ satisfying

$$\varsigma(T_1 a_1, T_1 a_2) \leq \delta \cdot [\varsigma(a_1, T_1 a_2) + \varsigma(a_2, T_1 a_1)]$$

with $0 \leq \delta < \frac{1}{2}$ and $\forall\ a_1, a_2 \in A_1$. Then T_1 has a unique fixed point". The mapping satisfying the above condition is known is Chetterjea type of mapping. Dass and Gupta [6] generalized Banach contraction principle by introducing rational contractive conditions in metric spaces.

The notion of b-metric space was introduced by Czerwik [7] in connection with some problems concerning with the convergence of n0n-measurable functions with respect to measure. Fixed point theorems regarding b-metric spaces was obtained in [8] and [9]. In 2013, Shukla [10] generalized the notion of b-metric spaces and introduced the concept of partial b-metric spaces. Recently, Rahman and Sarwar [11] further generalized the concept of b-metric space and initiated the notion of dislocated quasi-b-metric space.

In the present work, we have proved some fixed point theorems for generalized type contraction and for rational type contraction conditions in the setting of dislocated quasi b-metric spaces which improve, extend and

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generalize similar type of fixed point results in dislocated quasi-b-metric spaces.

2 Preliminaries

We need the following definitions which may be found in [11].

Definition 2.1. Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if $\forall\, x, y, z \in X$

(d_1) $d(x,y) = d(y,x) = 0$ implies that $x = y$;

(d_2) $d(x,y) \leq k[d(x,z) + d(z,y)]$.

The pair (X,d) is called dislocated quasi-b-metric space or shortly (dq b-metric) space.

Remark. In the definition of dislocated quasi-b-metric space if $k = 1$ then it becomes (usual) dislocated quasi-metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi-b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

Example 2.1. Let $X = \mathbb{R}$ and suppose

$$d(x,y) = |2x - y|^2 + |2x + y|^2.$$

Then (X,d) is a dislocated quasi-b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric space nor b-metric space.

Remark. Like dislocated quasi-metric space in dislocated quasi-b-metric space the distance between similar points need not to be zero necessarily as clear from the above example.

Definition 2.2. A sequence $\{x_n\}$ is called dq-$b-$convergent in (X,d) if for $n \geq N$ we have $d(x_n, x) < \varepsilon$ where $\varepsilon > 0$ then x is called the dq-b-limit of the sequence $\{x_n\}$.

Definition 2.3. A sequence $\{x_n\}$ in dq-b-metric space (X,d) is called Cauchy sequence if for $\varepsilon > 0$ there exists $n_0 \in N$, such that for $m, n \geq n_0$ we have $d(x_m, x_n) < \varepsilon$.

Definition 2.4. A dq-$b-$metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Definition 2.5. Let (X,d_1) and (Y,d_2) be two dq-b-metric spaces. A mapping $T : X \to Y$ is said to be continuous if for each $\{x_n\}$ which is dq-b convergent to x_0 in X, the sequence $\{Tx_n\}$ is dq-b convergent to Tx_0 in Y.

The following well-known results can be seen in [11].

Lemma 1. Limit of a convergent sequence in dislocated quasi-b-metric space is unique.

Lemma 2. Let (X,d) be a dislocated quasi-b-metric space and $\{x_n\}$ be a sequence in dqb-metric space such that

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \tag{1}$$

for $n = 1, 2, 3, \ldots$ and $0 \leq \alpha k < 1, \alpha \in [0,1)$, and k is defined in dq-b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 1. Let (X,d) be a complete dislocated quasi-b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0,1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Remark. Like b-metric space dislocated quasi-b-metric space is also continuous on its two variables.

Remark. In dislocated quasi-b-metric space the distance between similar points need not to be zero like usual dislocated quasi-metric space.

