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Stochastic Optimal Control of DC Pension Fund under the Fractional Brownian Motion

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Abstract: This paper considers that the goal of the fund manager is to minimize the expected utility loss function, and the noises involved in the dynamics of some wealth are fractional Brownian motions with short-range dependence. By applying Hamilton and Lagrange multiplier, the stochastic optimal control problem is converted into a non-random optimization. Furthermore, based on deterministic optimal control principle, it is obtained the explicit solution of the optimal strategies via moment equations. Finally, it is presented a simulation to analyze the dynamic behavior of the optimal portfolio strategy influenced by the orders of fractional Brownian motions.

Keywords: Dalgaard-Strulik Stochastic optimal control, Fractional Brownian motion, Defined-contribution pension scheme, Lagrange multiplier

1. Introduction

In a defined contribution (DC) pension plan, the financial risk is borne by the member: contributions are fixed in advance, and the benefits provided by the plan depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. Recently, due to the demographic evolution and the development of the equity market, DC schemes have become popular in global pension market.

A successful DC scheme will deliver good annuity at retirement, so the investment strategy for the accumulation phase in DC schemes is very critical. Literature about investment strategy of DC pension funds is prolific and from a methodological point of view, two approaches are exploited. The first one is stochastic control, used for the first time by [1]. Among the recent applications of this theory to DC pension fund profolio, see, e.g.,[2-4] pension fund portfolio, see, e.g.,[2-4]. The second method, also called the martingale method, was developed by[5] in the setting of complete markets and relies on the theory of Lagrange multipliers. In related literature, see, cf.,[6]. However, these studies generally supposed the risky asset price dynamics driven by a geometric Brownian motion (GBM,hereafter), which implies that the volatility of risky asset price is only a constant without considering the time-dependent in the market.

Whereas Lvy processes and stochastic volatility models are by now standard models for stock prices, more recently long memory processes like fractional Brownian motion (FBM) have attracted attention by stochastic analysts and mathematical finance researchers, cf. e.g. Hu and Oksendal[7] and references therein. For an introduction to FBM see [8]. Certain financial time series show long memory properties as observed since the 1980s; see [9-17].

The purpose of this paper is to extend the Brownian motion about some noises involved in the dynamics of wealth to fractional Brownian motion (FBM) with short-range dependence in pension fund.Instead of using the classical tool of optimal control as optimization engine (e.g.,[3]), we convert the stochastic optimal control problem into a non-random optimization, and try to find explicit solutions under the minimization of the expected utility loss function.

Finally, we provide a brief numerical example in order to analyze the dynamic behavior of the optimal portfo-

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lio strategy influenced by the orders of fractional Brownian motions. According to simulation, we derive six points conclusions. First, the investment trends of the assets are consistent with the portfolio managers experience and the conventional wisdom, i.e., during the beginning of the investment period, the fund manager realizes a more aggressive investment policy in order to boost the fund. Consistently, as the time approaches the deadline, a shift of wealth from the investment in higher risky asset to lower risky asset and riskless asset.Second, as long as one fractional order decreases and the other is fixed, the proportion invested in the risky asset involved in the decreased fractional order will increase. Third, as long as one fractional order increases and the other is fixed, the proportion invested in the risky asset involved in the increased fractional order will decrease.Fourth, when one fractional order increases and the other decreases, the proportion invested in the corresponding risky asset has the opposite change.Fifth, when the two fractional orders both decrease, the optimal proportion invested in the risky asset with the larger fractional order will increase and the other decrease. Sixth, when the two fractional orders both increase, the optimal proportion invested in the risky asset involved in the larger fractional order will decrease and the other increase.

