Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/030301

Fractional Derivatives of the Lauricella Function and the Multivariable I-Function Along with the General Class Polynomials

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Received: 5 Feb. 2016, Revised: 14 Aug. 2016, Accepted: 18 Aug. 2016
Published online: 1 Sep. 2016

Abstract: The objective of this paper is to explore the fractional derivative of the multivariable I-function Prasad [5], associated with a general class of multivariable polynomials of Srivastava [2] and the generalized Lauricella function of Srivastava and Daoust [9]. The results which are derived by using certain special cases are interesting and very general in nature. Each of these results is shown to apply to yield interesting new results for certain multivariable hypergeometric functions.

Keywords: Multivariable I-function, Fractional Derivative, General Class Polynomial, Lauricella function, Binomial expansion.
2010 Mathematics Subject Classification: 33C60, 33C20, 26A33, 33C65.

1 Introduction

In the present paper, the fractional derivatives of the product of the Lauricella functions, the general class polynomials and the multivariable I-function are derived. In recent years, several authors have found that derivatives and integrals of fractional order are suitable for description of properties of various real materials. The main advantages of fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of varies materials and process.

Recently, the multivariable I -function has been introduced and studied by Prasad [5] and Prasad and Yadav [8], which is a generalization of multivariable H-function and includes the generalized Lauricella functions of Srivastava and Daoust [7], Appell function etc.Therefore, the results established in this paper are of general character from which several known and new result can be deduced.

2 Preliminaries

In this section , we shall first recall some definitions and fundamental facts of Special functions and Fractional calculus.

Definition 1.

The multivariable I-function is represented in the following manner[5]:

$$I[z_1,\ldots,z_r] = H^{0,n_2;0,n_3;\ldots;0,n_r:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p_2,q_2;p_3,q_3;\ldots;p_r,q_r:(p',q');\ldots;(p^{(r)},q^{(r)})}$$

$$\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix}\left|\begin{matrix}\left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\left(a_{3j};\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3} \\ \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\left(b_{3j};\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3}\end{matrix}\right.\right.$$

$$;\ldots;\left(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj}\right)_{1,p_r},(a'_j,\alpha'_j)_{1,p^{(1)}};\ldots;$$

$$;\ldots;\left(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj}\right)_{1,q_r},(b'_j,\beta'_j)_{1,q^{(1)}};\ldots;$$

$$\left.\begin{matrix}(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\ell_1}\cdots\int_{\ell_r}\psi(s_1,\ldots,s_r)\left\{\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\right\}$$

$$ds_1\ldots ds_r \tag{1}$$

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where $\omega = \sqrt{-1}$,

$$\psi(s,...,s_r) = \frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_k}\Gamma\left(1-a_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}s_i\right)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_k+1}^{pk}\Gamma\left(a_{kj}-\sum_{i=1}^{k}\alpha_{kj}^{(i)}s_i\right)\right]}$$

$$\times\frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma(1-b_{kj}+\sum_{i=1}^{k}\beta_{kj}^{(i)}s_i)\right]} \qquad (2)$$

$$\phi_i(s_i) = \frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma(b_k^{(i)}-\beta_k^{(i)}s_i)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)\right]}$$

$$\times\frac{\left[\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)\right]}{\left[\prod_{m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_k^{(i)}+\beta_k^{(i)}s_i\right)\right]} \qquad (3)$$

$\forall i \in (1,...,r)$. Also,

$$\begin{aligned}
\{0,n_i\}_{2,r} &= 0,n_2:...:0,n_r,\\
\{p_i,q_i\}_{2,r} &= p_2,q_2:...:p_r,q_r,\\
\left\{(m^{(i)},n^{(i)})\right\}^{1,r} &= (m^{(1)},n^{(1)});...;(m^{(r)},n^{(r)}),\\
\left\{(p^{(i)},q^{(i)})\right\}^{1,r} &= (p^{(1)},q^{(1)});...;(p^{(r)},q^{(r)}),
\end{aligned}$$

