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Symbolic routes to BBP-type formulas of any degree in arbitrary bases

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Digit extraction (BBP-Type) formulas are usually discovered through computer searches. In this paper we present two alternative approaches which lead directly to the discovery of digit extraction formulas and thus remove the burden of finding the formulas first and finding the proofs later. We give various identities which generate a wide range of digit extraction formulas of any degree, in arbitrary bases.

Keywords: BBP-Type formulas, digit extraction, polylogarithms.

1 Introduction

It is known that formulas of the type

$$C = P(s,b,l,A) = \sum_{k=0}^{\infty} \left(\frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(lk+j)^s} \right) \tag{1.1}$$

allow for digit extraction — the i-th digit of a mathematical constant C in base b can be calculated directly, without needing to compute any of the previous $i-1$ digits, by means of simple algorithms that do not require multiple-precision arithmetic [1]. In Eq. (1.1), b, l and s are integers: the base, the length and the degree of the formula, respectively; while $A = (a_1, a_2, \ldots, a_l)$ is a vector of integers. Formula (1.1) is called a BBP-Type formula for C, after the authors (Bailey, Borwein and Plouffe) of the paper [2] in which such a formula was first presented. In general, any constant C that can be written in the form

$$C = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}\,,$$

where p and q are integer polynomials with degree of $p <$ degree of q, and $p(k)/q(k)$ is nonsingular for $k \geq 0$, possesses this digit extraction property [1]. A BBP algorithm for the extraction of the hexadecimal digits of π after the first i hex digits, based on the original

BBP formula can be found in [3]. Computing the binary digits of $\ln 2$ is also discussed in the cited reference.

Apart from digit extraction, another reason the study of BBP-Type formulas has continued to attract attention is that BBP-Type constants are conjectured to be either rational or normal to base b [1, 4–6], that is their base-b digits are randomly distributed.

David Bailey maintains a Compendium of BBP-Type formulas for Mathematical constants on his website [1]. A nice collection of such formulas may also be found in MathWorld [7] while there is a nice article on the subject in Wikipedia [8].

Experimentally, BBP-Type formulas are usually discovered by using Bailey and Ferguson's PSLQ (Partial Sum of Least Squares) algorithm [9] or its variations. A drawback is that PSLQ and other integer relation finding schemes typically do not suggest proofs [5,10]. Formal proofs must be developed after the formulas have been discovered. There have been attempts in the past to give general formulas which include the proofs [4, 11–16, 18].

In this paper we give various identities which generate a wide range of BBP-Type formulas.

2 Generators of BBP-Type formulas

In an earlier paper [15], we gave the following generators of degree 1 BBP-Type formulas:

$$-\frac{1}{2}\ln(1-2p\cos x+p^2)=\sum_{k=1}^{\infty}\frac{p^k\cos(kx)}{k}\,, \tag{2.1}$$

and

$$\arctan\left(\frac{p\sin x}{1-p\cos x}\right)=\sum_{k=1}^{\infty}\frac{p^k\sin(kx)}{k} \tag{2.2}$$

for x, p real and $|p|<1$.

We demonstrated that careful choices of p and x in Eq. (2.1) and Eq. (2.2) lead to interesting BBP-Type series.

In the present article we extend the discussion to higher degrees. In section 3 BBP-Type formulas obtained from polylogarithmic generalizations of equations (2.1) and (2.2) are discussed while section 4 focuses on BBP-Type formulas derived by repeated integration of degree 1 generators.

3 BBP-Type formulas generated by polylogarithm identities

One way to obtain higher degree formulas is to write a more general version of each of Eqs. (2.1) and (2.2) thus:

$$\operatorname{Re}\left(\operatorname{Li}_s(pe^{ix})\right)=\sum_{k=1}^{\infty}\frac{p^k\cos kx}{k^s} \tag{3.1}$$

and

$$\operatorname{Im}\left(\operatorname{Li_s}(pe^{ix})\right) = \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k^s}\,, \tag{3.2}$$

where Li_s is an index $s \in \mathbb{Z}^+$ polylogarithm function, and Re and Im denote real and imaginary parts respectively.