The rest of the paper is organized as follows: in section 2, we introduce the classical model related to the problem of the pension fund management. In section 3, we consider the extended model, and in which some of the classical Brownian motions are replaced by fractional Brownian motion with Hurst parameter lower than $\frac{1}{2}$, referred to as fractional Brownian motion with short-range dependence. In section 4, we convert the stochastic optimal control problem into a non-random optimization, and conclude the solution of the initial problem. Section 5 presents a simulation and calculates the sensitivity of the fractional orders to the optimal strategies and section 6 concludes.

2. Backgrounds on Classical Model of Pension Fund Management

Most of pension fund management models come from Mertons model[1]. Consider that the market structure consists of two financial assets, a risk-free asset and a risky asset. The risk-free asset (i.e., the bank account) whose price at time t, denoted by $S_0(t)$, $(t \geq 0)$ the evolution of $S_0(t)$ can be expressed as:

$$dS_0(t) = rS_0(t), \tag{1}$$

where r is a constant rate of interest.

The risky asset (called 'stock' hereafter) whose price at time t, denoted by $S_1(t)$, the dynamics of $S_1(t)$ are given by:

$$dS_1(t) = S_1(t)(\mu dt + \sigma dW(t)), \tag{2}$$

where μ and σ denote an expected instantaneous rate of return of the risky asset and instantaneous and volatility, respectively. $W(t)$ is a normalized Gaussian white noise with zero mean and unit variance (cf.[6]).

Suppose that the contribution rate of a DC pension plan is fixed and denoted by C, the proportion invested in the risky asset at time t is denoted by $y(t)$ (and thus $1-y(t)$ is the proportion of fund invested into the risk less asset). The wealth $x(t)$ at time t evolves according to the stochastic differential equation:

$$dx(t) = [y(t)x(t) + (1-y(t))rx(t) + c]dt + x(t)\sigma dW(t) \tag{3}$$

where $x(0) = x_0$, a constant number.

The aim of the pension fund management is to maximize the expected utility at the retirement time T, and then under the condition (3), the optimal control problem can be described by:

$$\max_{y(t)} E(u(x(T))), \tag{4}$$

The maximum principle leads to the following result (Hamilton-Jacobi method):

$$\max_{y}\{v_t + [y(t)(\mu - r)x + rx]v_x + \frac{1}{2}y(t)^2\sigma^2x^2v_{xx}\} = 0 \tag{5}$$

where $V(t,x)$ is the value function, and V_t, V_x, V_{xx} denote partial derivatives of first and second orders with respect to time and wealth, respectively.

The first order maximizing conditions for the optimal strategy $y^*(t)$ is:

$$y^*(t) = \frac{(r-\mu)V_x}{\sigma^2 x V_{xx}}. \tag{6}$$

In order to obtain an explicit solution to the problem, Devolder et al.[6] used Mertons method to choose a particular function: $\mu(x(T)) = \frac{x^y(T)}{\gamma}$, $\gamma \in (-\infty, 1)/0$ and the result is

$$y^*(t) = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma}. \tag{7}$$

In practice, the financial market is composed of numerous assets, in the next section, we shall consider $n+1$ kinds of assets, one is riskless asset and the rest are risky assets. Furthermore, considering that some important information may continuously influence the risky asset price volatility in many years, for example, the information of the government policy's adjustment to financial market may continuously influence some financial asset price's volatility in several years, we then assume some noises are no longer Gaussian white noises, but rather fractional white noises with short range dependence.

3. Pension Model with Fractional Noises

In this section, we first introduce the general properties for the fractional Brownian motion and then provide the optimization problem with fractional noise for risky assets.

3.1 Background on Fraction Brownian Motion.

The basic properties of the fractional Brownian motion defined as a fractional derivative of Gaussian White noise can be summarized as follows (cf. [11]).