$$\mathscr{A} = \left\{(a_{ji};\alpha_{ij}^{(1)},...,\alpha_{ij}^{(i)})_{1,p_i}^{2,r}\right\}$$
$$= (a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)})_{1,p_2};...;(a_{rj};\alpha_{rj}^{(1)},...,\alpha_{rj}^{(r)})_{1,p_r}, \qquad (4)$$

$$\mathscr{B} = \left\{(a_j^{(i)},\alpha_j^{(i)})_{1,p_(i)}^{1,r}\right\}$$
$$= (a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}};...;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}, \qquad (5)$$

$$\mathscr{C} = \left\{(b_{ij};\beta_{ij}^{(1)},...,\beta_{ij}^{(i)})_{1,q_i}^{2,r}\right\}$$
$$= (b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)})_{1,q_2};...;(b_{rj};\beta_{rj}^{(1)},...,\beta_{rj}^{(r)})_{1,q_r}, \qquad (6)$$

$$\mathscr{D} = \left\{(b_j^{(i)},\beta_j^{(i)})_{1,q^{(i)}}^{1,r}\right\}$$
$$= (b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}};...;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}, \qquad (7)$$

Such that $n_i,p_i,q_i,m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ are non-negative integers and all $a_{ij},b_{ij},\alpha_{ij},\beta_{ij},a_j^{(i)},b_j^{(i)},\alpha_j^{(i)},\beta_j^{(i)}$ are complex numbers and the empty product denotes unity .

The contour integral (1) convergence, if

$$|argz_i| < \frac{1}{2}U_i\pi,\ U_i > 0,\ i = 1,...,r \qquad (8)$$

where

$$U_i = \sum_{j=1}^{n^{(i)}}\alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}}\beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)} + \left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right)$$
$$+...+\left(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)}\right)$$
$$-\left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)}+...+\sum_{j=1}^{q_r}\beta_{rj}^{(i)}\right) \qquad (9)$$

and

$I[z_1,...,z_r] = \odot(|z_1|^{\alpha_1},...,|z_r|^{\alpha_r}), max\{|z_1|,...,|z_r|\}$

$\to 0$,
where

$$\alpha_i = min_{(1\le j\le m^{(i)})}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right),$$

$$\beta_i = max_{(1\le j\le n^{(i)})}\Re\left(\frac{a_j^{(i)}-1}{\alpha_j^{(i)}}\right), i = 1,..,r.$$

For the condition of convergence and analyticity of multivariable I-function we refer [5,8].

Definition 2.

The multidimensional analogue of a general class polynomials is defined by (see [11]):

$$S_n^{m_1,...,m_r}(x_1,...,x_r) = \sum_{k_1,...,k_r=0}^{m_1k_1+...+m_rk_r\le 0}$$
$$(-n)_{m_1k_1+\cdots+m_rk_r}A(n;k_1,...,k_r)\frac{{x_1}^{k_1}}{k_1!}\cdots\frac{{x_r}^{k_r}}{k_r!} \qquad (10)$$

where $m_1,...,m_r$ are arbitrary positive integer, $n = 0,1,2,...$ and the coefficients

$$A(n;k_1,...,k_r), k_i \ge 0, i = 1,...r$$

are arbitrary constants, real or complex.

The order of the highest degree of the variables $x_1,...,x_r$ of the multivariable polynomial (10) can be written as [12]

$$O(S_n^{m_1,...,m_r}(x_1,...,x_r)) = O\left(x_1^{[n/m_1]},...,x_r^{[n/m_r]}\right), \qquad (11)$$

where [x] denotes the greatest integer $\le x$.

Remarks:

(i) The case $r = 1$ of the multivariable polynomial (10) would give rise to the general class of polynomials S_n^m introduced by Srivastava[13].

(ii) For $m_i, i = 1,...,r$ and $A(n;k_1,...,k_r) = (1+\alpha_1+n_1)k_1,(1+\alpha_2+n_2)k_1...(1+\alpha_r+n_r)k_r$,the multivariable polynomial reduces to a multivariable Bessel polynomial [14].

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Definition 3.

