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Fractional Vector Calculus and Fractional Continuum Mechanics

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Abstract: Since modern continuum mechanics is mainly characterized by the strong influence of microstructure, Fractional Continuum Mechanics has been a promising research field, satisfying both experimental and theoretical demands. The geometry of the fractional differential is corrected and the geometry of the tangent spaces of a manifold is clarified providing the bases of the missing Fractional Differential Geometry. The Fractional Vector Calculus is revisited along with the basic field theorems of Green, Stokes and Gauss. New concepts of the differential forms, such as fractional gradient, divergence and rotation are introduced. Application of the Fractional Vector Calculus to Continuum Mechanics is presented. The Fractional right and left Cauchy-Green deformation tensors and Green (Lagrange) and Euler-Almanssi strain tensors are exhibited. The change of volume and the surface due to deformation (configuration change) of a deformable body are also discussed. Fractional stress tensors are also introduced. Further the Fractional Continuum Mechanics principles yielding the fractional continuity and motion equations are also derived.

Keywords: Fractional derivative, fractional deformation tensors, fractional deformation strain, fractional differential surface, fractional continuity equation.

1 Introduction

The mechanics researchers have been motivated by the mechanical behavior of disordered (non-homogeneous) materials with microstructure. Porous materials, Vardoulakis et al [1], Ma et al [2] , colloidal aggregates, Wyss et al [3], ceramics, etc, are materials with microstructure that exert strong influence in their deformation. Major factors in determining the material deformation are microcracks, voids, material phases etc. The non-homogeneity of the heterogeneous materials has been tackled by various homogenization theories Bakhalov & Panasenko [4]. Nevertheless, these materials require the lifting of the basic local action axiom of Continuum Mechanics, Truesdell [5], Truesdell et al. [6]. As defined by Noll [7] simple materials satisfy the three fundamental axioms:
(a) The principle of determinism.
(b) The principle of local action.
(c) The principle of material frame-indifference.
Truesdell [5] points out in his classic continuum mechanics book. The motion of body-points at a finite distance from a point x in some shape may be disregarded in calculating the stress at x. Material microstructure, inhomogeneities, microcracks etc., are some of the various important factors that affect the material deformation with non-local action. These factors are not considered in the simple materials formulation. Various theories have been proposed just to introduce a long distance action in the deformation of the materials. One direction considers Taylor's expansion of the strain tensor in the neighborhood of a point, taking in consideration one or two most important terms. Hence gradient strain theories have appeared in non-linear form, Toupin [8], and in linear deformation, Mindlin [9]. Eringen [10] has also proposed a theory dealing with micropolar elasticity. Mindlin introduced a more simple version of linear gradient theories and an even simpler model has been presented by Aifantis [11] with his GRADELA model. In these theories, the authors introduced intrinsic material lengths that accompany the higher order derivatives of the strain. Many problems have been solved employing those theories concerning size effects, lifting of various singularities, porous

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materials, Aifantis [11,12,13], Askes & Aifantis [14], mechanics of microbeams, microplates and microsheets, Lazopoulos [15,16]. Another non-local approach was introduced by Kunin [17,18] Lazopoulos [19] introduced fractional derivatives of the strain in the strain energy density function in an attempt to introduce non-locality in the elastic response of materials. Fractional calculus was used by many researchers, not only in the field of Mechanics but mainly in Physics and especially in Quantum Mechanics, to develop the idea of introducing non-locality through. In fact, the history of fractional calculus is dated since 17th century. Particle physics, electromagnetics, mechanics of materials, Hydrodynamics, fluid flow, rheology, viscoelasticity, optics, electrochemistry and corrosion, chemical physics are some fields where fractional calculus has been introduced. Fractional calculus in material deformations has been adopted in solving various types of problems. First we may consider the deformation problems with non-smooth strain field. Secondly heterogeneous material deformations may also be studied. Furthermore, time fractional derivative is proved to be more suitable in viscoelastic deformations, since viscoelastic deformations with retarded memory materials may also be discussed. The non-local strain effects of deformation problems are concerned by the last type of those problems. There are many studies considering fractional elasticity theory, introducing fractional strain, Drapaca et al [20], Carpinteri et al. [21,22], Di Paola et al. [23], Atanackovic et al. [24], Agrawal [25], Sumelka [26]. Baleanu and his co-workers [27,28,29] has presented along list of publications concerning various applications of Fractional Calculus in Physics, in control theory in solving differential equations and numerical solutions. In addition Tarasov [30,31] has presented a Fractional Vector Fields theory combining fractals, Feder [32] and fractional calculus. Lazopoulos [33,34] has clarified the geometry of the fractional differential resulting in fractional tangent spaces of the manifolds quite different from the conventional ones. Hence the Fractional Differential Geometry has been established, indispensable for the development of Fractional Mechanics. It is evident that the definition of the stress and the strain is greatly affected by the tangent spaces. Hence the fractional stress tensors and the fractional strain tensors are quite different from the conventional ones. The linear strain tensors are also revisited. Those basic concepts are important for establishing Fractional Continuum Mechanics. In the present work, Fractional Vector Calculus is revisited, since the fractional differential of a function is not linearly dependent upon the differential of the variables. Furthermore, the fractional derivative of a variable with respect to itself is different from one. The Fractional Vector Calculus is revisited along with the basic field theorems of Green, Stokes and Gauss. Application of the Fractional Vector Calculus to Continuum Mechanics is presented. The revision in the right and left Cauchy-Green deformation tensors and Green (Lagrange) and Euler-Almanssi strain tensors are exhibited. The change of volume and the surface due to deformation (change of configuration) of a deformable body is also discussed. Further the revisited Fractional Continuum Mechanics principles yielding the fractional continuity and motion equations are also discussed.