Definition 1. Let (Ω, F, P) denote a probability space, and a, $0 < a < 1$, referred to as the Hurst parameter. The stochastic process $W(t,a), t \geq 0$ defined on this probability space is a fractional Brownian motion $(fBm)_a$ of order a if

(1) $p\{W(0,a) = 0\} = 1$

(2) for each $t \in R_+$, $W(t,a)$ is an F-measurable random variable such that $E(W(t,a)) = 0$

(3) for $t, \tau \in R_+$,

$$E[W(t,a)W(\tau,a)] = \frac{\sigma^2}{2}(t^{2a} + \tau^{2a} - |t-\tau|^{2a}) \tag{8}$$

where σ^2 is the variance parameter.

It follows from (8) and from the kolmogorov's continuity criterion that, for $a > \frac{1}{2}$, the sample path of $W(t,a)$ are continuous with probability one, but nowhere differentiable.

Note that from Eq.(8), for $a = \frac{1}{2}$, $W(t,a)$ is a classical Brownian motion. Further comments about the fractional Brownian motion can be concluded (see e.g., Jumarie, 2001, 2005):

(1) Unlike the semblance, the equality (8) can be simply derived form the following equation:

$$W(\rho t, a) = \rho^a W(t,a), \quad \rho > 0 \tag{9}$$

(2) The $(fBm)_a$ can be constructed form the classical Brownian motion:

$$W(t) := \Gamma(a+1/2)^{-1}\int_0^t (t-\tau)^{a-1/2}dW(\tau) \tag{10}$$

which has been proposed by Mandelbrot and Van Ness (1968), where is the well-known gamma function.

(3) Using Maruyama' notation(see e.g.,Jumarie, 2005), it may be useful to write

$$dW(t,a) = w(t)(dt)^a \tag{11}$$

Definition 2. Let $f: R \to R, x \to f(x)$,denote a continuous function, its fractional derivative of order a is defined by the following expression (see, e.g., Jumarie, 2001).

$$f^a(x) = \frac{1}{\Gamma(-a)}\int_0^x (x-\xi)^{-a-1}f(\xi)d\xi, \quad a < 0 \tag{12}$$

For a positive a, one will set

$$f^a(x) = (f^{a-n})^n, \quad n-1 < a < n \tag{13}$$

Under this definition, Eqs.(9) and (10) can be re-written as:$W(t,a) = D^{-(a+\frac{1}{2})}W(t)$, where D denotes the derivative operator. Jumarie (2005) concludes the integral $(dt)^a$ of fractional Brownian motion for $0 < a < 1$.

Lemma 3. Let $f(t)$ denote a continuous function, and then

$$\int_0^t f(\tau)(d\tau)^a = a\int_0^t (t-\tau)^{a-1}f(\tau)d\tau, \quad 0 < a < 1$$

Proof. See appendix A.

B. The optimization program

Consider that the market structure consists of $n+1$ assets, one risk-free asset and n risky assets. We denote the price of the riskfree asset (i.e. the bank account) at time t by $S_0(t)$, which evolves according to the following equation:

$$dS_0(t) = rS_0(t)dt \tag{14}$$

where r is a constant rate of interest.

We denote the price of the risky asset i at time t by $S_i(t)(i = 1, \cdots, n)$, which is described by the following stochastic differential equation:

$$dS_i(t) = S_i(t)((r+\mu_i)dt + \sum_{j=1}^{m}\sigma_{ij}^{(1)}dW_j(t) + \sum_{j=m+1}^{n}\sigma_{ij}^{(2)}dW_j(t,a_j)) \tag{15}$$

where μ_i is an expected instantaneous rate of return of the risky asset i, $\sigma_{ij}^{(1)}$ the co-variance of the risky asset i and the asset $j(j = 1, \cdots, m)$ under the classical Brown motion, $\sigma_{ij}^{(2)}$ the co-variance of the risky asset i and the asset j, $(j = m+1, \cdots, n)$ under the fractional Brown motion.

