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Quantum Integral Inequalities via φ-Convex Functions

Muhammad Aslam Noor, Khalida Inayat Noor, Muhammad Uzair Awan and Farhat Safdar*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

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Abstract: In this paper, we consider the class of φ-convex functions, which was introduced and investigated by Noor [12] in 2006. We derive some quantum Hermite-Hadamard type inequalities for the φ-convex functions. Some special cases are discussed, which can be obtained from our results. The ideas and techniques of this paper may motivate further research in this field. It is expected that the readers may find the applications of the φ-convex functions and quantum integral inequalities in various fields of pure and applied sciences.

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Convexity theory has played an important and fundamental role in the developments of different fields of pure and applied sciences. In recent years, it received considerable attention. Several new generalizations and extensions of classical convexity have been introduced and investigated. For example see [1,2,9,11,12,13,**?**,15,14,17,19,24,**?**]. A significant generalization of classical convex functions is called φ-convex functions which was introduced by Noor [12]. Noor has shown that the optimality conditions of the differentiable φ-convex functions can be characterized by variational inequalities. Noor [12] investigated some basic properties of φ-convex functions and showed that φ-convex functions are nonconvex functions. Noor [13] established some Hermite-Hadamard type results for φ-convex functions.
An other importance of theory of convexity is its close relationship with theory of inequalities. A wide class of inequalities have been derived via convex functions, see [2,3,4,7,11,13,**?**,14,16,17,19,22,23,24]. In past few years, several authors have used the concepts of quantum calculus to obtain integral inequalities for different classes of convex functions, see [6,18,20,21,25,27].
In this chapter, we again consider the class of φ-convex functions. We obtain some new Hermite-Hadamard like inequalities for φ-convex functions using quantum calculus. These quantum Hermite-Hadamard inequalities and their variant forms are useful for quantum physics where lower and upper bounds of natural phenomena described by integrals are frequently required. In passing, we would like to point out that study of the quantum calculus was initiated by Euler (1707-1783). He introduced the q in tracks of Newton infinite series. In quantum calculus, we obtain the q-analogues of mathematical objects which can be recaptured as $q \to 1$. In fact, quantum calculus has emerge as fascinating and dynamic field. We also discuss some special cases which can be deduced from the main results. This is the main motivation of this chapter. The interested readers are encouraged to find the applications of quantum calculus and φ-convexity in other fields of pure and applied sciences.

1 Preliminaries of quantum calculus

In this section, we discuss some basic concepts and results pertaining to quantum calculus. For more details interested readers may consult [5,10].
Let us start with q-analogue of differentiation. For that matter, consider

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \frac{\mathrm{d}f}{\mathrm{d}x},$$

the above expression gives the derivative of a function $f(x)$ at $x = x_0$.
If we take $x = qx_0$ where $0 < q < 1$ is a fixed number and do not take limits, then we enter in the world of Quantum

* Corresponding author e-mail: noormaslam@gmail.com

calculus. The q-derivative of x^n is $[n]x^{n-1}$, where

$$[n]=\frac{q^n-1}{q-1},$$

is the q-analogue of n in the sense that n is the limit of $[n]$ as $q\to 1$.
Now we give the formal definition of q-derivative of a function f.

Definition 1. *The q-derivative is defined as*

$$\mathrm{D}_q f(x)=\frac{f(qx)-f(x)}{(q-1)x}. \tag{1}$$

Note that when $q\to 1$, then we have ordinary derivative.

Now we move to words q-antiderivatives of a function.

Definition 2. *The function $F(x)$ is a q-antiderivative of $f(x)$ if $D_qF(x)=f(x)$. It is denoted by*

$$\int f(x)\mathrm{d}_q x. \tag{2}$$

Our next definition is due to Jackson.