In recent years, by several authors (see [3,9]), the fractional derivatives of a function f(t) of complex order have been established as follows;

$$ {}_{\alpha}D_t^{\gamma}\{f(t)\} = \begin{cases} \frac{1}{\Gamma(-\gamma)}\int_0^t (t-x)^{-\gamma-1} f(x)dx, (Re(\gamma)<0), \\ \frac{d^m}{dt^m}\, {}_{\alpha}D_t^{\gamma-m}\{f(t)\}, (0\le Re(\gamma)<m). \end{cases} \tag{12}$$

where m is a positive integer.

Remarks: (i)The special case of the fractional derivative (see [9]) is

$$D_t^{\mu}(t^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu}, (Re(\lambda)>-1). \tag{13}$$

(ii)For simplicity, the special case of the fractional derivative operator ${}_{\alpha}D_t^{\gamma}\{f(t)\}$, when $\alpha=0$ will be written as D_t^{μ}.

Thus we have

$$D_t^{\mu} = {}_{\alpha}D_t^{\gamma}\{f(t)\}. \tag{14}$$

Definition 4.

Binomial expansion is given by

$$(x+\zeta)^{\lambda} = \zeta^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(\frac{x}{\zeta}\right)^m, \left(|\frac{x}{\zeta}|<1\right). \tag{15}$$

Also the generalized Leibniz rue for fractional derivatives

$$D_x^{\mu}\{f(x)g(x)\} = \sum_{m=0}^{\infty}\binom{\mu}{m} D_x^{\mu-m}\{f(x)g(x)\}. \tag{16}$$

Definition 5.

The generalized Lauricella function of several complex variables is given by [7]:

$$F_{H:W^1,\ldots,W^r}^{G:V^1,\ldots,V^r}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{pmatrix} \equiv F\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{pmatrix}. \tag{17}$$

3 Required Result

we need to recall the following result (see[10]):
If$\mu\ge 0, 0<x<1, Re(1+\rho)>0, Re(q)>-1, \mu_i>0$ and $\Omega_i>0$ as $\Omega_i=0$ and $|z_i|<\rho, i=1,2,\ldots,r$,then

$$x^{\mu}F\begin{pmatrix}z_1x^{\mu_1}\\ \vdots\\ z_rx^{\mu_r}\end{pmatrix} = \sum_{M=0}^{\infty}\frac{(1+p+q+2M)}{M!} \times\frac{(-\mu)_M(1+p)_{\mu}}{(1+p+q+M)_{\mu+1}}F_M[z_1,\ldots,z_r] \times {}_2F_1\left[\begin{matrix}-M,1+p+q+M;\\ 1+p;\end{matrix} x\right] \tag{18}$$

where

$$F_M[z_1,\ldots,z_r] = F_{H+2:W';\ldots;W^r}^{G+2:V';\ldots;V^r}\left[\begin{matrix}\left[c:\eta^{(1)};\ldots;\eta^{(r)}\right],\\ \left[d:\sigma^{(1)};\ldots;\sigma^{(r)}\right],\end{matrix}\right.$$

$$[1+p+\mu:\mu_1,\ldots,\mu_r],[\mu+1;\ldots;\mu_r]$$

$$[2+p+q+M+\mu:\mu_1;\ldots;\mu_r],[\mu-M+1:\mu_1;\ldots;\mu_r]$$

$$\left.\begin{matrix}:[y':x'];\ldots;\left[y^{(r)}:x^{(r)}\right];\\ :[V':t'];\ldots;\left[V^{(r)}:t^{(r)}\right];\end{matrix} z_1,\ldots,z_r\right] \tag{19}$$

where $M\ge 0$

4 The Main Result

In this section,we have evaluated two theorems involving Lauricella function,the product of the multidimensional analogue of a general class polynomials with multivariable I-function.