Let $x(t)$ denote the wealth of pension fund at time $t \in [0,T]$ and y_i denote the proportion of the pension fund invested in the risky asset i. Correspondingly, $1 - \sum_{i=1}^n y_i$ denotes the proportion of the pension fund invested in the bank. The dynamics of the pension fund are given by:

$$dx(t) = x(t)((r + \sum_{i=1}^n y_i\mu_i)dt + \sum_{i=1}^n\sum_{j=1}^m y_i\sigma_{ij}^{(1)}dW_j(t) + \sum_{i=1}^n\sum_{j=m+1}^n y_i\sigma_{ij}^{(2)}dW_j(t,\alpha_j)) \tag{16}$$

and $x(0) = x_0$, where x_0 stands for an initial weal.

Suppose that the goal of the pension fund manager is to choose portfolio strategies in order to minimize the expected value of utility loss function. Under the wealth process denoted by (14), the investor looks for a strategy y_t^* minimizing the utility function:

$$\min_{y(t)} E[\int_0^T e^{-\rho t}U(F(t) - x(t))dt] \tag{17}$$

where ρ is a discounted factor, and $F(t)$ is the aim fund level of the pension fund manager at time .That is to say, $F(t)$ is in advance given before investment. $U(\cdot)$ is a strictly concave function and satisfies the Inada conditions $\mu'(0) = +\infty$ and $\mu'(+\infty) = 0$.

In this paper, we describe the pension fund investor's objective function with a power utility function, that is,

$$U(x) = x^{\gamma}, \gamma \in (-\infty, 1).$$

The choice of the power utility function is motivated by three reasons.

First, pension funds are in general large companies who define their strategies with respect to the amount of money they are managing, more or less in a scaling way. This feature is well capture by the use of the power utility function.

Second, pension funds are managed in such a way that they cannot reach negative values. This is true also in the logarithm utility case, thanks to the infinite marginal utility at zero.

Note that the diffusion equation (16) has fractional Brownian motions; it is difficult to solve the explicit solution of this problem. In the next section, we shall show how one can obtain the closed form solution of this problem of fractional stochastic optimal control by using the dynamic equation of the moments.

4. Variational Approach Via Moment Equations

We introduce the value function of the problem (17):

$$J(t,x) = E[\int_t^T e^{-\rho s} U(F(s) - x(s)) ds | x(t) = x] \quad (18)$$

For the sake of simplicity, we define:

$$f := r + \sum_{i=1}^{n} y_i \mu_i,\ g_j := \sum_{i=1}^{n} y_i \sigma_{ij}^{(1)},\ h_j := \sum_{i=1}^{n} y_i \sigma_{ij}^{(2)} \quad (19)$$

Then Eq.(16) can be re-written as:

$$dx(t) = x(t)(f dt + \sum_{j=1}^{m} g_j dW_j(t) + \sum_{j=m+1}^{n} h_j dW_j(t, \alpha_j)) \quad (20)$$

Let

$$F(t) - x(t) = v(t) x(t) \quad (21)$$

and then Eq.(18) may be re-written:

$$J(t,x) = \int_t^T [e^{-\rho s} v^{\gamma}(s) E(x^{\gamma}(s)) ds | x(t) = x] \quad (22)$$

Let σ be the matrix $(\sigma_m, \sigma_{n-m})_{n\times n}$, (which we assume is non-singular), where

$$\begin{cases} \sigma_m = (\sigma_{ij}^{(1)}), \\ (i = 1, \cdots, n; j = 1, \cdots, m). \\ \sigma_{n-m} = (\sigma_{ij}^{(2)})_{n\times(n-m)}, \\ (i = 1, \cdots, n; j = m+1, \cdots, n). \end{cases}$$

Note that Eq.(22) includes the term $E(x^{\gamma}(s))$. Thus we can define a new state variable: $k(t) = E(x^{\gamma}(t))$.

The dynamical equation satisfied by $k(t)$ is:

$$dk(t) = \gamma f k(t) dt + \frac{\gamma(\gamma-1)k(t)}{2} \sum_{j=1}^{m} g_j^2 dt + \frac{\gamma(\gamma-1)k(t)}{2} \sum_{j=m+1}^{n} h_j^2 (dt)^{2\alpha_j} \quad (23)$$

with $k(0) = x_0^{\gamma}$.