2 Basic Properties of Fractional Calculus

Fractional Calculus were introduced by Leibniz, who pointed out, in his letters to l'Hospital in 1695 and Wallis in 1725, the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n = \frac{1}{2}$. Lately fractional calculus has become a branch of pure mathematics with many applications in Physics and Engineering, Tarasov [30,31], Baleanu et al. [35], Golmankhaneh et al. [36]. There exist many definitions of fractional derivatives with some advantages of the one over the others. Nevertheless they all share one common property. They are not local, contrary to the conventional ones. Details concerning the properties of fractional derivatives may be found in Kilbas et al. [39], Podlubny [37], Samko et al. [38]. Starting from Cauchy formula for the n-fold integral of a primitive function *f(x)*

$$ {}_{\mathrm{a}}I_x^n f(x) = \frac{1}{(n-1)!}\int_{\mathrm{a}}^{x} (x-s)^{n-1} f(s)ds, \quad x>0, n\in N \tag{1} $$

and

$$ {}_{x}I_b^n f(x) = \frac{1}{(n-1)!}\int_{x}^{b} (s-x)^{n-1} f(s)ds, \quad x>0, n\in N \tag{2} $$

we define the left and right fractional integral of f as:

$$ {}_{\mathrm{a}}I_x^a f(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathrm{a}}^{x} \frac{f(s)}{(x-s)^{1-a}} ds, \tag{3} $$

$$ {}_{x}I_b^a f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b} \frac{f(s)}{(s-x)^{1-a}} ds. \tag{4} $$

In Eqs.(3,4) we assume that is the order of fractional integrals with $0 < a \leq 1$, considering $\Gamma(x) = (x-1)!$ with $\Gamma(\alpha)$ Eulers Gamma function. Thus the left and right Riemann-Liouville (R-L) derivatives are defined by:

$$ {}_{\mathrm{a}}D_x^a f(x) = \frac{d}{dx}({}_{\mathrm{a}}I_x^{1-a}\ f(x)) \tag{5} $$

and

$$ {}_xD_b^a f(x) = -\frac{d}{dx}({}_bI_x^{1-a}\ f(x)). \tag{6} $$

Nevertheless the fact that the R-L derivatives of a constant c are not zero, imposed the need for Jumarie derivative that is more friendly in the description of physical systems, although it is more restrictive. In fact Caputo derivatives are defined by:

$$ {}_{\mathrm{a}}^{c}D_x^a f(x) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^{x}\frac{f'(s)}{(x-s)^a}ds \tag{7} $$

and

$$ {}_{x}^{c}D_b^a f(x) = -\frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}\frac{f'(s)}{(s-x)^a}ds. \tag{8} $$

Evaluating Caputo derivatives for functions of the type

$$ f(x) = (x-a)^n \ \ or \ \ f(x) = (b-x)^n $$

we get:

$$ \mathrm{a}^c D_x^a(x-\mathrm{a})^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(x-\mathrm{a})^{\nu-\alpha}. \tag{9} $$

and for the corresponding right Caputo derivative:

$$ {}_{x}^{c}D_b^a(b-x)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(b-x)^{\nu-\alpha}. \tag{10} $$

Likewise, Caputo derivatives are zero for constant functions:

$$ {}_{x}^{c}D_b^a c = 0. \tag{11} $$

3 The Fractional Tangent Plane of a Surface

It is reminded that the n-fold integral of the primitive function f(x), Eq.(1), is

$$ I^n f(x) = \int_{\mathrm{a}}^{x} f(s)\,(ds)^n \tag{12} $$

that is real for any positive or negative increment ds. Passing to the fractional integral

$$ I^\alpha\,(f(x)) = \int_{\mathrm{a}}^{x} f(s)\,(ds)^\alpha \tag{13} $$

simply the integer n is substituted by the fractional number α . Nevertheless, that substitution is not at all straightforward. The major difference between passing from Eq.(11) to Eq.(12) is that although $(ds)^n$ is real for negative values of ds ds, $(ds)^\alpha$ is complex. Therefore, the fractional integral, Eq.(13), is not compact for any increment ds. Hence the integral of Eq.(13) is misleading. In other words, the differential that is necessary for the existence of the fractional integral, Eq.(13), is wrong. Therefore, a new fractional differential that is real and valid for positive and negative values of the increment ds should be established. It is reminded that the a-Fractional differential of a function $f(x)$ is defined by, [28]:

$$ d^a f(x) = {}_{\mathrm{a}}^{c}D_x^a f(x)(dx)^a. \tag{14} $$

It is evident that the fractional differential defined by Eq.(14) is valid for positive incremental dx, whereas for negative ones that differential might be complex. Hence considering for the moment that the increment dx is positive, and recalling that ${}_{\mathrm{a}}^{c}D_{x}^{a}x \neq 1$, the a-fractional differential of the variable x is:

$$d^{a}x = {}_{\mathrm{a}}^{c}D_{x}^{a}x(dx)^{a}. \tag{15}$$

Hence,

$$d^{a}f(x) = \frac{{}_{\mathrm{a}}^{c}D_{x}^{a}f(x)}{{}_{\mathrm{a}}^{c}D_{x}^{a}x}d^{a}x. \tag{16}$$

It is evident that $d^{a}f(x)$ is a non-linear function of dx, although it is a linear function of $d^{a}x$. That fact suggests the consideration of the fractional tangent space that we propose. Now, the definition of fractional differential, Eq.(16), is imposed either for positive or negative variable differentials $d^{\alpha}x$. In addition the proposed L-fractional (in honour of Leibniz) derivative ${}_{0}^{L}D_{x}^{a}f(x)$ is defined by,

$$d^{a}f(x) = {}_{\mathrm{a}}^{L}D_{x}^{a}f(x)d^{a}x \tag{17}$$

with the Leibniz L-fractional derivative,

$${}_{\mathrm{a}}^{L}D_{x}^{a}f(x) = \frac{{}_{\mathrm{a}}^{c}D_{x}^{a}f(x)}{{}_{\mathrm{a}}^{c}D_{x}^{a}x}. \tag{18}$$