Theorem 4.1. If $n_i,p_i,q_i,m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ are non-negative integers and $a_{ij},b_{ij},\alpha_{ij},\beta_{ij},a_j^{(i)},b_j^{(i)},\alpha_j^{(i)},\beta_j^{(i)}$ are complex numbers, where $\alpha_i = min_{(1\le j\le m^{(i)})}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)$, and $\beta_i = max_{(1\le j\le n^{(i)})}\Re\left(\frac{a_j^{(i)}-1}{\alpha_j^{(i)}}\right)$, also $max\{|z_1|,\ldots,|z_r|\}\to 0$,then the fractional derivative formula involving the Lauricella functions,generlized polynomial and the I-function as follows:

$$D_t^{\mu}(t-\zeta)^{\lambda}x^{\lambda}(\eta-t)^{\lambda+\tau}F\begin{pmatrix}\gamma_1\{x(\eta-t)\}^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \gamma_r\{x(\eta-t)\}^{\lambda_r}\end{pmatrix}$$

$$\times S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{pmatrix}(t-\zeta)^{a_1}(\eta-t)^{b_1}\\ \cdot\\ \cdot\\ \cdot\\ (t-\zeta)^{a_s}(\eta-t)^{b_s}\end{pmatrix}$$

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$$\times I\begin{pmatrix} z_1\{t(t-\zeta)\}^{\lambda_1}\{t(\eta-t)\}^{\tau_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r\{t(t-\zeta)\}^{\lambda_r}\{t(\eta-t)\}^{\tau_r} \end{pmatrix} = \sum_{m,l=0}^{\infty}\sum_{k,M=0}^{\infty}$$

$$\sum_{k_1,\ldots,k_s=0}^{M_1k_1+\ldots+M_sk_s\leq N}(-N)_{M_1k_1+\ldots+M_sk_s}A(N;k_1,\ldots,k_s)\Omega$$

$$\times I^{\{0,n_i+3\}_{2,r}:\{m^{(i)},n^{(i)}\}^{1,r}}_{\{p_i+3,q_i+3\}_{2,r}:\{p^{(i)},q^{(i)}\}^{1,r}}$$

$$\left(\begin{matrix} z_1(-\zeta)^{\lambda_1}(\eta)^{\tau_1}t^{\lambda_1+\tau_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(-\zeta)^{\lambda_r}(\eta)^{\tau_r}t^{\lambda_r+\tau_r} \end{matrix}\middle| \begin{matrix} A_1,A_2,A_3, \\ \\ \\ \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\ldots; \end{matrix}\right.$$

$$\left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\ldots;\left(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj}\right)_{1,p_r},$$

$$\left(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj}\right)_{1,q_r},B_1,B_2,B_3:(b'_j,\beta'_j)_{1,q^{(1)}};$$

$$\left.\begin{matrix} (a'_j,\alpha'_j)_{1,p^{(1)}};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ \ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{matrix}\right). \quad (20)$$

where

$$A_1=\left(-\lambda-\sum_{i'=1}^{s}a'_ik'_i;\lambda_1,\ldots,\lambda_r\right),$$

$$A_2=\left(\tau-k-\sum_{i'=1}^{s}b'_ik'_i;\tau_1,\ldots,\tau_r\right),$$

$$A_3=(-m-l;\lambda_1+\tau_1,\ldots,\lambda_r+\tau_r),$$

$$B_1=\left(m-\lambda-\sum_{i'=1}^{s}a'_ik'_i;\lambda_1,\ldots,\lambda_r\right),$$

$$B_2=\left(\eta-\lambda-k-\sum_{i'=1}^{s}b'_ik'_i;\tau_1,\ldots,\tau_r\right),$$

$$B_3=(-m+l-\mu;\lambda_1+\tau_1,\ldots,\lambda_r+\tau_r).$$

$$\Omega=\frac{(-1)^m(-1)^n(-\zeta)^{-l}(1+\alpha+\beta+2M)(-\lambda)_M}{k!M!(1+a+b+M)_{\lambda+1}(1+\alpha)_k}$$

$$\frac{(1+\alpha+\beta+M)_k(1+\alpha)_\lambda}{(m+1)!(n+1)!}x^k\times(-\zeta)^{\lambda-m+\sum_{i'}^{s}a'_ik'_i}$$

$$(\eta)^{\tau+k+\sum_{i'}^{s}b'_ik'_i}t^{m+l-\mu}F_M[z_1,\ldots,z_r];$$

$\lambda_i>0,\tau_i>0,i=1,2,\ldots,r.$

Also

$$Re(\tau)+\sum_{i=1}^{r}\tau_i\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>-1,$$

$$Re(\lambda)+\sum_{i=1}^{r}\lambda_i\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>-1$$

Proof. For a simple (direct) proof of the result (20), we first express the Lauricella function by eq.(18) and the multivariable $I-$function by eq.(1) and general class of polynomial (10) respectively and collecting the powers of $(t-\zeta)$ and $(\eta-t)$, and apply the binomial expansion

$$(x+\zeta)^\lambda=\zeta^\lambda\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(\frac{x}{\zeta}\right)^m,\left(|\frac{x}{\zeta}|<1\right)$$

by making use of the familiar formula

$$D_t^\mu(t^\lambda)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu},(Re(\lambda)>-1),$$

we get the desired result (20).

