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New Identities for Carlitz's Twisted (h,q) -Euler Polynomials under Symmetric Group of Degree n

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Abstract: In this paper, we consider the Carlitz's twisted (h,q)-Euler polynomials and give some new symmetric identities of these polynomials arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree n.

Keywords: Symmetric identities; Carlitz's twisted (h,q)-Euler polynomials; Fermionic p-adic q-integral on $\mathbb{Z}_p$; Invariant under S_n.
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1 Introduction

In the last years, symmetric identities of some special polynomials, such as q-Genocchi polynomials of higher order under third Dihedral group D_3 in [1], q-Genocchi polynomials under the symmetric group of degree four in [4], weighted q-Genocchi polynomials under the symmetric group of degree four in [5], q-Frobenius-Euler polynomials under symmetric group of degree five in [3], Carlitz's-type q-Euler polynomials invariant under the symmetric group of degree five in [11], q-Euler polynomials derived from fermionic integral on $\mathbb{Z}_p$ under S_3 in [6], q-Bernoulli polynomials under the symmetric group of degree n in [8], q-Euler polynomials under the symmetric group of degree n in [12] have been studied extensively.

Let $\mathbb{N}=\{1,2,3,\cdots\}$, $\mathbb{N}^*=\mathbb{N}\cup\{0\}$ and p be a prime number with $p\equiv 1\,(\text{mod}\,2)$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote, respectively, the ring of p-adic rational integers, the field of rational numbers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p=p^{-1}$. The notation "q" can be considered as an indeterminate, a complex number $q\in\mathbb{C}$ with $|q|<1$, or a p-adic number $q\in\mathbb{C}_p$ with $|q-1|_p<p^{-\frac{1}{p-1}}$ and $q^x=\exp(x\log q)$ for $|x|_p\leq 1$. For any x, q number of x (or q analog of x) is defined as $[x]_q=\frac{1-q^x}{1-q}=1+q+q^2+\cdots q^{x-1}$. Expressly $\lim_{q\to 1}[x]_q=x$ (see [1-13]).

The p-adic q-integral on $\mathbb{Z}_p$ of a function $f\in UD(\mathbb{Z}_p)$ is defined by Kim [11]:

$$I_q(f)=\int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) \tag{1}$$
$$=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)\,q^x.$$

In [10], Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\lim_{q\to -q} I_q(f)\ :=I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) \tag{2}$$
$$=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)\,(-q)^x.$$

Further, in the special case $q\to 1$ in Eq. (2), the integral

$$I_{-1}(f)=\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) \tag{3}$$
$$=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\,(-1)^x$$

is called as the fermionic p-adic invariant integral on $\mathbb{Z}_p$, see [6] and [7].

By the Eq. (3), it can be derived easily that

$$I_{-1}(f_n)=(-1)^n I_{-1}(f)+2\sum_{r=0}^{n-1}(-1)^{n-r-1} f(r)$$

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where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$, one can refer [1], [3], [4], [5], [6], [8], [9], [12], [14] and [15].

The Euler polynomials $E_n(x)$ are defined by the exponential generation function to be

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad (|t| < \pi). \tag{4}$$

When we take $x = 0$ in the Eq. (4), we then get $E_n(0) := E_n$ that is widely known n-th Euler number (see, e.g., [6], [9], [10], [12], [13], [14], [15], [16]).

As a q-generalization of $E_n(x)$, Kim defined the q-Euler polynomials with Witt's formula by using fermionic p-adic q-integral, in [9]:

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_{-q}(y) = E_{n,q}(x),$$

also, putting $x = 0$ in the above equation gives $E_{n,q}(0) := E_{n,q}$ known as n-*th* q-Euler polynomials.

Let $h \in \mathbb{Z}$ and $T_p = \bigcup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \left\{ w : w^{p^N} = 1 \right\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to C_p$ the locally constant function $x \to w^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $w \in T_p$, the Carlitz's twisted (h,q)-Euler polynomials are defined by the following p-adic fermionic q-integral on $\mathbb{Z}_p$ in [14]:

$$\mathscr{E}_{n,q,w}^{(h)}(x) = \int_{\mathbb{Z}_p} w^y q^{hy} [x+y]_q^n \, d\mu_{-1}(y) \quad (n \geq 0). \tag{5}$$

Letting $x = 0$ into the Eq. (5), we get $\mathscr{E}_{n,q,w}^{(h)}(0) := \mathscr{E}_{n,q,w}^{(h)}$ called n-th Carlitz's twisted (h,q)-Euler numbers.