Definition 3. *The Jackson integral of $f(x)$ is defined as*

$$\int f(x)\mathrm{d}_q x=(1-q)x\sum_{j=0}^{\infty}q^j f(q^j x). \tag{3}$$

It is evident from above definition, that

$$\begin{aligned}&\int f(x)D_q g(x)\mathrm{d}_q x\\ &=(1-q)x\sum_{j=0}^{\infty}q^j f(q^j x)D_q g(q^j x)\\ &=(1-q)x\sum_{j=0}^{\infty}q^j f(q^j x)\frac{g(q^j x)-g(q^{j+1}x)}{(1-q)q^j x}.\end{aligned}$$

Definite q-integrals are defined as:

Definition 4([8]). *Let $0<a<b$. The definite q-integral is defined as*

$$\int_0^b f(x)\mathrm{d}_q x=(1-q)b\sum_{j=0}^{\infty}q^j f(q^j b), \tag{4}$$

provided the sum converge absolutely.

A more general formula for definite integrals is given as

$$\int_0^b f(x)\mathrm{d}_q x=\sum_{j=0}^{\infty}f(q^j b)(g(q^j b)-g(q^{j+1}b)).$$

Remark. From above definition of definite q-integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)\mathrm{d}_q x=\int_0^b f(x)\mathrm{d}_q x-\int_0^a f(x)\mathrm{d}_q x.$$

We now recall some basic concepts of quantum calculus on finite intervals. These results are mainly due to Tariboon et al. [26,27].

Let $J=[a,b]\subseteq\mathbb{R}$ be an interval and $0<q<1$ be a constant. The q-derivative of a function $f:J\to\mathbb{R}$ at a point $x\in J$ on $[a,b]$ is defined as follows.

Definition 5. *Let $f:J\to\mathbb{R}$ be a continuous function and let $x\in J$. Then q-derivative of f on J at x is defined as*

$$\mathscr{D}_q f(x)=\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\quad x\neq a. \tag{5}$$

A function f is q-differentiable on J if $\mathscr{D}_q f(x)$ exists for all $x\in J$.

Example 1. Let $x\in[a,b]$ and $0<q<1$. Then, for $x\neq a$, we have

$$\begin{aligned}\mathscr{D}_q x^2&=\frac{x^2-(qx+(1-q)a)^2}{(1-q)(x-a)}\\ &=\frac{(1+q)x^2-2qax-(1-q)x^2}{x-a}\\ &=(1+q)x+(1-q)a.\end{aligned}$$

Note that when $x=a$, we have $\lim_{x\to a}(\mathscr{D}_q x^2)=2a$.

Definition 6. *Let $f:J\to\mathbb{R}$ is a continuous function. A second-order q-derivative on J, which is denoted as $\mathscr{D}_q^2 f$, provided $\mathscr{D}_q f$ is q-differentiable on J is defined as $\mathscr{D}_q^2 f=\mathscr{D}_q(\mathscr{D}_q f):J\to\mathbb{R}$. Similarly higher order q-derivative on J is defined by $\mathscr{D}_q^n f=:J\to\mathbb{R}$.*

Lemma 1. *Let $\alpha\in\mathbb{R}$, then*

$$\mathscr{D}_q(x-a)^\alpha=\left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}.$$

Tariboon et al. [26,27] defined the q-integral as:

Definition 7. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$\int_a^x f(t)\mathrm{d}_q t=(1-q)(x-a)\sum_{n=0}^{\infty}q^n f(q^n x+(1-q^n)a), \tag{6}$$

for $x\in J$.