Hence only Leibniz derivative has any geometrical of physical meaning. In addition, Eq. 3, is deceiving and the correct form of Eq. (3) should be substituted by,

$$f(x) - f(\mathrm{a}) = {}_{\mathrm{a}}^{L}I_{x}^{a}\left({}_{\mathrm{a}}^{L}D_{x}^{\alpha}f(x)\right) = \frac{1}{\Gamma(\alpha)\Gamma(2-a)}\int_{\mathrm{a}}^{x}\frac{(s-\mathrm{a})^{1-\alpha}}{(x-s)^{1-a}}\,{}_{\mathrm{a}}^{L}D_{x}^{\alpha}f(s)ds. \tag{19}$$

It should be pointed out that the correct forms are defined for the fractional differential by Eq.(17), the Leibniz derivative, Eq.(18) and the fractional integral by Eq.(19). All the other forms are misleading. Configuring the fractional differential along with the first fractional differential space (fractional tangent space), the function $y = f(x)$ has been drawn in Fig.1, with the corresponding first differential space at a point x according to Addas definition, Eq.(14).

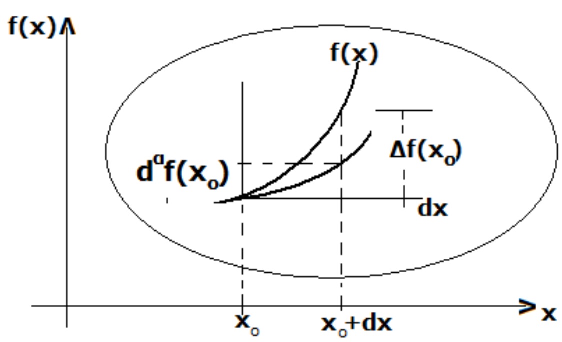

Fig. 1: The non-linear differential of $f(x)$

The tangent space , according to Adda [40] definition, Eq. (14), is configured by the nonlinear curve $d^{\alpha}f(x)$ versus dx. Nevertheless, there are some questions concerning the correct picture of the configuration, Fig. 1, concerning the fractional differential presented by Adda [40]. Indeed,

(a)The tangent space should be linear. There is not conceivable reason for the nonlinear tangent spaces.
(b)The differential should be configured for positive and negative increments dx. However, the tangent spaces, in the present case, do not exist for negative increments dx.

(c)The axis $d^{\alpha}f(x)$, in Fig.1, presents the fractional differential of the function f(x), however the axis dx denotes the conventional differential of the variable x. It is evident that both axes along x and $f(x)$ should correspond to differentials of the same order.

Therefore, the tangent space (first differential space), should be configured in the coordinate system with axes $(d^{\alpha}x,\ d^{\alpha}f(x))$ Hence, the fractional differential, defined by Eq. (17), is configured in the plane $(d^{\alpha}x,\ d^{\alpha}f(x))$ as a line, as it is shown in Fig.

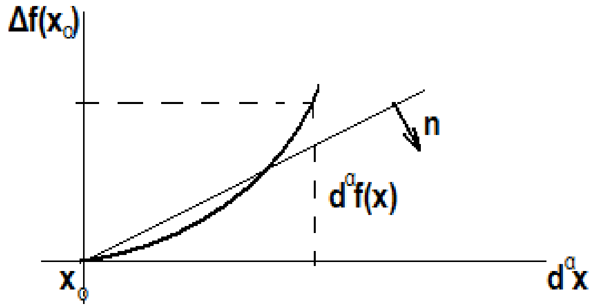

Fig. 2: The virtual tangent space of the f(x) at the point $x = x_0$

It is evident that the differential space is not tangent (in the conventional sense) to the function at x_0, but intersects the figure $y = f(x)$ at least at one point x_0. This space, we introduce, is the tangent space, that is updated at any point x. Likewise, we may consider in addition to the Fractional tangent space, the normal at any point. The normal is perpendicular to the line of the fractional tangent. Hence we are able to establish Fractional Differential Geometry of curves and surfaces with the Fractional Field Theory. It is evident that when $\alpha = 1$, the tangent spaces we propose, coincide with the conventional tangent spaces. Let us consider a manifold with points $M(u,v)$ defined by the vectors

$$M(u,v) = \mathbf{x}(u,v) \tag{20}$$

with

$$x_i = x_i(u,v), \qquad u_1 \leqslant u \leqslant u_2, \quad v_1 \leqslant v \leqslant v_2, \qquad i = 1,2,3. \tag{21}$$

The infinitesimal distance between two points P and Q on the manifold M is defined by,

$$d^{\alpha}\mathbf{x} = \frac{{}_{\text{a}}^{c}D_u^{\alpha}\mathbf{x}}{{}_{\text{a}}^{c}D_u^{\alpha}u}d^{\alpha}u + \frac{{}_{\text{a}}^{c}D_u^{\alpha}\mathbf{x}}{{}_{\text{a}}^{c}D_u^{\alpha}v}d^{\alpha}v. \tag{22}$$

In fact for the surface

$$z = u^2v^2, \tag{23}$$

(see, Fig. 3, Fig. 4)show the surface defined by Eq.(23) with its fractional tangent plane (space) at the point $(u,v) = (0.5,0.5)$ for two fractional dimensions, $\alpha = 1$ (the conventional case) and $\alpha = 0.3$. It is clear that the fractional tangent plane is different from the conventional one $(\alpha = 1)$.