The initial stochastic optimal control problem (17) is then converted into a non-random optimal control involving the dynamics (23). Finally, we can obtain the optimal strategy of problem (17).

Proposition 4. The optimal portfolio proportions Y for the risky assets are given by:

$$Y = \frac{(\sigma')^{-1} \sigma_*^{-1} U}{1-\gamma},$$

with the residual fractional of wealth $1 - E.\,Y$ allocated to the bank, where

$$\begin{cases} \sigma_* = (\sigma_m, \sigma_{n-m}^0)_{n\times n}, \\ Y = (y_1, \cdots, y_n)', \\ U = (\mu_1, \cdots, \mu_n)', \\ \sigma_{n-m}^0 = (2a_j (T-t)^{2a_j - 1}, \sigma_{ij}^{(2)})_{n\times(n-m)}, \\ (i = 1, \cdots, n; j = 1, \cdots, m), \\ E = (1, \cdots, 1)_{n\times 1}. \end{cases}$$

Proof. Introducing the Lagrange parameter function $\lambda(t)$, we consider the augmented gain function:

$$J(0,x) = \int_0^T \{[e^{-\rho t} v^{\gamma}(t) k(t) + \lambda \gamma f k(t) + \frac{\lambda\gamma(\gamma-1)k(t)}{2} \sum_{j=1}^{m} g_j^2] dt + \frac{\lambda\gamma(\gamma-1)k(t)}{2} \sum_{j=m+1}^{n} h_j^2 dt^{2\alpha_j} - \lambda dk(t)\} \quad (24)$$

According Lemma 3, we have:

$$\int_0^t f(\tau)(d\tau)^{2\alpha} = 2\alpha \int_0^t (t-\tau)^{2\alpha-1} f(\tau) d\tau, \quad (25)$$

Submitting Eq. (25) into Eq.(24), we derive:

$$J(0,x) = \int_0^T [e^{-\rho t} v^{\gamma}(t) k(t) + \lambda\gamma f k(t) + \lambda\gamma(\gamma-1)k(t) \sum_{j=m+1}^{n} h_j^2 \alpha_j (T-t)^{2\alpha_j - 1} + \frac{\lambda\gamma(\gamma-1)k(t)}{2} \sum_{j=1}^{m} g_j^2] dt - \int_0^T \lambda dk(t) \quad (26)$$

Now the Hamiltonian operator is:

$$H = k(t)[e^{-\rho t} v(t)^{\gamma} + \lambda\gamma f + \frac{\lambda\gamma(\gamma-1)}{2} \sum_{j=1}^{m} g_j^2 + \lambda\gamma(\gamma-1) \sum_{j=m+1}^{n} \alpha_j h_j^2 (T-t)^{2\alpha_j - 1}]. \quad (27)$$

Therefore, according to the optimal control conditions:

$$u_i+(\gamma-1)\sum_{j=1}^{m}\sigma_{ij}^{(1)}g_j+(\gamma-1)$$

$$\sum_{j=m+1}^{n}2\alpha_j(T-t)^{2\alpha_j-1}\sigma_{ij}^{(2)}h_j=0,(\frac{\partial H}{\partial y_i}=0) \tag{28}$$

Namely,

$$\sum_{j=1}^{m}\sigma_{ij}^{(1)}g_j+\sum_{j=m+1}^{n}2\alpha_j(T-t)^{2\alpha_j-1}\sigma_{ij}^{(2)}h_j=\frac{u_i}{1-\gamma} \tag{29}$$