In general, $\operatorname{Li}_s\left(pe^{ix}\right)$ cannot be expressed in terms of elementary or tabulated functions. For $s = 1$, $\operatorname{Li}_s\left(pe^{ix}\right)$ is elementary, evaluating to the logarithm and arctangent functions. The BBP-Type formulas resulting from case $s = 1$ have been discussed in reference [15]. For $s = 2$ (the dilogarithm function) the imaginary part of $\operatorname{Li}_s\left(pe^{ix}\right)$ can always be expressed in terms of the Clausen integral [17], which is a tabulated function, while the real part can be expressed as an integral which can sometimes be evaluated in closed form or in terms of simpler polylogarithm terms. For a general s, polylogarithm ladders may be employed to discover BBP-Type formulas as was done for example in references [6, 12, 16, 18]. The focus in this present paper is to obtain BBP-Type formulas directly from the polylogarithm identities Eqs. (3.1) and (3.2). This way, the formulas are very general and in some lucky instances the polylogarithm forms can be evaluated for some special values of p and x.

When $s = 2$, we have [17]

$$\operatorname{Re}\left[\operatorname{Li}_2\left(pe^{ix}\right)\right] = -\frac{1}{2}\int_0^p \frac{\ln(1-2y\cos x+y^2)}{y}\mathrm{d}y \tag{3.3}$$

and

$$\operatorname{Im}\left[\operatorname{Li}_2\left(pe^{ix}\right)\right] = \omega\ln p + \frac{1}{2}\operatorname{Cl}_2(2\omega) - \frac{1}{2}\operatorname{Cl}_2(2\omega+2x) + \frac{1}{2}\operatorname{Cl}_2(2x)\,, \tag{3.4}$$

where

$$\omega = \arctan\left(\frac{p\sin x}{1-p\cos x}\right)$$

and Cl_2 is Clausen integral defined by

$$\operatorname{Cl}_2(\theta) = -\int_0^\theta \ln\left(2\sin\tfrac{1}{2}y\right)\mathrm{d}y\,.$$

The BBP-Type formulas resulting from Eq. (3.1) and Eq. (3.2) are presented in the following sections.

3.1 BBP-Type formulas generated by $x = \pi/2$ in Eqs. (3.1) and (3.2)

With $x = \pi/2$ in Eqs. (3.1) and (3.2), we have

$$\operatorname{Re}\left[\operatorname{Li_s}\left(pe^{\frac{i\pi}{2}}\right)\right] = \sum_{k=1}^{\infty} p^k \frac{\cos(\frac{k\pi}{2})}{k^s} \tag{3.5}$$

and

$$\operatorname{Im}\left[\mathrm{Li_s}\left(pe^{\frac{i\pi}{2}}\right)\right] = \sum_{k=1}^{\infty} p^k \frac{\sin(\frac{k\pi}{2})}{k^s}\,. \tag{3.6}$$

By noting that

$$\sum_{k=1}^{\infty} p^k \frac{\cos(k\pi/2)}{k^s} = \frac{p^2}{2^s}\sum_{k=0}^{\infty} p^{4k}\left[-\frac{1}{(2k+1)^s} + \frac{p^2}{(2k+2)^s}\right]$$

and

$$\sum_{k=1}^{\infty} p^k \frac{\sin(k\pi/2)}{k^s} = p\sum_{k=0}^{\infty} p^{4k}\left[\frac{1}{(4k+1)^s} - \frac{p^2}{(4k+3)^s}\right],$$

and upon setting $p = 1/\sqrt{n}$, n a positive integer, we are able to write Eqs. (3.5) and (3.6) respectively as

$$\operatorname{Re}\left[\mathrm{Li_s}\left(\frac{i}{\sqrt{n}}\right)\right] = \frac{1}{2^s n^2}\sum_{k=0}^{\infty}\frac{1}{(n^2)^k}\left[-\frac{n}{(2k+1)^s} + \frac{1}{(2k+2)^s}\right] \tag{3.7}$$

and

$$\operatorname{Im}\left[\mathrm{Li_s}\left(\frac{i}{\sqrt{n}}\right)\right] = \frac{1}{n\sqrt{n}}\sum_{k=0}^{\infty}\frac{1}{(n^2)^k}\left[\frac{n}{(4k+1)^s} - \frac{1}{(4k+3)^s}\right]. \tag{3.8}$$