4 Fractional Vector Calculus

For Cartesian coordinates, fractional generalizations of the divergence or gradient operators are defined by:

$$\nabla^{(\alpha)}f(\mathbf{x}) = grad^{(a)}f(\mathbf{x}) = \nabla_i^{(\alpha)}f(x)\mathbf{e}_i = \frac{{}_{\omega}^{c}D_i^{a}f(\mathbf{x})}{{}_{\omega}^{c}D_i^{a}x_i}\mathbf{e}_i = {}_{\omega}^{L}D_i^{a}f(\mathbf{x})\mathbf{e}_i, \tag{24}$$

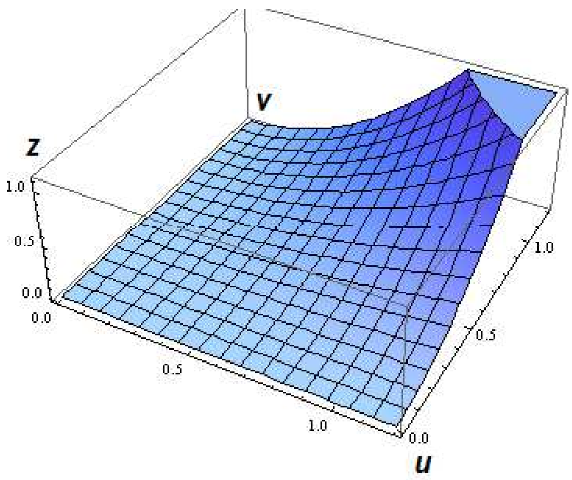

Fig. 3: The surface $z = u^2 v^2$

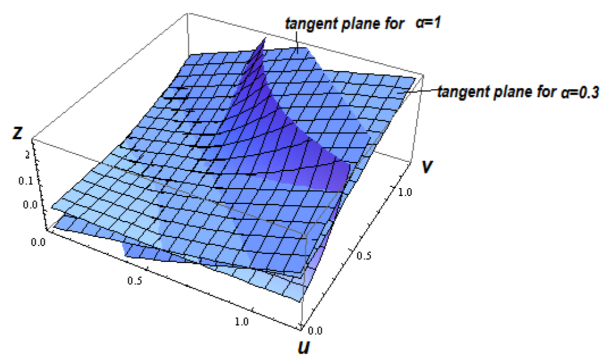

Fig. 4: The tangent planes for various values of the fractional dimension α

where ${}_{\omega}^{c}D_{i}^{a}$ are Caputo fractional derivatives of order and the sub line meaning no contraction. Hence, the gradient of the vector x is

$$\nabla^{(\alpha)}\mathbf{x} = \mathbf{I}$$

with I denoting the identity matrix.

Consequently for a vector field

$$\mathbf{F}(x_1,x_2,x_3) = \mathbf{e}_1 F_1(x_1,x_2,x_3) + \mathbf{e}_2 F_2(x_1,x_2,x_3) + \mathbf{e}_3 F_3(x_1,x_2,x_3), \tag{25}$$

where $F_i(x_1,x_2,x_3)$ are absolutely integrable, the circulation is defined by:

$$C_L^{(\alpha)}(\mathbf{F}) = ({}_{\omega}I_L^{(a)}, \mathbf{F}) = \int\limits_L (dL, \mathbf{F}) = {}_{\omega}I^{(a)}{}_L\left(F_1 d^{\alpha} x_1\right) + {}_{\omega}I^{(a)}{}_L\left(F_2 d^{\alpha} x_2\right) + {}_{\omega}I^{(a)}{}_L\left(F_3 d^{\alpha} x_3\right). \tag{26}$$

It should be pointed out that line, surface and volume fractional integrals are different from the one, two or three dimensional fractional integrals of the function respectively, since the fractional derivatives of the variables are different from one. Furthermore, the divergence of a vector $F(x)$ is defined by:

$$\nabla^{(a)} \cdot \mathbf{F}(x) = div^{(a)}\mathbf{F}(x) = \frac{{}_{\omega}^{c}D_k^a F_k(\mathbf{x})}{{}_{\omega}^{c}D_k^a x_k} = {}_{\omega}^{L}D_k^a F_k(\mathbf{x}), \tag{27}$$

where the sub-line denotes no contraction.
Likewise, the fractional curl F $(curl(a)F(x))$ of a vector F is defined by,

$$curl^{(a)}\mathbf{F} = \mathbf{e}_l\, \varepsilon_{lmn} \frac{{}^{c}_{\omega}D^{a}_{m}F_n}{{}^{c}_{\omega}D^{a}_{m}x_m} = \mathbf{e}_l\, \varepsilon_{lmn}\, {}^{L}_{\omega}D^{a}_{m}F_n. \tag{28}$$

The flux in common vector calculus is defined by

$$\Phi_s(\mathbf{F}) = (I_s, \mathbf{F}) = \iint_S (F_1 dx_2 dx_3 + F_2 dx_3 dx_1 + F_3 dx_1 dx_2).$$

Since

$$\mathbf{dS} = \mathbf{e}_1 dx_2 dx_3 + \mathbf{e}_2 dx_3 dx_1 + \mathbf{e}_3 dx_1 dx_2$$

,

a fractional flux of the vector F expressed in Cartesian coordinates across surface S is a fractional surface integral of the field with:

$$\Phi^{\alpha}_s(\mathbf{F}) = ({}_{\omega}I^{\alpha}_s, \mathbf{F}) = {}_{\omega}^{(\alpha)}\iint_S (F_1 d^{\alpha}x_2 d^{\alpha}x_3 + F_2 d^{\alpha}x_3 d^{\alpha}x_1 + F_3 d^{\alpha}x_2 d^{\alpha}x_3). \tag{29}$$

A fractional volume integral of a triple fractional integral of a scalar field $f = f(x_1, x_2, x_3)$ is defined by:

$${}_{\omega}V^{(a)}_{\Omega}[f] = {}_{\omega}I^{(a)}_{\Omega}[x_1, x_2, x_3]\, f(x_1, x_2, x_3) = {}_{\omega}^{(\alpha)}\iiint_{\Omega} f(x_1, x_2, x_3)\, d^{\alpha}x_1 d^{\alpha}x_2 d^{\alpha}x_3. \tag{30}$$

It should be pointed out that the triple fractional integral is not a volume integral, since the fractional derivative of a variable with respect to itself is different from one. So there is a clear distinction between the simple, double or triple integrals and the line, surface and volume integrals respectively.