These integrals can be viewed as Riemann-type q-integral. If $a=0$ in (6), then we have the classical q-integral, that is

$$\int_0^x f(t)\mathrm{d}_q t=(1-q)x\sum_{n=0}^{\infty}q^n f(q^n x),\quad x\in[0,\infty).$$

Moreover, if $c\in(a,x)$, then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\mathrm{d}_q t&=\int_a^x f(t)\mathrm{d}_q t-\int_a^c f(t)\mathrm{d}_q t\\ &=(1-q)(x-a)\sum_{n=0}^{\infty}q^n f(q^n x+(1-q^n)a)\\ &\quad-(1-q)(c-a)\sum_{n=0}^{\infty}q^n f(q^n c+(1-q^n)a).\end{aligned}$$

Theorem 1. *Let $f : I \to \mathbb{R}$ be a continuous function, then*

1. $\mathscr{D}_q \int_a^x f(t)\,\mathrm{d}_q t = f(x)$
2. $\int_c^x \mathscr{D}_q f(t)\mathrm{d}_q t = f(x) - f(c)$ *for* $x \in (c,x)$.

Theorem 2. *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$, then $x \in J$*

1. $\int_a^x [f(t) + g(t)]\,\mathrm{d}_q t = \int_a^x f(t)\,\mathrm{d}_q t + \int_a^x g(t)\,\mathrm{d}_q t$
2. $\int_a^x (\alpha f(t))(t)\,\mathrm{d}_q t = \alpha \int_a^x f(t)\,\mathrm{d}_q t$
3. $$\int_a^x f(t)\,{}_a\mathscr{D}_q g(t)\,\mathrm{d}_q t$$ $$= (fg)|_c^x - \int_c^x g(qt + (1-q)a)\mathscr{D}_q f(t)\mathrm{d}_q t$$ *for* $c \in (a,x)$.

Lemma 2. *Let $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (t-a)^{\alpha}\mathrm{d}_q t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}.$$

2 φ-convexity

In this section, we recall the concept of φ-convex sets and φ-convex functions respectively.

Definition 8([12]). *Let $K_\varphi \subset H$ be a set. Then the set K_φ is said to be φ-convex, if*

$$u + te^{i\varphi}(v-u) \in K_\varphi, \quad \forall u, v \in K_\varphi, t \in [0,1].$$

We would like to point out that the definition of the $\varphi-$convex set has a clear geometric interpretation. This definition says that there is a path starting from a point u which is contained in K_φ. We do not required that the point v should be one of the end point of the path. This observation plays crucial part in our studies. If we demand that v should be an end point of the path, then obviously, $u + e^{i\varphi}(v-u) = v$. This implies that $\varphi = 0$. Consequently, $\varphi-$convex set reduces to the convex set. That is,

$$u + t(v-u) \in K, \quad \forall u, v \in K, t \in [0,1].$$

Definition 9([12]). *A function $f : K_\varphi \to H$ is said to be φ-convex with respect to φ, if*

$$f(u + te^{i\varphi}(v-u)) \le (1-t)f(u) + tf(v), \quad \forall u, v \in K_\varphi, t \in [0,1].$$

Note that if $\varphi = 0$ in the above definition, then, we have definition of classical convex functions.

Definition 10([12]). *A function $f : K_\varphi \to H$ is said to be quasi φ-convex with respect to φ, if*

$$f(u + te^{i\varphi}(v-u)) \le \max\{f(u), f(v)\}, \quad \forall u, v \in K_\varphi, t \in [0,1].$$

For the applications and other properties of the φ-convex sets and φ-convex functions, see [12].

3 Main Results

We are now ready to prove our main results. For simplicity of the notations, we take $I_\varphi = [a, a + te^{i\varphi}(b-a)]$ be the interval and I_φ^0 be the interior of I_φ.

Theorem 3(Hermite-Hadamard type inequality). *Let $f : I_\varphi \to \mathbb{R}$ be integrable φ-convex function with respect to φ, if*

$$f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \le \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)\,\mathrm{d}_q x \le \frac{qf(a) + f(b)}{2}. \tag{7}$$

Proof. Let f be a φ-convex function, then

$$f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \le \frac{1}{2}\left[f(a + te^{i\varphi}(b-a)) + f(a + (1-t)e^{i\varphi}(b-a))\right].$$

q-integrating above inequality with respect to t on $[0,1]$, we have

$$f\left(\frac{2a + te^{i\varphi}(b-a)}{2}\right) \le \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)\,\mathrm{d}_q x. \tag{8}$$