With $s = 2$ and using Eqs. (3.3) and (3.4), we have

$$\mathrm{Li}_2\left(-\frac{1}{n}\right) = \frac{1}{n^2}\sum_{k=0}^{\infty}\frac{1}{(n^2)^k}\left[-\frac{n}{(2k+1)^2} + \frac{1}{(2k+2)^2}\right] \tag{3.9}$$

and

$$\begin{aligned} &-\frac{1}{2}\omega\ln n + \frac{1}{2}\mathrm{Cl}_2(2\omega) - \frac{1}{2}\mathrm{Cl}_2(2\omega+\pi) \\ &\quad = \frac{1}{n\sqrt{n}}\sum_{k=0}^{\infty}\frac{1}{(n^2)^k}\left[\frac{n}{(4k+1)^2} - \frac{1}{(4k+3)^2}\right], \end{aligned} \tag{3.10}$$

where $\omega = \arctan(1/\sqrt{n})$. In particular, when $n = 3$, then $\omega = \pi/6$ and Eqs. (3.9) and (3.10) give

$$\mathrm{Li}_2\left(-\frac{1}{3}\right) = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{-3}{(2k+1)^2} + \frac{1}{(2k+2)^2}\right] \tag{3.11}$$

$$-\frac{\pi}{4}\ln 3 + \frac{5}{2}\mathrm{Cl}_2(\frac{\pi}{3}) = \frac{1}{\sqrt{3}}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{3}{(4k+1)^2} - \frac{1}{(4k+3)^2}\right]. \tag{3.12}$$

In obtaining Eq. (3.12) we used the identity [17]

$$\frac{1}{2}\mathrm{Cl}_2(2\theta) = \mathrm{Cl}_2(\theta) - \mathrm{Cl}_2(\pi-\theta)\,. \tag{3.13}$$

Using reasoning that is completely analogous to that in the present section, we now present the various formulas generated by $x = \pi/3$, $x = \pi/4$ and $x = \pi/6$ respectively.

3.2 BBP-Type formulas generated by $x = \pi/3$ in Eqs. (3.1) and (3.2)

$$\mathrm{Re}\left[\mathrm{Li}_s\left(\frac{1}{n}\exp(\frac{i\pi}{3})\right)\right] = \frac{1}{2n^3}\sum_{k=0}^{\infty}\frac{1}{(-n^3)^k}\left[\frac{n^2}{(3k+1)^s} - \frac{n}{(3k+2)^s} - \frac{2}{(3k+3)^s}\right] \tag{3.14}$$

and

$$\mathrm{Im}\left[\mathrm{Li}_s\left(\frac{1}{n}\exp(\frac{i\pi}{3})\right)\right] = \frac{\sqrt{3}}{2n^2}\sum_{k=0}^{\infty}\frac{1}{(-n^3)^k}\left[\frac{n}{(3k+1)^s} + \frac{1}{(3k+2)^s}\right] \tag{3.15}$$

$n \in \mathbb{Z}$, $n \neq 0$ and $s \in \mathbb{Z}^+$.