5 Fractional Vector Field Theorems

(a)Fractional Green formula.

Green theorem relates a line integral around a simple closed curve ∂B and a double integral over the plane region B with boundary ∂B. With positively oriented boundary , the conventional Green theorem for a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2$ is expressed by:

$$\int_{\partial B} (F_1 dx_1 + F_2 dx_2) = \int\int_B \left(\frac{\partial (F_1)}{\partial x_2} - \frac{\partial (F_2)}{\partial x_1}\right) dx_1 dx_2. \tag{31}$$

Recalling that:

$$d^a\mathbf{x} = (d^a x_1, d^a x_2) = ({}_{\mathrm{a}}D^{a}_{x_1}[x_1]\,(dx_1)^{\alpha}, {}_{\mathrm{a}}D^{a}_{x_2}[x_2]\,(dx_2)^{\alpha})$$

and substituting into conventional Green theorem Eq.(31) we get:

$$\begin{aligned} {}_{\omega}\int^{(\alpha)}_{\partial W} (F_1\, d^{\alpha}x_1 + F_2\, d^{\alpha}x_2) &= {}_{\omega}\int^{(\alpha)}\int_W \left(\frac{{}^{c}_{\omega}D^{a}_{x_2}(F_1)}{{}^{c}_{\omega}D^{a}_{x_2}(x_2)} - \frac{{}^{c}_{\omega}D^{a}_{x_1}(F_2)}{{}^{c}_{\omega}D^{a}_{x_1}(x_1)}\right) d^{\alpha}x_1\, d^{\alpha}x_2 \\ &= {}_{\omega}\int^{(\alpha)}\int_W \left({}^{L}_{\omega}D^{a}_{x_2}(F_1) - {}^{L}_{\omega}D^{a}_{x_1}(F_2)\right) d^{\alpha}x_1\, d^{\alpha}x_2. \end{aligned} \tag{32}$$

(b)Fractional Stokes formula:
Restricting in the consideration of a simple surface W, if we denote its boundary by ∂W and if F is a vector field defined on W, then the conventional Stokes Theorem asserts that:

$$\oint_{W} \mathbf{F}\cdot\mathbf{dL} = \oiint_{W} curl\mathbf{F}\cdot\mathbf{dS}. \tag{33}$$

In Cartesian coordinates it yields,

$$\int_{\partial W} (F_1 dx_1 + F_2 dx_2 + F_3 dx_3) = \int\int_{W} \left(\frac{\partial (F_3)}{\partial x_2} - \frac{\partial (F_2)}{\partial x_3}\right) dx_2 dx_3 + \left(\frac{\partial (F_1)}{\partial x_3} - \frac{\partial (F_3)}{\partial x_1}\right) dx_3 dx_1 + \left(\frac{\partial (F_2)}{\partial x_1} - \frac{\partial (F_1)}{\partial x_2}\right) dx_1 dx_2, \tag{34}$$

where $F(x_1,x_2,x_3) = e_1 F_1(x_1,x_2,x_3) + e_2 F_2(x_1,x_2,x_3) + e_3 F_3(x_1,x_2,x_3)$.
In this case the fractional curl operation is defined by:

$$\begin{aligned} curl_w^{\alpha}(\mathbf{F}) &= e_l \varepsilon_{lmn} {}^{c}_{\omega}D^{a}_{x_m}(F_n) / {}^{c}_{\omega}D^{a}_{x_k}(x_i)\, \delta_{mk} = \mathbf{e}_1 \left(\frac{{}^{c}_{\omega}D^{a}_{x_2}F_3}{{}^{c}_{\omega}D^{a}_{x_2}x_2} - \frac{{}^{c}_{\omega}D^{a}_{x_3}F_2}{{}^{c}_{\omega}D^{a}_{x_3}x_3}\right) \\ &+ \mathbf{e}_2 \left(\frac{{}^{c}_{\omega}D^{a}_{x_3}F_1}{{}^{c}_{\omega}D^{a}_{x_3}x_3} - \frac{{}^{c}_{\omega}D^{a}_{x_1}F_3}{{}^{c}_{\omega}D^{a}_{x_1}x_1}\right) + \mathbf{e}_3 \left(\frac{{}^{c}_{\omega}D^{a}_{x_1}F_2}{{}^{c}_{\omega}D^{a}_{x_1}x_1} - \frac{{}^{c}_{\omega}D^{a}_{x_2}F_1}{{}^{c}_{\omega}D^{a}_{x_2}x_2}\right) \\ &= \mathbf{e}_1 \left({}^{L}_{\omega}D^{a}_{x_2}F_3 - {}^{L}_{\omega}D^{a}_{x_3}F_2\right) + \mathbf{e}_2 \left({}^{L}_{\omega}D^{a}_{x_3}F_1 - {}^{L}_{\omega}D^{a}_{x_1}F_3\right) + \mathbf{e}_3 \left({}^{L}_{\omega}D^{a}_{x_1}F_2 - {}^{L}_{\omega}D^{a}_{x_2}F_1\right). \end{aligned} \tag{35}$$