Theorem 4.2.

If $\lambda_i>0,\tau_i>0,i=1,2,\ldots,r$ and $n_i,p_i,q_i,m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ are non-negative integers and $a_{ij},b_{ij},\alpha_{ij},\beta_{ij},a_j^{(i)},b_j^{(i)},\alpha_j^{(i)},\beta_j^{(i)}$ are complex numbers, where $\alpha_i=min_{(1\leq j\leq m^{(i)})}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)$, $\beta_i=max_{(1\leq j\leq n^{(i)})}\Re\left(\frac{a_j^{(i)}-1}{\alpha_j^{(i)}}\right)$, and $max\{|z_1|,\ldots,|z_r|\}\to 0$,then

$$D_t^\mu(t-\zeta)^\lambda x^\lambda(\eta-t)^{\lambda+\tau}F\begin{pmatrix}\gamma_1\{x(\eta-t)\}^{\lambda_1} \\ \cdot \\ \cdot \\ \cdot \\ \gamma_r\{x(\eta-t)\}^{\lambda_r}\end{pmatrix}$$

$$\times S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix}(t-\zeta)^a(\eta-t)^{b_1} \\ (\eta-t)^{b_2} \\ \cdot \\ \cdot \\ \cdot \\ (\eta-t)^{b_s}\end{pmatrix}$$

$$\times I\begin{pmatrix} z_1\{t(t-\zeta)\}^c\{t(\eta-t)\}^{\tau_1} \\ z_2\{t(\eta-t)\}^{\tau_2} \\ \cdot \\ \cdot \\ \cdot \\ z_r\{t(\eta-t)\}^{\tau_r}\end{pmatrix}=\sum_{m,l=0}^{\infty}\sum_{k,M=0}^{\infty}$$

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$$\times \sum_{k_1,\ldots,k_s=0}^{M_1k_1+\ldots+M_sk_s\leq N} (-N)_{M_1k_1+\ldots+M_sk_s}$$

$$.A(N;k_1,\ldots,k_s)\Omega I^{\{0,n_i+2\}_{2,r}:\{m^{(i)}+1,n^{(i)}\}^{1,r}}_{\{p_i+2,q_i+2\}_{2,r}:\{p^{(i)}+1,q^{(i)}+1\}^{1,r}}$$

$$\left(\begin{array}{c|c} z_1(-\zeta)^{\lambda}(\eta)^{\tau_1}(t)^{\lambda+\tau_1} & C_1,C_2, \\ z_2(\eta)^{\tau_2}(t)^{\tau_2} & \\ \cdot & \\ \cdot & \\ \cdot & \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\ldots; \\ z_r(\eta)^{\tau_r}(t)^{\tau_r} & \end{array}\right.$$

$$\left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\ldots;\left(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj}\right)_{1,p_r},$$

$$\left(b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj}\right)_{1,q_r},D_1,D_2;(b'_j,\beta'_j)_{1,q^{(1)}},$$

$$\left.\begin{array}{c}(a'_j,\alpha'_j)_{1,p^{(1)}};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}},C_3 \\ \\ \ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}},D_3\end{array}\right). \qquad (21)$$

where
$C_1=\left(-\tau-k-\sum_{i'=1}^{s} b'_i k'_i;\lambda_1,\ldots,\lambda_r\right)$,
$C_2=(-m-l;\tau_1-c;\tau_2,\ldots,\tau_r)$,
$C_3=(1+\lambda+ak_1-m,c)$,
$D_1=\left(-\tau-k+l+\sum_{i'=1}^{s} b'_i k'_i;\tau_1,\ldots,\tau_r\right)$,
$D_2=(-m-l+\mu;\tau-c;\tau_1,\ldots,\tau_r)$,
$D_3=(1+\lambda+ak_1,c)$.