For the sake of simplicity, we define the variables:

$$\begin{cases}\sigma_{n-m}^{0}=(2\alpha_j(T-t)^{2\alpha_j-1}\sigma_{ij}^{(2)})_{n\times(n-m)},\\(i=1,\cdots,n;\quad j=1,\cdots m)\\\sigma_*=(\sigma_m\ \sigma_{n-m}^{0})_{n\times n},\\A=(g_1,\cdots,g_m,h_{m+1},\cdots,h_n)'\\Y=(y_1,\cdots,y_n)'\\U=(u_1,\cdots,u_n)'\end{cases} \tag{30}$$

Note that $(2\alpha_j(T-t)^{2\alpha_j-1}\neq 0)$,and then σ_* is a non-singular matrix. From Eq.(29), we derive:

$$\sigma_*A=\frac{U}{1-\gamma} \tag{31}$$

According to (19), we have:

$$Y^{/}\sigma=A' \tag{32}$$

Combining with Eqs.(30)and (31), we derive the proportion invested risky assets:

$$Y=\frac{(\sigma')^{-1}\sigma_*^{-1}U}{1-\gamma} \tag{33}$$

Correspondingly, the proportion of pension fund invested in the bank such that:

$$1-E\cdot Y \tag{34}$$

where$E=(1,\cdots,1)_{n\times 1}$.

Remark 5. Note that from Eqs. (11), (24), (27) and lemma3 we can conclude:

(1) If we set $a=\frac{1}{2}$,we get the optimal Merton's portfolio for the classical Brownian motion.

(2) When $\frac{1}{2}<a<1$, according to the lemma 3,clearly we have

$$\int_0^t f(\tau)(d\tau)^{2a}=a^2\left[\int_0^t(t-\tau)^{a-1}f^{\frac{1}{2}}(\tau)d\tau\right]^2,$$

Then the Eq.(24) can be changed :

$$\begin{aligned}J(0,x)\ =&\ \int_0^T[e^{-\rho t}v^{\gamma}(t)k(t)+\lambda\gamma k(t)(f-v(t))\\&+\ \frac{\lambda\gamma(\gamma-1)k(t)}{2}\sum_{j=1}^{m}g_j^2]dt+\frac{\lambda\gamma(\gamma-1)k(t)}{2}\\&\sum_{j=1}^{m}\left(\int_0^T(T-\tau)^{\alpha_j-1}h_j\sqrt{k(t)}d\tau\right)^2\end{aligned} \tag{35}$$

Since the above equation contains quadratic integral, it is difficult to solve.

5. Simulation

In this section, we give a brief numerical example in order to analyze the dynamic behavior of the optimal portfolio strategy influenced by the orders of fractional Brownian motions.

We assume that there are three assets, one is riskless asset, and the other two are a lower risky asset (risky 1) and a higher risky asset (risky 2), respectively. The key parameters take the following values: (formula)

The eight parameters have been taken from Deestra et al. (2003), who illustrate the cash, bond and stock dynamic models. We consider the investment period equal to 30 years, the value x(0)=1 is normalization. We suppose two fractional Brownian motions with parameters $a_1=0.4,a_2=0.333$.

According to the proposition 4, we simulate the optimal strategies illustrated in Fig.1, which shows that the optimal proportion invested in the risky 1 starts from about 25% to 40%, and the trend slowly decreases before 25 years but abruptly increases during the last few years. The proportion invested in the riskless asset increases from an initial value around 52% to about 63%. On the other hand, the proportion invested in the risky 2 gradually decreases from an initial value close to 21% to zero. The investment trends of the three assets are consistent with the portfolio manager's experience and the conventional wisdom. During the beginning of the investment period, the fund manager realizes a more aggressive investment policy in order to boost the fund. Consistently, as the time approaches the deadline T, Fig.1 shows a shift of wealth from the investment in higher risky asset to lower risky asset and riskless asset.