With $s = 2$ in Eq. (3.15) and using Eq. (3.4), we have

$$\begin{aligned} &-\omega\ln n + \frac{1}{2}\mathrm{Cl}_2(2\omega) - \frac{1}{2}\mathrm{Cl}_2\left(2\omega + \frac{2\pi}{3}\right) + \frac{1}{2}\mathrm{Cl}_2(\frac{2\pi}{3}) \\ &= \frac{\sqrt{3}}{2n^2}\sum_{k=0}^{\infty}\frac{1}{(-n^3)^k}\left[\frac{n}{(3k+1)^2} + \frac{1}{(3k+2)^2}\right] \end{aligned} \tag{3.16}$$

where

$$\omega = \arctan\left(\frac{\sqrt{3}}{2n-1}\right).$$

The particular case $n = 2$ ($\omega = \pi/6$) in Eq. (3.16) gives

$$-\frac{\pi}{3}\ln 2 + \frac{5}{3}\mathrm{Cl}_2(\frac{\pi}{3}) = \frac{\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(-8)^k}\left[\frac{2}{(3k+1)^2} + \frac{1}{(3k+2)^2}\right], \tag{3.17}$$

where we again made use of formula (3.13).

3.3 BBP-Type formulas generated by $x = \pi/4$ in Eqs. (3.1) and (3.2)

$$\begin{aligned} &\mathrm{Re}\left[\mathrm{Li}_s\left(\frac{1}{n\sqrt{2}}\exp(\frac{i\pi}{4})\right)\right] \\ &= \frac{1}{4n^4}\sum_{k=0}^{\infty}\frac{1}{(-4n^4)^k}\left[\frac{2n^3}{(4k+1)^s} - \frac{n}{(4k+3)^s} - \frac{1}{(4k+4)^s}\right] \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\mathrm{Im}\left[\mathrm{Li}_s\left(\frac{1}{n\sqrt{2}}\exp(\frac{i\pi}{4})\right)\right] \\ &= \frac{1}{4n^3}\sum_{k=0}^{\infty}\frac{1}{(-4n^4)^k}\left[\frac{2n^2}{(4k+1)^s} + \frac{2n}{(4k+2)^s} + \frac{1}{(4k+3)^s}\right] \end{aligned} \tag{3.19}$$

$n \in \mathbb{Z}$, $n \neq 0$ and $s \in \mathbb{Z}^+$With $s = 2$ in Eq. (3.19) and using Eq. (3.4), we have

$$
\begin{aligned}
&-\omega\ln(2n^2) + \mathrm{Cl}_2(2\omega) - \mathrm{Cl}_2\left(2\omega + \frac{\pi}{2}\right) + \mathrm{G} \\
&= \frac{1}{2n^3}\sum_{k=0}^{\infty}\frac{1}{(-4n^4)^k}\left[\frac{2n^2}{(4k+1)^2} + \frac{2n}{(4k+2)^2} + \frac{1}{(4k+3)^2}\right]
\end{aligned} \tag{3.20}
$$

where G is Catalan's constant and $\omega = \arctan(1/(2n-1))$. In obtaining Eq. (3.20) we used $\mathrm{Cl}_2(\pi/2) = \mathrm{G}$ [19].

The particular case $n = 1$ in Eqs. (3.18) and (3.19) give the BBP-Type formulas

$$
-\frac{1}{2}\ln^2 2 + \frac{5\pi^2}{24} = \sum_{k=0}^{\infty}\frac{1}{(-4)^k}\left[\frac{2}{(4k+1)^2} - \frac{1}{(4k+3)^2} - \frac{1}{(4k+4)^2}\right] \tag{3.21}
$$

and

$$
\mathrm{G} - \frac{\pi\ln 2}{8} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(-4)^k}\left[\frac{2}{(4k+1)^2} + \frac{2}{(4k+2)^2} + \frac{1}{(4k+3)^2}\right] \tag{3.22}
$$

In obtaining Eqs. (3.21) and (3.22) we made use of the formula [20]

$$
\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] = \mathrm{Li}_2\left[\frac{(1+i)}{2}\right] = \left(\frac{5\pi^2}{96} - \frac{\ln^2 2}{8}\right) + i\left(\mathrm{G} - \frac{\pi\ln 2}{8}\right) .
$$

Note that, alternatively, formula (3.22) can also be obtained directly from Eq. (3.20) with $n = 1$. Eq. (3.22) is the alternating version of formula (25) of Bailey's Compendium.