Therefore, transforming the conventional Stokes theorem into the fractional form we get:

$$\begin{aligned} {}^{(\alpha)}_{\omega}\int_{W} (F_1 d^{\alpha}x_1 + F_2 d^{\alpha}x_2 + F_3 d^{\alpha}x_3) &= {}^{(\alpha)}_{\omega}\iint_{W} \left\{ \left(\frac{{}^{c}_{\omega}D^{a}_{x_2}F_3}{{}^{c}_{\omega}D^{a}_{x_2}x_2} - \frac{{}^{c}_{\omega}D^{a}_{x_3}F_2}{{}^{c}_{\omega}D^{a}_{x_3}x_3}\right) d^{\alpha}x_2 d^{\alpha}x_3 \right. \\ &\left. + \left(\frac{{}^{c}_{\omega}D^{a}_{x_3}F_1}{{}^{c}_{\omega}D^{a}_{x_3}x_3} - \frac{{}^{c}_{\omega}D^{a}_{x_1}F_3}{{}^{c}_{\omega}D^{a}_{x_1}x_1}\right) d^{\alpha}x_3 d^{\alpha}x_1 + \left(\frac{{}^{c}_{\omega}D^{a}_{x_1}F_2}{{}^{c}_{\omega}D^{a}_{x_1}x_1} - \frac{{}^{c}_{\omega}D^{a}_{x_2}F_1}{{}^{c}_{\omega}D^{a}_{x_2}x_2}\right) d^{\alpha}x_1 d^{\alpha}x_2 \right\} \\ &= {}^{(\alpha)}_{\omega}\iint_{W} \left\{ \left({}^{L}_{\omega}D^{a}_{x_2}F_3 - {}^{L}_{\omega}D^{a}_{x_3}F_2\right) d^{\alpha}x_2 d^{\alpha}x_3 \right. \\ &\left. + \left({}^{L}_{\omega}D^{a}_{x_3}F_1 - {}^{L}_{\omega}D^{a}_{x_1}F_3\right) d^{\alpha}x_3 d^{\alpha}x_1 + \left({}^{L}_{\omega}D^{a}_{x_1}F_2 - {}^{L}_{\omega}D^{a}_{x_2}F_1\right) d^{\alpha}x_1 d^{\alpha}x_2 \right\}. \end{aligned} \tag{36}$$

(c)Fractional Gauss formula
For the conventional fields theory, let $F = e_1F_1 + e_2F_2 + e_3F_3$. be a continuously differentiable real-valued function in a domain W with boundary ∂W .Then the conventional divergence Gauss theorem is expressed by,

$$\iint_{\partial W} \mathbf{F}\cdot\mathbf{dS} = \iiint_{W} div\mathbf{F}dV. \tag{37}$$

Since

$$\mathbf{d}^{(\alpha)}\mathbf{S} = \mathbf{e}_1 d^{\alpha}x_2 d^{\alpha}x_3 + \mathbf{e}_2 d^{\alpha}x_3 d^{\alpha}x_1 + \mathbf{e}_3 d^{\alpha}x_1 d^{\alpha}x_2, \tag{38}$$

where $d^{a}x_i$ i=1,2,3 is expressed by Eq.(15),

$$d^{(\alpha)}V = d^{\alpha}x_1 d^{\alpha}x_2 d^{\alpha}x_3. \tag{39}$$

Furthermore, see Eq.(27),

$$div^{(a)}\mathbf{F}(x) = \frac{{}^{c}_{\omega}D^{a}_{k}F_k(\mathbf{x})}{{}^{c}_{\omega}D^{a}_{k}x_m}\delta_{km} = {}^{L}_{\omega}D^{a}_{k}F_k(\mathbf{x}). \tag{40}$$

The Fractional Gauss divergence theorem becomes:

$$ {}_{\omega}^{(\alpha)} \iint_{\partial W} \mathbf{F} \cdot d^{(\alpha)} \mathbf{S} = {}_{\omega}^{(\alpha)} \iiint_{w} div^{(\alpha)} \mathbf{F} d^{(\alpha)} V. \quad (41) $$

Remember that the differential $d^{\alpha}\mathbf{S} = \mathbf{n}^{\alpha} d^{\alpha} S$, where $\mathbf{n}^{\alpha}$ is the unit normal of the fractional tangent space as it has been defined in section 3.

6 Fractional Deformation Geometry

Assuming the description in the Euclidean space, we consider the reference configuration B with the boundary ∂B of a body displaced to its current configuration b with the boundary ∂b, see Truesdell [5], Ogden [41]. The points in the reference placement B, defining the material points, are described by X, whereas the set of the displaced points y describe the current configuration b with the boundary ∂b . The coordinate system in B is denoted by X_A, while the corresponding to the current configuration B are reflected to y_i. In the present description both systems have the same axial directions with base vectors e_A , e_i whether the reference concerns the current or the initial (unstressed) configuration. The motion of a reference point $\mathbf{X}$ is described by the function:

$$ \mathbf{y} = \Psi(\mathbf{X}, t). \quad (42) $$

The conventional gradient of the deformation is defined by:

$$ \mathbf{F}(\Psi, t) = \frac{\partial \Psi(\mathbf{X}, t)}{\partial \mathbf{X}} \text{ or } F_{iA} = \frac{\partial \Psi_i}{\partial X_A} \mathbf{e}_i \otimes \mathbf{e}_A. \quad (43) $$

Therefore the differential line element in the current placement is described by:

$$ \mathbf{dy} = \mathbf{F} \mathbf{dX} \text{ or } dy_i = F_{iA} dX_A \quad (44) $$

with its inverse form

$$ \mathbf{F}^{-1}(\mathbf{y}, t) = \frac{\partial \Psi^{-1}(\mathbf{y}, t)}{\partial \mathbf{y}} \text{ or } F_{iA}{}^{-1} = \frac{\partial \Psi_A^{-1}}{\partial y_i} e_A \otimes e_i. \quad (45) $$