Since f is φ-convex function, then, $\forall\, t \in [0,1]$, we have

$$f(a + te^{i\varphi}(b-a)) \le (1-t)f(a) + tf(b).$$

q-integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)\,\mathrm{d}_q x \le \frac{qf(a) + f(b)}{2}. \tag{9}$$

Combining (8) and (9) completes the proof. □

Remark. If $q \to 1$, then, Theorem 3 reduces to Theorem 2.1 [13]. If $q \to 1$ and $\varphi = 0$, then, Theorem 3 reduces to classical Hermite-Hadamard inequality.

Theorem 4. *Let $f,g: I_\varphi \to \mathbb{R}$ be integrable and φ-convex functions, then, for $0<q<1$, we have*

$$2f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)g\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) -\frac{1}{2}[K_1M(a,b)+K_2N(a,b)] \leq \frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)\,\mathrm{d}_qx,$$

where

$$K_1=\frac{q^2}{(1+q)(1+q+q^3)},$$
$$K_2=\frac{1+2q+q^3}{(1+q)(1+q+q^2)},$$
$$M(a,b)=f(a)g(a)+f(b)g(b),$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

Proof. Since f and g be φ-convex functions, then

$$\begin{aligned}
&f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)g\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\\
&= f\left(\frac{a+te^{i\varphi}(b-a)+a+(1-t)e^{i\varphi}(b-a)}{2}\right)\\
&\quad\times g\left(\frac{a+te^{i\varphi}(b-a)+a+1-te^{i\varphi}(b-a)}{2}\right)\\
&\leq \frac{1}{4}\left[\left\{f(a+te^{i\varphi}(b-a))+f(a+(1-t)e^{i\varphi}(b-a))\right\}\right.\\
&\quad\left.\left\{g(a+te^{i\varphi}(b-a))+g(a+(1-t)e^{i\varphi}(b-a))\right\}\right]\\
&\leq \frac{1}{4}\left[\left\{f(a+te^{i\varphi}(b-a))g(a+te^{i\varphi}(b-a))\right.\right.\\
&\quad\left.+f(a+(1-t)e^{i\varphi}(b-a))g(a+(1-t)e^{i\varphi}(b-a))\right\}\\
&\quad\left.+\left\{2t(1-t)M(a,b)+(t^2+(1-t)^2)N(a,b)\right\}\right].
\end{aligned}$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$2f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)g\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) -\frac{2q^2M(a,b)+(1+2q+q^3)N(a,b)}{2(1+q)(1+q+q^2)} \leq \frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)\,\mathrm{d}_qx.$$

This completes the proof. □

Theorem 5. *Let $f,g: I_\varphi \to H$ be integrable and φ-convex function, then, for $0<q<1$, we have*

$$\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)\,\mathrm{d}_qx \leq P_1f(a)g(a)+P_2\left[q(1+q^2)f(b)g(b)+q^2N(a,b)\right],$$

where

$$P_1=\frac{1}{1+q+q^2},$$
$$P_2=\frac{1}{(1+q)(1+q+q^2)},$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

Proof. Since f and g are φ-convex functions, then

$$f(a+te^{i\varphi}(b-a))\leq(1-t)f(a)+tf(b), \tag{10}$$

and

$$g(a+te^{i\varphi}(b-a))\leq(1-t)g(a)+tg(b). \tag{11}$$

Multiplying (10) and (11), we have

$$\begin{aligned}
&f(a+te^{i\varphi}(b-a))g(a+te^{i\varphi}(b-a))\\
&\leq(1-t)^2f(a)g(a)+t(1-t)f(a)g(b)\\
&\qquad+t(1-t)f(b)g(a)+t^2f(b)g(b).
\end{aligned}$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f(a+te^{i\varphi}(b-a))g(a+te^{i\varphi}(b-a))\,\mathrm{d}_qt\\
&\leq f(a)g(a)\int_0^1(1-t)^2\,\mathrm{d}_qt+f(a)g(b)\int_0^1 t(1-t)\,\mathrm{d}_qt\\
&\qquad+f(b)g(a)\int_0^1 t(1-t)\,\mathrm{d}_qt+f(b)g(b)\int_0^1 t^2\,\mathrm{d}_qt.
\end{aligned}$$