Formulas (3.21) and (3.22) can also be found in reference [18].

3.4 BBP-Type formulas generated by $x = \pi/6$ in Eqs. (3.1) and (3.2)

$$
\begin{aligned}
\mathrm{Re}\left[\mathrm{Li}_s\left(\frac{1}{n\sqrt{3}}\exp(\frac{i\pi}{6})\right)\right] = \frac{1}{54n^6}\sum_{k=0}^{\infty}\frac{1}{(-27n^6)^k}&\left[\frac{27n^5}{(6k+1)^s} + \frac{9n^4}{(6k+2)^s}\right. \\
&\left.-\frac{3n^2}{(6k+4)^s} - \frac{3n}{(6k+5)^s} - \frac{2}{(6k+6)^s}\right]
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
\mathrm{Im}\left[\mathrm{Li}_s\left(\frac{1}{n\sqrt{3}}\exp(\frac{i\pi}{6})\right)\right] = \frac{\sqrt{3}}{54n^5}\sum_{k=0}^{\infty}\frac{1}{(-27n^6)^k}&\left[\frac{9n^4}{(6k+1)^s} + \frac{9n^3}{(6k+2)^s}\right. \\
&\left.+\frac{6n^2}{(6k+3)^s} + \frac{3n}{(6k+4)^s} + \frac{1}{(6k+5)^s}\right] ,
\end{aligned} \tag{3.24}
$$

$n \in \mathbb{Z}$, $n \neq 0$ and $s \in \mathbb{Z}^+$.

With $s = 2$ and using Eq. (3.4), Eq. (3.24) can be written

$$\begin{aligned}
&-\omega\ln(3n^2) + \mathrm{Cl}_2(2\omega) - \mathrm{Cl}_2\left(2\omega + \frac{\pi}{3}\right) + \mathrm{Cl}_2(\frac{\pi}{3})\\
&\quad = \frac{\sqrt{3}}{27n^5}\sum_{k=0}^{\infty}\frac{1}{(-27n^6)^k}\left[\frac{9n^4}{(6k+1)^2} + \frac{9n^3}{(6k+2)^2} + \frac{6n^2}{(6k+3)^2}\right.\\
&\qquad\left. + \frac{3n}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right],
\end{aligned} \tag{3.25}$$

where

$$\omega = \arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2n-1}\right).$$

The particular case $n = 1$ gives

$$\begin{aligned}
-\frac{\pi}{4}\ln 3 + 2\mathrm{Cl}_2(\frac{\pi}{3}) = \frac{\sqrt{3}}{9}\sum_{k=0}^{\infty}\frac{1}{(-27)^k}&\left[\frac{9}{(6k+1)^2} + \frac{9}{(6k+2)^2} + \frac{6}{(6k+3)^2}\right.\\
&\left. + \frac{3}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right].
\end{aligned} \tag{3.26}$$

4 Higher degree BBP-Type formulas by integration

Obviously, repeated integration of Eqs. (2.1) and (2.2) with respect to x, with p as some special function (usually trigonometric) of x will also lead to higher degree BBP-Type formulas. Presently we look at some instances.

4.1 BBP-Type formulas generated by substituting $p = 1/(2\cos x)$ in Eq.(2.1) and Eq.(2.2)

The use of $2p\cos x = 1$ in Eq. (2.1) and Eq. (2.2) give:

$$\ln(2\cos x) = \sum_{k=1}^{\infty}\left(\frac{1}{2\cos x}\right)^k\frac{\cos kx}{k}, \tag{4.1}$$

and

$$x = \sum_{k=1}^{\infty}\left(\frac{1}{2\cos x}\right)^k\frac{\sin kx}{k}. \tag{4.2}$$

The degree 1 BBP-type formulas generated by Eq (4.1) and Eq (4.2) have already been discussed elsewhere [15]. Generators of higher degree BBP-type formulas may be obtained by integrating Eq (4.1) and Eq (4.2) repeatedly with respect to x. The integration is justified by uniform convergence.