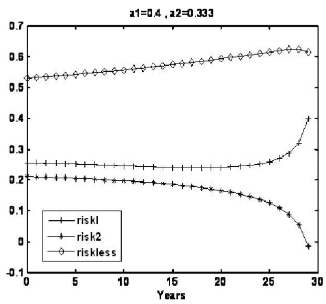

Figure1 $a_1=0.4,a_2=0.333$

The sensitivity of the fractional orders to the optimal investment strategies is shown as in Figs.2 to 6.

In Fig.2, we observe that, as a1 is equal to 0.392 (i.e., decreasing 2 percent) and the value of a_2 is fixed, compared with Fig.1, the proportion invested in the risky 1 increases with respect to the same time (in Fig.2, the proportion starts from about 38% to around 45%), moreover, the trend apparently decreases before 25 years but abruptly increases during the last few years. However, from Fig.2, we notice that the optimal percentages invested in the riskless asset and in risky 2 are all lower than those in fig.1 with respect to the same time, respectively.

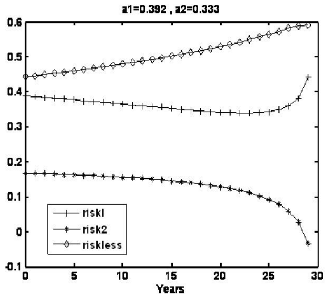

Figure 2 $a_1 = 0.392, a_2 = 0.333$

Fig.3 shows that as a2 decreases 2 percent and a1 is fixed, compared with Fig.1, the proportion invested in risky 1 decreases with respect to the same time. The proportion invested in the risky 2 increases, but the downtrend becomes faster. The optimal proportion in riskless asset increases.

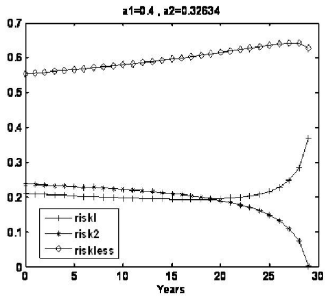

Figure 3 $a_1 = 0.4, a_2 = 0.32634$

In Figs. 4 to 6, we let the two fractional orders synchronously change in order to find the influence on the optimal strategies.

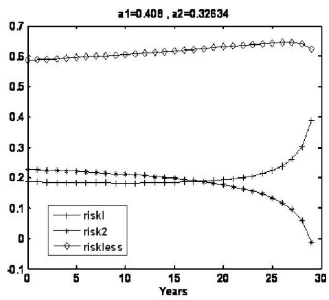

Figure 4 $a_1 = 0.4, a_2 = 0.32634$

In Fig.4, we notice that as $a_1 = 0.408$ and $a_2 = 0.32634$ (that is, a_1 increases 2 percent, a_2 decreases 2 percent), compared with Fig.1, the proportion invested in the risky 1 decreases (in Fig.4, starting from about 18% to 40%). The proportion invested in the risky 2 increases (in Fig.4, from about 23% to zero). The optimal percentage invested in the riskless asset increases. In Fig.4, we observe that, compared with Fig.3, the proportion invested in the risky 1decreases, however, the proportions invested in the riskless asset and risky 2 all increase, moreover, the downtrend of the risky 2 becomes slow. Comparing Fig.2 with Fig.4, we can conclude that, in Fig.4, the proportion invested in the risky 1 is lower than that in Fig.2, while the proportions in riskless asset and risky 2 are all higher than those in Fig.2, respectively. Form Figs.3 and 4, we find that as long as one fractional order increases and the other is fixed, the proportion invested in the risky asset involved in the increased fractional order will decrease. Furthermore, form Figs.1 and 4 (or, Figs.2 and 4, Figs.2 and 3), we conclude that when one fractional order increases and the other decreases, the proportion invested in the corresponding risky asset has the opposite change.