Nevertheless, the main difference between the common calculus and the fractional ones is given by the differential form. In the fractional calculus the differential $d^{\alpha}\mathbf{y}$ given by Eq.(44) in the conventional case takes the form,

$$ \mathbf{d}^{\alpha} \mathbf{y} = \overset{\alpha}{\mathbf{F}} (\mathbf{dX})^{\alpha}, \quad (46) $$

where

$$ \overset{\alpha}{\mathbf{F}} = \overset{\alpha}{F}_{ij} = \overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{y} = {}_{a}^{L} D_{X_j}^{\alpha} \mathbf{y}_i = \overset{\alpha}{\nabla}_{\mathbf{X}} (\mathbf{X} + \mathbf{u}) = \overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} + \overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{u} \quad (47) $$

and

$$ (\mathbf{dX})^{\alpha} = (dX_A)^{\alpha} \mathbf{e}_A. \quad (48) $$

Applying Eq.(17) the vector $(\mathbf{dX})^{\alpha}$ may be expressed by the fractional differential $d^{\alpha}\mathbf{X}$, since

$$ (\mathbf{dX})^{\alpha} = \Gamma(2-a) X_A^{\alpha-1} d^{\alpha} X_L \delta_{AL} \mathbf{e}_A = \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \cdot d^{\alpha} \mathbf{X} \quad (49) $$

with

$$ (\overset{\alpha}{\nabla}_{\mathbf{x}} \mathbf{X})^{-1} = \Gamma(2-\alpha) \begin{bmatrix} X_1^{\alpha-1} & 0 & 0 \\ 0 & X_2^{\alpha-1} & 0 \\ 0 & 0 & X_3^{\alpha-1} \end{bmatrix}. \quad (50) $$

Substituting for $(\mathbf{dX})^{\alpha}$ into Eq.(46) that is a nonlinear function of dX we get the relation

$$ d^{\alpha} \mathbf{y} = \overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \cdot d^{\alpha} \mathbf{X}. \quad (51) $$

That is a linear function of the fractional differential $d^{\alpha}\mathbf{X}$. Likewise, the fractional differential $d^{\alpha}\mathbf{y}$ may be expressed in the current placement by,

$$d^{\alpha}\mathbf{y} = (\overset{\alpha}{\nabla}_{\mathbf{y}}\mathbf{y}) \cdot (\mathbf{dy})^{\alpha} \tag{52}$$

with

$$\overset{\alpha}{\underset{\mathbf{y}}{\nabla}}\mathbf{y} = \frac{1}{\Gamma(2-\alpha)} \begin{bmatrix} y_1^{1-\alpha} & 0 & 0 \\ 0 & y_2^{1-\alpha} & 0 \\ 0 & 0 & y_3^{1-\alpha} \end{bmatrix}. \tag{53}$$

Considering the reference state B and its current state b, the infinitesimal linear lengths dS and ds of and in the conventional differential geometry, are defined respectively by

$$dS^2 = dX_A dX_A = d\mathbf{X} \cdot d\mathbf{X} \text{ and } (ds)^2 = dy_i \cdot dy_i = d\mathbf{y} \cdot d\mathbf{y}. \tag{54}$$

However, in fractional differential geometry the infinitesimal lengths, $d^{\alpha}S$ and $d^{\alpha}S$ are defined as

$$d^{a}S^2 = d^{\alpha}\mathbf{X}^T \cdot d^{\alpha}\mathbf{X} \text{ and } d^{\alpha}s^2 = d^{a}\mathbf{y} \cdot d^{\alpha}\mathbf{y}. \tag{55}$$

7 Fractional Strain Tensors

In classical formulation the strain tensors were defined considering the difference of the square of the infinitesimal lengths before and after deformation. In fact:

$$ds^2 - dS^2 = d\mathbf{x}^T \cdot d\mathbf{x} - d\mathbf{X}^T \cdot d\mathbf{X} = d\mathbf{X}^T \cdot (\mathbf{C} - \mathbf{I})d\mathbf{X}, \tag{56}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the right Cauchy-Green deformation tensor. It is pointed out that the identity tensor indicates the Cauchy- Green deformation tensor for the reference placement. Furthermore

$$ds^2 - dS^2 = d\mathbf{x}^T \cdot (\mathbf{I} - \mathbf{B}^{-1})d\mathbf{x}, \tag{57}$$

where $B = FF^T$ is the left Cauchy-Green deformation tensor. That formulation is useful in conventional deformation analysis, since B^{-1} may be expressed with respect to current placement coordinates. In the case of fractional deformation analysis

$$(d^{\alpha}s)^2 - (d^{\alpha}S)^2 = (d^{\alpha}\mathbf{X})^T \cdot (\overset{\alpha}{\mathbf{C}} - \overset{\alpha}{\mathbf{I}}_X)d^{\alpha}\mathbf{X} \tag{58}$$

with $\overset{\alpha}{\underset{x}{\mathbf{I}}}$ the fractional identity matrix expressed by

$$\overset{\alpha}{\underset{X}{\nabla}}\mathbf{X} = \frac{1}{\Gamma(2-\alpha)} \begin{bmatrix} X_1^{1-\alpha} & 0 & 0 \\ 0 & X_2^{1-\alpha} & 0 \\ 0 & 0 & X_3^{1-\alpha} \end{bmatrix}$$

and the right Cauchy-Green fractional deformation tensor,

$$\overset{\alpha}{\mathbf{C}} = \left(\overset{\alpha}{\mathbf{F}}(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}\right)^T \left(\overset{\alpha}{\mathbf{F}}(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}\right) \tag{59}$$