$$\Omega=\frac{(-1)^{m+l}(-\zeta)^{-l}(1+\alpha+\beta+2M)(-\lambda)_M}{k!M!(1+a+b+M)_{\lambda+1}(1+\alpha)_k}$$

$$\times\frac{(-M)_k(1+\alpha+\beta+M)_k(1+\alpha)_\lambda}{(m+1)!(n+1)!}x^k$$

$$\times(-\zeta)^{\lambda-m+ak_1}(\eta)^{\tau+k+\sum_{i'}^{s} b'_i k'_i}t^{m+l-\mu}F_M[\gamma_1,\ldots,\gamma_r];$$

$(\lambda_i>0,\tau_i>0),i=1,2,\ldots,r.$

Also

$$Re(\tau)+\sum_{i=1}^{r}\tau_i\left(\frac{b^{(i)}_j}{\beta^{(i)}_j}\right)>-1,$$

$$Re(\lambda)+\sum_{i=1}^{r}\lambda_i\left(\frac{b^{(i)}_j}{\beta^{(i)}_j}\right)>-1$$

Proof. In order to prove (21), we first express the Lauricella function by eq. (18) and the multivariable $I-$function by eq. (1) and general class of polynomial (10) respectively and collecting the powers of $(t-\zeta)$ and $(\eta-t)$, and apply the binomial expansion (15),making use of the familier formula (12),we get the desired result (21).

5 Special cases

(i) $\eta_i=0;i=1,2,\ldots,r-1$,the result (20) as given in section 4, reduces to the known main result given by Chaurasia and Shekhawat (see [[1] p. 497, Equation 3.1]).

(ii) When $a^{(i)}_j,b^{(i)}_j,\alpha^{(i)}_j,\beta^{(i)}_j$'s are real numbers and $n_i,p_i,q_i,i=1,2,\ldots,r$, the multivariable I-function breaks in to product of H-function of r-variable and equation (21) as given in section 4, reduces to the known result given by Chaurasia and Shekhawat (see [[1], p.498, Equation (4.1)]).

(iii) If $\alpha^{(i)}_j=\beta^{(i)}_j=1,i=1,2,\ldots,s$ in case (2), then the result of case (2) reduces to Meijers G-function, given by Chaurasia and Shekhawat (see [[1],p.499, Equation (4.2)]).

(iv) Letting $N_i=0(i=1,\ldots,s)$, the result in case (1), reduces to the known result given by Sharma and Singh [4], after a little simplification.

(v)Replacing $N_1,\ldots,N_s$ by N in case (1), we have a known result obtained by Chaurasia and Singhal [15].

If $A(N;k_1,\ldots,k_s)=I(Q)$, then $S^{M_1,\ldots,M_s}_N[w_1,\ldots,w_s]$ reduces to the generalized Lauricella function of Srivastava and Daoust

$$S^{M_1,\ldots,M_s}_N[w_1,\ldots,w_s]=F^{1+E:U^{(1)};\ldots;U^{(s)}}_{G:V^{(1)};\ldots;V^{(s)}}\left[\begin{array}{c}w_1\\ \cdot\\ \cdot\\ \cdot\\ w_s\end{array}\right|$$

$$[-N;M_1,\ldots,M_s],\left[(e);\eta^{(1)},\ldots,\eta^{(s)}\right]:$$

$$\left[(g);\zeta^{(1)},\ldots,\zeta^{(s)}\right]:$$

$$\left.\begin{array}{c}\left[U^{(1)}:x^{(1)}\right];\ldots;\left[U^{(s)}:x^{(s)}\right]\\ \left[V^{(1)}:t^{(1)}\right];\ldots;\left[V^{(s)}:x^{(s)}\right]\end{array}\right] \qquad (22)$$

$$D^{\mu}_t(t-\zeta)^{\lambda}x^{\lambda}(\eta-t)^{\lambda+\tau}F\left(\begin{array}{c}\gamma_1\{x(\eta-t)\}^{\lambda_1}\\ \cdot\\ \cdot\\ \cdot\\ \gamma_r\{x(\eta-t)\}^{\lambda_r}\end{array}\right.$$