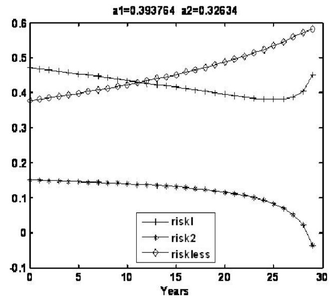

Figure 5 $a_1 = 0.393764, a_2 = 0.32634$

In Fig.5, let a_1 be equal to 0.39374 and a_2 equal to 0.32634, that is, compared with the values in Fig.1, a_1 decreases 1.5 percent, a_2 decreases 2 percent, we can observe that in Fig.5 the proportion invested in the risky 1 is higher than that in Fig.1 (in Fig.5, from an initial value close to 47% to about 45%). While the optimal percentages invested in the riskless asset and in risky 2 are all lower than those in Fig.1, respectively.

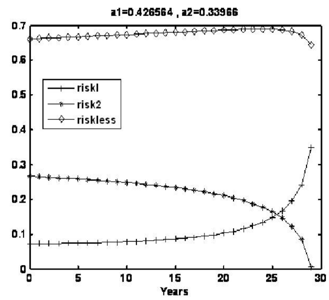

Figure 6 $a_1 = 0.426564, a_2 = 0.33966$

Fig.6 shows that a_1 is equal to 0.426564 and a_2 equal to 0.33966 (that is, a_1 increases 7 percent, a_2 increases 2 percent), compared with Fig.1, in Fig.6, the proportion invested in the risky 1 is lower. However, the optimal percentages invested in the riskless asset and in risky 2 are all higher than those in Fig.1, respectively. Form Figs.1 and 5, we can derive when the two fractional orders both decrease, the optimal proportion invested in the risky asset with the larger fractional order will increase and the other decrease. Compared with Figs.6 and 1(Figs.2, 3, 4, 5, etc.), it can be concluded that when the two fractional orders increase, the optimal proportion invested in the risky asset involved in the larger fractional order will decrease and the other increase.

6. Conclusion

In this paper, assuming that the goal of the fund manager is to minimize the expected utility loss function, we extend some noises involved in the dynamics of the wealth satisfy fractional Brownian motions with short-range dependence. Instead of using the dynamic programming approach, we convert the stochastic optimal control problem into a non-random optimization. Based on deterministic optimal control principle, we obtain the explicit solution of the optimal strategies. The mathematical framework is essentially engineering mathematics, and mainly one will work formally by using the Maruyama notation of fractional order. In future research about the pension fund investment field, it would be significant to consider fractional Brownian motions with long-range dependence, that is to say, with a Hurst parameter higher than $\frac{1}{2}$.

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Appendix Lemma 3.3

Proof. Let us consider the fractional differential equation

$$x^a(t) = f(t), \quad 0 < a < 1$$

In a straightforward manner, its solution is obtained as

$$x(t) = D^{-a} f(t)$$

By using the fractional derivative, namely, Eq. (12), one has

$$x(t) = \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-1} f(\tau) d\tau \qquad (36)$$

Using Taylor expansion of fraction order derivative(see, e.g., [12]).

$$f^a(x) = \lim_{h\to 0} \frac{\Delta^a f(x)}{h^a} = \Gamma(1+a) \lim_{h\to 0} \frac{\Delta f(x)}{h^a}, \quad 0 < a < 1$$

We can conclude: $d^a f = \Gamma(1+a) df$.
From $x^a = f(t)$, so that, $dx^a = f(t)(dt)^a$ and then

$$\Gamma(1+a) dx = f(t)(dt)^a.$$

So

$$x(t) = \frac{1}{\Gamma(1+a)} \int_0^t f(\tau) d\tau. \qquad (37)$$

Compared with (36) and (37), we shall obtain the equality:

$$\frac{1}{\Gamma(1+a)} \int_0^t f(\tau) d\tau = \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-1} f(\tau) d\tau$$

Therefore,

$$\int_0^t f(\tau) d\tau = a \int_0^t (t-\tau)^{a-1} f(\tau) d\tau, \quad 0 < a < 1 \qquad (38)$$

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