3 Main Results

Theorem 3.1. Let (X,d) be a complete dq-b-metric space with coefficient $k \geq 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot [d(x,Tx) + d(y,Ty)] +$$
$$\mu \cdot [d(x,Ty) + d(y,Tx)]$$

$\forall\, x, y \in X$, where $\alpha, \beta, \mu \geq 0$, with $k\alpha + (1+k)\beta + 2(k^2 + k)\mu < 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots\ldots, x_{n+1} = Tx_n \text{ for } n \in N.$$

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using given condition in the theorem we have

$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$
$$+\mu \cdot [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$
$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$+\mu \cdot [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)].$$

Using triangular inequality in 3rd term we get

$$(1 - (\beta + 2k\mu))d(x_n, x_{n+1}) \leq (\alpha + \beta + 2k\mu)d(x_{n-1}, x_n).$$

By given restrictions on the constants we have

$$\lambda = \frac{\alpha + \beta + 2k\mu}{1 - (\beta + 2k\mu)} < \frac{1}{k}.$$

Therefore

$$d(x_n, x_{n+1}) \leq \lambda \cdot d(x_{n-1}, x_n).$$

Now using Lemma 2 we get that $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now to show that u is the fixed point of T. Since $x_n \to u$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tu$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tu.$$

Thus $Tu = u$. Hence u is the fixed point of T.

Uniqueness. Let u, v be two distinct fixed points of T. Then

$$d(u,v) = d(Tu,Tv) \le \alpha \cdot d(u,v) + \beta \cdot [d(u,Tu) + d(v,Tv)]$$
$$+\mu \cdot [d(u,Tv) + d(v,Tu)].$$

Since u, v are fixed points of T and using given condition in the theorem one can easily get that $d(u,u) = d(v,v) = 0$ so finally we get

$$d(u,v) \le (\alpha+\mu)\cdot d(u,v) + \mu d(v,u). \tag{2}$$

Similarly we can show that

$$d(v,u) \le \mu \cdot d(u,v) + (\alpha+\mu) d(v,u). \tag{3}$$

Adding (2) and (3) we get

$$[d(u,v) + d(v,u)] \le (\alpha + 2\mu)[d(u,v) + d(v,u)].$$

The above inequality is possible only if $d(u,v) + d(v,u) = 0$ which is again possible if $d(u,v) = d(v,u) = 0$. So, by (d_2) $u = v$. Hence fixed point of T is unique in X.

The following corollaries are deduced from Theorem 3.1.

Corollary 3.1. Let (X,d) be a complete dq-b-metric space with coefficient $k \ge 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \le \mu \cdot [d(x,Ty) + d(y,Tx)]$$

$\forall\, x, y \in X$, where $\mu \ge 0$ with $2(k^2+k)\mu < 1$. Then T has a unique fixed point.

Corollary 3.2. Let (X,d) be a complete dq-b-metric space with coefficient $k \ge 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \le \beta \cdot [d(x,Tx) + d(y,Ty)]$$

$\forall\, x, y \in X$, where $\beta \ge 0$ with $(1+k)\beta < 1$. Then T has a unique fixed point.

Corollary 3.3. Let (X,d) be a complete dq-b-metric space with coefficient $k \ge 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot d(x,Tx) + \gamma \cdot d(y,Ty) +$$
$$\mu \cdot d(x,Ty) + \delta \cdot d(y,Tx)$$

$\forall\, x, y \in X$, where $\alpha, \beta, \gamma, \mu, \delta \ge 0$ with $k\alpha + \beta + k\gamma + 2k^2\mu + 2k\delta < 1$. Then T has a unique fixed point.

Remark.

1. Corollary 3.1 generalize the result of Chetterjea [5] in dq-b-metric space.
2. Corollary 3.2 generalize the result of Kannan [4] in dq-b-metric space.
3. Corollary 3.3 generalize the result of Hardy-Rogeres [13] in dq-b-metric space.

Our next theorem is about rational type contraction conditions in the frame work of dq-b-metric spaces.