This implies

$$\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+te^{i\varphi}(b-a)} f(x)g(x)\,\mathrm{d}_qx \leq P_1f(a)g(a)+P_2\left[q(1+q^2)f(b)g(b)+q^2N(a,b)\right].$$

This completes the proof. □

We now give an auxiliary result which will be helpful in obtaining our next results.

Lemma 3. *Let $f: I_\varphi \to H$ be a continuous function and $0<q<1$. If $\mathscr{D}_qf$ is an integrable function on I_φ^0, then*

$$\Omega_f(a,b;q;\varphi) =\frac{qe^{i\varphi}(b-a)}{1+q}\int_0^1(1-(1+q)t)\,\mathscr{D}_qf(a+te^{i\varphi}(b-a))\,\mathrm{d}_qt,$$

where

$$\Omega_f(a,b;q;\varphi)=\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)\,\mathrm{d}_qx -\frac{qf(a)+f(a+e^{i\varphi}(b-a))}{1+q}.$$

Proof. The proof is left on interested readers. □

Theorem 6. *Let $f: I_\varphi \to \mathbb{R}$ be a q-differentiable function on I_φ° with $\mathscr{D}_q$ be continuous and integrable on I_φ where $0<q<1$. If $|\mathscr{D}_q f|$ is φ-convex function, then*

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&\le \Psi_\varphi(a,b;q)\left[(1+3q^2+2q^3)|\mathscr{D}_q f(a)|\right.\\&\qquad\left.+(1+4q+q^2)|\mathscr{D}_q f(b)|\right],\end{aligned}$$

where

$$\Psi_\varphi(a,b;q)=\frac{q^2 e^{i\varphi}(b-a)}{(1+q)^2(1+q+q^2)}.$$

Proof. Using Lemma 3, property of modulus and the fact that $|\mathscr{D}_q f|$ is φ-convex function, then

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&=\left|\frac{qe^{i\varphi}(b-a)}{1+q}\int_0^1(1-(1+q)t)\,\mathscr{D}_q f(a+te^{i\varphi}(b-a))\,\mathrm{d}_q t\right|\\&\le\frac{qe^{i\varphi}(b-a)}{1+q}\left[|\mathscr{D}_q f(a)|\int_0^1|1-(1+q)t|(1-t)\,\mathrm{d}_q t\right.\\&\qquad\left.+|\mathscr{D}_q f(b)|\int_0^1|1-(1+q)t|t\,\mathrm{d}_q t\right]\\&=\frac{q^2e^{i\varphi}(b-a)}{(1+q)^4(1+q+q^2)}\\&\quad\times\left[(1+3q^2+2q^3)|\mathscr{D}_q f(a)|+(1+4q+q^2)|\mathscr{D}_q f(b)|\right].\end{aligned}$$

This completes the proof. □

Theorem 7. *Let $f: I_\varphi \to \mathbb{R}$ be a q-differentiable function on I_φ° with $\mathscr{D}_q$ be continuous and integrable on I_φ where $0<q<1$. If $|\mathscr{D}_q f|^r$ is φ-convex function, where $r\ge 1$, then*

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&\le\Theta_\varphi(a,b;q)\\&\times\left[\frac{(1+3q^2+2q^3)|\mathscr{D}_q f(a)|^r+(1+4q+q^2)|\mathscr{D}_q f(b)|^r}{(1+q+q^2)(2+q+q^3)}\right]^{\frac{1}{r}},\end{aligned}$$

where

$$\Theta_\varphi(a,b;q)=\frac{q^2(2+q+q^2)e^{i\varphi}(b-a)}{(1+q)^4}.$$

Proof. Using Lemma 3, property of modulus, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is φ-convex function, then