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$$F^{1+E:U^{(1)};...;U^{(s)}}_{G:V^{(1)};...;V^{(s)}}\left[\begin{matrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_s \end{matrix}\middle| \begin{matrix}[-N;M_1,...,M_s], \\ \\ ... \end{matrix}\right.$$

$$\left.\begin{matrix}\left[(e);\eta^{(1)},...,\eta^{(s)}\right]:\left[U^{(1)}:x^{(1)}\right];...;\left[U^{(s)}:x^{(s)}\right] \\ \left[(g);\zeta^{(1)},...,\zeta^{(s)}\right]:\left[V^{(1)}:t^{(1)}\right];...;\left[V^{(s)}:x^{(s)}\right]\end{matrix}\right]$$

$$\times I\begin{pmatrix} z_1\{t(t-\zeta)\}^{\lambda_1}\{t(\eta-t)\}^{\tau_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r\{t(t-\zeta)\}^{\lambda_r}\{t(\eta-t)\}^{\tau_r}\end{pmatrix}$$

$$\times \sum_{m,l=0}^{\infty}\sum_{k,M=0}^{\infty}\sum_{k_1,...,k_s=0}^{M_1k_1+...+M_sk_s\leq N}\Delta$$

$$\times I^{\{0,n_i+3\}_{2,r}:\{m^{(i)},n^{(i)}\}^{1,r}}_{\{p_i+3,q_i+3\}_{2,r}:\{p^{(i)},q^{(i)}\}^{1,r}}$$

$$\left(\begin{matrix} z_1(-\zeta)^{\lambda_1}(\eta)^{\tau_1}t^{\lambda_1+\tau_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(-\zeta)^{\lambda_r}(\eta)^{\tau_r}t^{\lambda_r+\tau_r}\end{matrix}\middle| \begin{matrix} A_1,A_2,A_3, \\ \\ \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};...;\end{matrix}\right.$$

$$\left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};...;\left(a_{rj};\alpha'_{rj},...,\alpha^{(r)}_{rj}\right)_{1,p_r},$$

$$\left(b_{rj};\beta'_{rj},...,\beta^{(r)}_{rj}\right)_{1,q_r},B_1,B_2,B_3;(b'_j,\beta'_j)_{1,q^{(1)}};$$

$$\left.\begin{matrix}(a'_j,\alpha'_j)_{1,p^{(1)}};...;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ ...;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right). \tag{23}$$

where

$$A_1=\left(-\lambda-\sum_{i'=1}^{s}a'_ik'_i;\lambda_1,...,\lambda_r\right),$$

$$A_2=\left(\tau-k-\sum_{i'=1}^{s}b'_ik'_i;\tau_1,...,\tau_r\right),$$

$$A_3=(-m-l;\lambda_1+\tau_1,...,\lambda_r+\tau_r),$$

$$B_1=\left(m-\lambda-\sum_{i'=1}^{s}a'_ik'_i;\lambda_1,...,\lambda_r\right),$$

$$B_2=\left(\eta-\lambda-k-\sum_{i'=1}^{s}b'_ik'_i;\tau_1,...,\tau_r\right),$$

$$B_3=(-m+l-\mu;\lambda_1+\tau_1,...,\lambda_r+\tau_r).$$

Also Δ is defined as Ω according to main result. Equation(23) is valid under the same conditions as those in main result.

6 Conclusion

The I-function of r-variable defined by Prasad [6]and Prasad and Yadav [3] in terms of the Mellin-Barnes type of basic integrals is most general character which involves a number of special functions. The fractional integral and derivative operators, especially, involving various special functions have found significant importance and application in various field applied mathematics. The results deuced in the present paper may provide better fractional derivative and general polynomial of some simpler multivariable special function, which are expressible in terms of the H-function, the I-function, the G-function of one variable and their special cases. Thus the results presented in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problems of science, engineering and mathematically physics etc.

Acknowledgement

The authors are thankful to anonymous referees for their useful suggestions and helpful comments towards improvement of the paper.

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