Theorem 3.2. Let (X,d) be a complete dq-b-metric space with coefficient $k \ge 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$$
$$+\gamma \cdot \frac{d(y,Ty) + d(y,Tx)}{1+d(y,Ty)\cdot d(y,Tx)} \tag{4}$$

$\forall\, x, y \in X$ where $\alpha, \beta, \gamma \ge 0$ and $k\alpha + \beta + (1+k+k^2)\gamma < 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots\ldots, x_{n+1} = Tx_n \quad \text{for all} \quad n \in N.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X then consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using Eq. (4) and definition of the defined sequence we have

$$d(x_n,x_{n+1}) \le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}$$
$$+\gamma \cdot \frac{d(x_n,x_{n+1}) + d(x_n,x_n)}{1+d(x_n,x_{n+1})\cdot d(x_n,x_n)}$$
$$\le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot \frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}$$
$$+\gamma \cdot [d(x_n,x_{n+1}) + kd(x_{n-1},x_n) + kd(x_n,x_{n+1})].$$

Therefore

$$d(x_n,x_{n+1}) \le \frac{\alpha + k\gamma}{1-(\beta+(1+k)\gamma)} \cdot d(x_{n-1},x_n) = h \cdot d(x_{n-1},x_n).$$

Where $h = \frac{\alpha+k\gamma}{1-(\beta+(1+k)\gamma)}$ with $h < \frac{1}{k}$ because $k\alpha + \beta + (1+k+k^2)\gamma < 1$. Now using Lemma 2 we get that $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now to show that u is the fixed point of T. Since $x_n \to u$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tu$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tu.$$

Thus $Tu = u$. Hence u is the fixed point of T.

Uniqueness. Suppose that T has two distinct fixed points u and v. Consider

$$d(u,v) = d(Tu,Tv) \leq \alpha \cdot d(u,v) + \beta \cdot \frac{d(v,Tv)[1+d(u,Tu)]}{1+b(u,v)}$$

$$+\gamma \cdot \frac{d(v,Tv)+d(v,Tu)}{1+d(v,Tv)\cdot d(v,Tu)}.$$

Since u,v are fixed points of T and using Eq. (4) one can easily get that $d(u,u) = d(v,v) = 0$ so finally we get

$$d(u,v) \leq \alpha \cdot (u,v) + \gamma \cdot d(v,u). \quad (5)$$

Similarly

$$d(v,u) \leq \alpha \cdot (v,u) + \gamma \cdot d(u,v). \quad (6)$$

Adding (5) and (6) we get

$$d(u,v) + d(v,u) \leq (\alpha + \gamma)[d(u,v) + d(v,u)].$$

The above inequality is possible only if $d(u,v) + d(v,u) = 0$ which is again possible if $d(u,v) = d(v,u) = 0$. So, by (d_2) $u = v$. Hence fixed point of T is unique in X.

One can easily deduced the following corollaries from Theorem 3.2.

Corollary 3.4. Let (X,d) be a complete dq-b-metric space with coefficient $k \geq 1$ and T be a continuous self-mapping $T: X \to X$ satisfying the condition

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$$

$\forall\, x,y \in X$ where $\alpha, \beta \geq 0$ and $k\alpha + \beta < 1$. Then T has a unique fixed point.

Corollary 3.5. Let (X,d) be a complete dq-b-metric space with coefficient $k \geq 1$ and T be a continuous self-mapping $T: X \to X$ satisfying the condition

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \gamma \cdot \frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)\cdot d(y,Tx)}$$

$\forall\, x,y \in X$ where $\alpha, \gamma \geq 0$ and $k\alpha + (1+k+k^2)\gamma < 1$. Then T has a unique fixed point.

Remark. Theorem 3.2 and the corollaries deduced from it generalize and extend several fixed point results in the setting of dislocated quasi-b-metric space.

Conclusion

Our established results generalize and extend the results Kannan [4], Chetterjea [5], Hardy Rogeres [13] and many other several fixed point results in the frame work of dislocated quasi-b-metric spaces.

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