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&=\left|\frac{qe^{i\varphi}(b-a)}{1+q}\int_0^1(1-(1+q)t)\,\mathscr{D}_q f(a+te^{i\varphi}(b-a))\,\mathrm{d}_q t\right|\\&\le\left(\int_0^1|1-(1+q)t|\,\mathrm{d}_q t\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1|1-(1+q)t|[(1-t)|\mathscr{D}_q f(a)|^r+t\mathscr{D}_q f(b)|^r]\,\mathrm{d}_q t\right)^{\frac{1}{r}}\\&=\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{1-\frac{1}{r}}\\&\quad\times\left(\frac{q}{(1+q)^3(1+q+q^2)}\left[(1+3q^2+2q^3)|\mathscr{D}_q f(a)|^r\right.\right.\\&\qquad\left.\left.+(1+4q+q^2)|\mathscr{D}_q f(b)|^r\right]\right)^{\frac{1}{r}}.\end{aligned}$$

This completes the proof. □

Now, we derive some q-analogues of Iyengar type inequalities.

Theorem 8. *Let $f: I_\varphi \to \mathbb{R}$ be a q-differentiable function on I_φ° with $\mathscr{D}_q$ be continuous and integrable on I_φ where $0<q<1$. If $|\mathscr{D}_q f|^r$ is quasi φ-convex function where $r\ge 1$, then*

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&\le\frac{q^2e^{i\varphi}(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\{|\mathscr{D}_q f(a)|^r,|\mathscr{D}_q f(b)|^r\}\right)^{\frac{1}{r}}.\end{aligned}$$

Proof. Using Lemma 3, property of modulus, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is quasi φ-convex function, we have

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&=\left|\frac{qte^{i\varphi}(b-a)}{1+q}\int_0^1(1-(1+q)t)\,{}_a\mathrm{D}_q f(a+te^{i\varphi}(b-a))_0\mathrm{d}_q t\right|\\&\le\frac{qe^{i\varphi}(b-a)}{1+q}\left(\int_0^1|1-(1+q)t|\mathrm{d}_q t\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1|1-(1+q)t||{}_a\mathrm{D}_q f(a+te^{i\varphi}(b-a))|^r{}_0\mathrm{d}_q t\right)^{\frac{1}{r}}\\&=\frac{q^2e^{i\varphi}(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\{|\mathscr{D}_q f(a)|^r,|\mathscr{D}_q f(b)|^r\}\right)^{\frac{1}{r}}.\end{aligned}$$

This completes the proof. □

Theorem 9. *Under the conditions of Theorem 8, if $r=1$, then, we have*

$$\begin{aligned}&\left|\Omega_f(a,b;q;\varphi)\right|\\&\le\frac{q^2e^{i\varphi}(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\{|\mathscr{D}_q f(a)|,|\mathscr{D}_q f(b)|\}\right).\end{aligned}$$

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Muhammad Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics(Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 800 research papers to his credit which were published in leading world class journals.

Khalida Inayat Noor is a leading world-known figure in mathematics and is presently employed as HEC Foreign Professor at CIIT, Islamabad. She obtained her PhD from Wales University (UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: Presidents Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at CIIT. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred) research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.

Muhammad Uzair Awan has earned his Ph.D degree from COMSATS Institute of Information Technology, Islamabad, Pakistan under the supervision of Prof. Dr. Muhammad Aslam Noor. His field of interest is Convex Analysis and Numerical Optimization. He has published several papers in outstanding International journals of Mathematics and Engineering sciences.

Farhat Safdar has earned her MS degree from COMSATS Institute of Information Technology, Islamabad, Pakistan. Currently she is Ph.D Scholar in Mathematics Department, COMSATS Institute of Information Technology, Islamabad. Her field of interest is Convex and Numerical Analysis.