Integrating once, we obtain the following generators:

$$-x\ln(2\cos x)+2\int_0^x \ln(2\cos y)dy=\sum_{k=1}^\infty\left(\frac{1}{2\cos x}\right)^k\frac{\sin kx}{k^2} \tag{4.3}$$

and

$$-\ln^2(2\cos x)-x^2+\pi^2/6=2\sum_{k=1}^\infty\frac{1}{(2\cos x)^k}\frac{\cos kx}{k^2}\,, \tag{4.4}$$

for $x\in[0,\pi/2)$.

Setting $x=0$, $x=\pi/4$ and $x=\pi/6$, respectively, in Eq. (4.4) lead to the following interesting BBP-Type formulas:

$$-\ln^2 2+\frac{\pi^2}{6}=\sum_{k=0}^\infty\frac{1}{2^k}\frac{1}{(k+1)^2}\,, \tag{4.5}$$

$$-\frac{1}{2}\ln^2 2+\frac{5\pi^2}{24}=\sum_{k=0}^\infty\frac{1}{(-4)^k}\left[\frac{2}{(4k+1)^2}-\frac{1}{(4k+3)^2}-\frac{1}{(4k+4)^2}\right] \tag{4.6}$$

and

$$-\frac{1}{4}\ln^2 3+\frac{5\pi^2}{36}=\frac{1}{27}\sum_{k=0}^\infty\frac{1}{(-27)^k}\left[\frac{27}{(6k+1)^2}+\frac{9}{(6k+2)^2}-\frac{3}{(6k+4)^2}-\frac{3}{(6k+5)^2}-\frac{2}{(6k+6)^2}\right]. \tag{4.7}$$

Note that Eq. (4.6) is identical to Eq. (3.21) obtained directly from a polylogarithm identity.

Note also that Eq. (4.7) may also be obtained by setting $n=1$ and $s=2$ in Eq. (3.23).

Eqs. (4.5) and (4.6) can be written in base 16, length 8 thus:

$$-\frac{5}{16}\ln^2 2+\frac{5\pi^2}{96}=\frac{5}{2^5}\sum_{k=0}^\infty\frac{1}{2^{4k}}\left[\frac{2^3}{(8k+2)^2}+\frac{2^2}{(8k+4)^2}+\frac{2^1}{(8k+6)^2}+\frac{1}{(8k+8)^2}\right] \tag{4.8}$$

$$-\frac{1}{8}\ln^2 2+\frac{5\pi^2}{96}=\frac{1}{2^5}\sum_{k=0}^\infty\frac{1}{2^{4k}}\left[\frac{2^4}{(8k+1)^2}-\frac{2^3}{(8k+3)^2}-\frac{2^3}{(8k+4)^2}-\frac{2^2}{(8k+5)^2}+\frac{2}{(8k+7)^2}+\frac{2}{(8k+8)^2}\right]. \tag{4.9}$$

Subtracting Eq. (4.8) from Eq. (4.9) gives

$$\ln^2 2=\frac{1}{6}\sum_{k=0}^\infty\frac{1}{2^{4k}}\left[\frac{2^4}{(8k+1)^2}-\frac{5\cdot 2^3}{(8k+2)^2}-\frac{2^3}{(8k+3)^2}-\frac{7\cdot 2^2}{(8k+4)^2}-\frac{2^2}{(8k+5)^2}-\frac{5\cdot 2^1}{(8k+6)^2}+\frac{2}{(8k+7)^2}-\frac{3}{(8k+8)^2}\right]. \tag{4.10}$$

Eq. (4.10) is formula (23) of Bailey's Compendium, listed as not proved.