Taking $w = 1$ and $q \to 1$ in the Eq. (5) yields to

$$\mathscr{E}_{n,q,w}^{(h)}(x) \to E_n(x) := \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_{-1}(y).$$

In the next section, we give new symmetric identities of Carlitz's twisted (h,q)-Euler polynomials associated with the fermionic p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree n shown by S_n.

2 New identities for $\mathscr{E}_{n,q,w}^{(h)}(x)$ under S_n

Let $h \in \mathbb{Z}$, $w \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ and $w_i \in \mathbb{N}$ be a natural number which satisfies the condition $w_i \equiv 1 (\bmod 2)$, in which $i \in \mathbb{Z}$ lies in $1 \leq i \leq n$. From the Eqs. (3) and (5), we get ;

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y + \left(\prod_{j=1}^{n} w_j\right)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} \\
&\quad \times w^{y\prod_{j=1}^{n-1} w_j} q^{hy\prod_{j=1}^{n-1} w_j} d\mu_{-1}(y) \\
&\quad = \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-1)^y w^{y\prod_{j=1}^{n-1} w_j} q^{hy\prod_{j=1}^{n-1} w_j} \\
&\quad \times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y + \left(\prod_{j=1}^{n} w_j\right)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} \\
&= \lim_{N\to\infty} \sum_{m=0}^{w_n-1} \sum_{y=0}^{p^N-1} (-1)^{m+y} w^{(m+w_n y)\prod_{j=1}^{n-1} w_j} q^{h(m+w_n y)\prod_{j=1}^{n-1} w_j} \\
&\quad \times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y) + \left(\prod_{j=1}^{n} w_j\right)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.
\end{aligned}$$

Applying

$$\begin{aligned}
&\prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)} \\
&\quad \times w^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} q^{hw_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}
\end{aligned}$$

to the both sides of the above gives

$$\begin{aligned}
I &= \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)} \\
&\quad \times w^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} q^{hw_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \\
&\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y + \left(\prod_{j=1}^{n} w_j\right)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} \\
&\quad \times w^{y\prod_{j=1}^{n-1} w_j} q^{hy\prod_{j=1}^{n-1} w_j} d\mu_{-1}(y) \\
&= \lim_{N\to\infty} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{l=0}^{w_n-1} \sum_{y=0}^{p^N-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)+m+y} \\
&\quad w^{\prod_{j=1}^{n-1} w_j m + \prod_{j=1}^{n} w_j y + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \\
&\quad \times q^{h\left(\prod_{j=1}^{n-1} w_j m + \prod_{j=1}^{n} w_j y + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right)} \\
&\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y) + \left(\prod_{j=1}^{n} w_j\right)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.
\end{aligned} \tag{6}$$

We see that the Eq. (6) is invariant under any permutation $\sigma \in S_n$. Thus, this equation can be stated as follows:

$$\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}k_s\right)}$$
$$\times w^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j} q^{hw_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j}$$
$$\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n}w_{\sigma(j)}\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t}$$
$$\times w^{y\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} q^{hy\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} d\mu_{-1}(y)$$

in which σ lies in S_n. Therefore, we acquire the following theorem.

Theorem 1. *Let $h\in\mathbb{Z}$, $w\in T_p$, $q\in\mathbb{C}_p$ with $|q-1|_p<1$ and $w_i\in\mathbb{N}$ be a natural number which satisfies the condition $w_i\equiv 1(\mathrm{mod}\,2)$, in which $i\in\mathbb{Z}$ lies in $1\le i\le n$. Then the following*

$$\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}k_s\right)}$$
$$\times w^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j} q^{hw_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j}$$
$$\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n}w_{\sigma(j)}\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t}$$
$$\times w^{y\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} q^{hy\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)} d\mu_{-1}(y)$$

holds true for any $\sigma\in S_n$.

We derive by using the definition of q-number that

$$\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q \tag{7}$$
$$=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1w_2\cdots w_{n-1}}}$$
$$=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}.$$

It is observed from the Eq. (7) that

$$\int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q t} \tag{8}$$
$$\times w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$$
$$=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1}w_j\right]_q^m\left(\int_{\mathbb{Z}_p}w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}\right.$$
$$\left.\times\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m d\mu_{-1}(y)\right)\frac{t^m}{m!}$$
$$=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1}w_j\right]_q^m$$
$$\mathscr{E}_{m,q^{w_1w_2\cdots w_{n-1}},w^{w_1w_2\cdots w_{n-1}}}^{(h)}\left(w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right)\frac{t^m}{m!}.$$

From Eq. (8), for $m\ge 0$, we have

$$\int_{\mathbb{Z}_p}\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q^m \tag{9}$$
$$\times w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$$
$$=\left[\prod_{j=1}^{n-1}w_j\right]_q^m\mathscr{E}_{m,q^{w_1w_2\cdots w_{n-1}},w^{w_1w_2\cdots w_{n-1}}}^{(h)}\left(w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right).$$

Therefore, by Theorem 1 and Eq. (9), we derive the following theorem.

Theorem 2. *Let $h\in\mathbb{Z}$, $w\in T_p$, $q\in\mathbb{C}_p$ with $|q-1|_p<1$ and $w_i\in\mathbb{N}$ be a natural number which satisfies the condition $w_i\equiv 1(\mathrm{mod}\,2)$, in which $i\in\mathbb{Z}$ lies in $1\le i\le n$. For $m\ge 0$, the following*

$$\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}k_s\right)}\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q^m$$
$$\times w^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j} q^{hw_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j}$$
$$\times\mathscr{E}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}},w^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}^{(h)}\left(w_{\sigma(n)}x+\sum_{j=1}^{n-1}\frac{w_{\sigma(n)}}{w_{\sigma(j)}}k_j\right)$$

holds true for any $\sigma\in S_n$.

By using the definitions of $[x]_q$ and binomial theorem, we can write :

$$\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m \tag{10}$$
$$=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$
$$\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}^l.$$

Applying $\int_{\mathbb{Z}_p}w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$ to the both sides of the above equation gives

$$\int_{\mathbb{Z}_p}\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m \tag{11}$$
$$\times w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$$
$$=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$
$$\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}$$
$$\times\int_{\mathbb{Z}_p}w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}^l.d\mu_{-1}(y)$$
$$=\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$
$$\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\mathscr{E}_{l,q^{w_1w_2\cdots w_{n-1}},w^{w_1w_2\cdots w_{n-1}}}^{(h)}(w_nx).$$

As a result of the Eq. (11), we obtain

$$\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q^m\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\left(\sum_{i=1}^{n-1}k_i\right)} \tag{12}$$
$$w^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}q^{hw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}$$
$$\times\int_{\mathbb{Z}_p}\left[y+w_nx+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}$$
$$\times w^{y\prod_{j=1}^{n-1}w_j}q^{hy\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$$
$$=\sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_j\right]_q^l[w_n]_q^{m-l}$$
$$\times\mathscr{E}_{l,q^{w_1w_2\cdots w_{n-1}},w^{w_1w_2\cdots w_{n-1}}}^{(h)}(w_nx)$$
$$\times\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\left(\sum_{i=1}^{n-1}k_i\right)}w^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}$$
$$\times q^{(h+l)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_{q^{w_n}}^{m-l}$$
$$=\sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_j\right]_q^l[w_n]_q^{m-l}$$
$$\times\mathscr{E}_{l,q^{w_1w_2\cdots w_{n-1}},w^{w_1w_2\cdots w_{n-1}}}^{(h)}(w_nx)$$
$$\times U_{m,q^{w_n},w^{w_n}}(w_1,w_2,\ldots,w_{n-1}\mid l),$$

where

$$U_{m,q,w}(w_1,w_2,\ldots,w_{n-1}\mid l)$$
$$=\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\left(\sum_{l=1}^{n-1}k_l\right)}$$
$$\times w^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}q^{(h+l)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1}w_i\right)k_j}$$
$$\times\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q^{m-l}.$$

Therefore, by (11), we obtain the following theorem.

Theorem 3. *Let $h\in\mathbb{Z}$, $w\in T_p$, $q\in\mathbb{C}_p$ with $|q-1|_p<1$, $w_i\in\mathbb{N}$ be a natural number which satisfies the condition $w_i\equiv 1(\mathrm{mod}\,2)$, in which $i\in\mathbb{Z}$ lies in $1\leq i\leq n$ and let $m\geq 0$. Then the following expression*

$$\sum_{l==0}^{m}\binom{n}{m}\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^{l}\left[w_{\sigma(n)}\right]_q^{m-l}$$
$$\times\mathscr{E}^{(h)}_{l,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}},w^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x\right)$$
$$\times U_{m,q^{w_{\sigma(n)}},w^{w_{\sigma(n)}}}\left(w_{\sigma(1)},w_{\sigma(2)},\ldots,w_{\sigma(n-1)}\mid l\right)$$

holds true for some $\sigma\in S_n$.

3 Conclusion

In this study, we have obtained some symmetric identities for Carlitz's twisted (h,q)-Euler polynomials associated with the p-adic invariant integral on $\mathbb{Z}_p$ under the symmetric group of degree n. Note that in the case $n=3$, for $w=1$ and $h=0$, all our results in this paper reduce to the results in [6]. Moreover, in the case $n=3$ and for $w=q^{-1}$, all our results in this paper reduce to the results in [7].

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