Incidentally, since the sum in Eq. (4.4) converges uniformly, termwise differentiation is allowed, and gives the following generator of degree 1 BBP-Type formulas:

$$-\tan x\ln(2\cos x)+x=\frac{1}{2\cos^2 x}\sum_{k=1}^{\infty}\frac{1}{(2\cos x)^k}\frac{\sin kx}{k+1}\,. \tag{4.11}$$

Setting $x=\pi/4$ and $x=\pi/6$ respectively in Eq. (4.11) yield the following BBP-Type formulas:

$$\pi-2\ln 2=\sum_{k=0}^{\infty}\frac{1}{(-4)^k}\left[\frac{2}{4k+2}+\frac{2}{4k+3}+\frac{1}{4k+4}\right] \tag{4.12}$$

and

$$9\pi\sqrt{3}-27\ln 3=2\sum_{k=0}^{\infty}\frac{1}{(-27)^k}\left[\frac{9}{6k+2}+\frac{9}{6k+3}+\frac{6}{6k+4}+\frac{3}{6k+5}+\frac{1}{6k+6}\right]. \tag{4.13}$$

Combining equations (46) and (47) from Bailey's Compendium with Eq. (4.13) leads to the ternary zero relation

$$\begin{aligned}0=\sum_{k=0}^{\infty}\frac{1}{(729)^k}&\left[\frac{729}{12k+1}-\frac{1134}{12k+2}-\frac{162}{12k+3}-\frac{243}{12k+4}-\frac{54}{12k+5}\right.\\&\left.-\frac{180}{12k+6}-\frac{27}{12k+7}-\frac{18}{12k+8}+\frac{6}{12k+9}-\frac{11}{12k+10}+\frac{2}{12k+11}\right].\end{aligned} \tag{4.14}$$

Eq. (4.14) is a simple linear combination of relations (78) and (79) in the Compendium, both of which are listed as not yet proved.

4.2 BBP-Type formulas generated by substituting $p=\cos x$ in Eq.(2.1) and Eq.(2.2)

Setting $p=\cos x$ in Eq. (2.1) and Eq. (2.2) gives the following:

$$-\ln\sin x=\sum_{k=1}^{\infty}\frac{\cos^k x\cos kx}{k} \tag{4.15}$$

and

$$\frac{\pi}{2}-x=\sum_{k=1}^{\infty}\frac{\cos^k x\sin kx}{k}\,. \tag{4.16}$$

Integration with respect to x gives:

$$(\frac{\pi}{2}-x)\ln(\cos x)-\int_0^x\ln\tan y\,dy=\sum_{k=1}^{\infty}\frac{\cos^k x\sin kx}{k^2} \tag{4.17}$$

and

$$\frac{\pi^2}{8} - \frac{\pi x}{2} + \frac{x^2}{2} + \frac{1}{4}\mathrm{Li}_2(\cos^2 x) = \sum_{k=1}^{\infty} \frac{\cos^k x \cos kx}{k^2} \,. \tag{4.18}$$

Setting $x = \pi/3$ in Eq. (4.18) gives the following degree 2 BBP-Type formula:

$$\frac{\pi^2}{72} + \frac{1}{4}\mathrm{Li}_2(\frac{1}{4}) = \frac{1}{16}\sum_{k=0}^{\infty} \frac{1}{(-8)^k}\left[\frac{4}{(3k+1)^2} - \frac{2}{(3k+2)^2} - \frac{2}{(3k+3)^2}\right], \tag{4.19}$$

which can be rewritten

$$\pi^2 = \frac{9}{8}\sum_{k=0}^{\infty} \frac{1}{2^{6k}}\left[\frac{2^4}{(6k+1)^2} - \frac{3\cdot 2^3}{(6k+2)^2} - \frac{2^3}{(6k+3)^2} - \frac{3\cdot 2^1}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right].$$

5 Conclusion

In this paper we have discussed two main approaches to discovering BBP-Type formulas: derivations from polylogarithm identities and repeated integration of lower order BBP-Type formulas. Many BBP-Type formulas in arbitrary bases were discovered, without doing any computer searches. Many known BBP-Type formulas were rediscovered and many new ones discovered.

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