and

$$(d^{\alpha}s)^2 - (d^{\alpha}S)^2 = (d^{\alpha}\mathbf{y})^T \cdot (\mathbf{I} - (\overset{\alpha}{\mathbf{B}})^{-1})d^{\alpha}\mathbf{y}, \tag{60}$$

where,

$$\overset{\alpha}{\mathbf{B}} = \left(\overset{\alpha}{\mathbf{F}}(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}\right) \left(\overset{\alpha}{\mathbf{F}}(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}\right)^T \tag{61}$$

is the left Cauchy-Green fractional deformation tensor. Likewise, the fractional non-linear fractional Green-Lagrange strain tensor $\overset{\alpha}{\mathbf{E}}$ may be defined as

$$\underset{\mathbf{N}}{\overset{\alpha}{E}} = \frac{d^{\alpha}s - d^{\alpha}S}{d^{\alpha}S} \approx \frac{1}{2}\frac{d^{a}s\cdot d^{a}s - d^{\alpha}S\cdot d^{\alpha}S}{d^{\alpha}S\cdot d^{\alpha}S}. \tag{62}$$

Recalling Eqs.(58), Eq.(62) becomes,

$$\underset{\mathbf{N}}{\overset{\alpha}{E}} = \mathbf{N}\cdot\overset{\alpha}{\mathbf{E}}\mathbf{N}, \tag{63}$$

where N is the unit vector of the considered fiber in the reference placement, with

$$\overset{a}{\mathbf{E}} = \frac{1}{2}(\overset{\alpha}{\mathbf{C}} - \mathbf{I}). \tag{64}$$

Recalling that the current placement $\mathbf{y} = \mathbf{X} + \mathbf{u}$, where u denotes the displacement vector, the fractional Green-Lagrange deformation tensor becomes

$$\overset{\alpha}{\mathbf{E}} = \frac{1}{2}\left[(\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T}\cdot(\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^{T} + \overset{a}{\nabla}_{\mathbf{X}}\mathbf{u}\cdot(\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1} + (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T}\cdot(\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^{T}\,\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u}\cdot(\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}\right]. \tag{65}$$

It is pointed out that $\overset{a}{\nabla}\mathbf{X}$ is diagonal, hence

$$\underset{\mathbf{X}}{\overset{a}{\nabla}}\mathbf{X} = \underset{\mathbf{X}}{\overset{a}{\nabla}}\mathbf{X}^{T}. \tag{66}$$

Proceeding to define the corresponding Euler-Almansi strain tensor we define the strain by:

$$\varepsilon_{\mathbf{n}} = \frac{d^{\alpha}s - d^{\alpha}S}{d^{\alpha}s} = \frac{1}{2}\frac{d^{\alpha}s\cdot d^{a}s - d^{\alpha}S\cdot d^{a}S}{d^{\alpha}s\cdot d^{a}s}. \tag{67}$$

Recalling Eq.(60), we derive the fractional strain tensor referred to the current placement, i.e. Euler-Almansi strain tensor,

$$\overset{\alpha}{\mathbf{A}} = \frac{1}{2}(\mathbf{I} - \left(\overset{a}{\mathbf{B}}\right)^{-1}) \tag{68}$$

with the strain

$$\varepsilon_{\mathbf{n}} = \mathbf{n}^{T}\cdot\overset{\alpha}{\mathbf{A}}\mathbf{n}, \tag{69}$$

where n is the unit vector along the deformed fiber, corresponding to the N unit vector along the reference placement fiber . It is evident that in the conventional case with $a = 1$ the fractional Euler Almansi strain tensor $\overset{a}{\mathbf{A}}$ reduces to the conventional strain tensor A. It should be pointed out that the fractional stretches $\overset{\alpha}{\lambda}$ adopting the right Cauchy-Green strain tensor $\overset{\alpha}{\mathbf{C}}$ are defined by:

$$\overset{\alpha}{\lambda} = \frac{\overset{\alpha}{ds}}{\overset{\alpha}{dS}} \tag{70}$$

as the ratio of the measures of the final infinitesimal length over the corresponding length of the fractional differential $d^{\alpha}\mathbf{X}$ vector with $d^{\alpha}S^{a} = (d^{\alpha}\mathbf{X}\cdot d^{\alpha}\mathbf{X})^{\frac{1}{2}}$. In fact the stretches $\overset{a}{\lambda}$ are defined by:

$$\left(\overset{\alpha}{\lambda}\right)^{2} = \mathbf{N}^{T}\cdot\overset{\alpha}{\mathbf{C}}\mathbf{N}, \tag{71}$$

where $\mathbf{N} = N_{A}\mathbf{e}_{A}$ is the unit vector directed along the material reference fiber. Furthermore for

$$\frac{1}{\left(\overset{\alpha}{\lambda}\right)^{2}} = \left(\frac{\overset{\alpha}{dS}}{\overset{\alpha}{ds}}\right)^{2} = \mathbf{n}^{T}\cdot\left(\overset{\alpha}{B}\right)^{-1}\mathbf{n}, \tag{72}$$

where n the unit vector along the deformed fiber.

8 Polar Decomposition of the Deformation Gradient

It is well known that every matrix may be decomposed in a product of an orthogonal and a symmetric positive tensor. Applying the property to the deformation gradient we get

$$\overset{a}{\mathbf{F}} = \mathbf{R}\overset{a}{\mathbf{U}} = \overset{a}{\mathbf{V}}\mathbf{R}, \tag{73}$$

where R is orthogonal $R = R^{-T}$ and U and V are symmetric positive $(U = U^T \ and \ V = V^T)$. Therefore, we have

$$\overset{a}{\mathbf{C}} = \left(\overset{\alpha}{\mathbf{U}}\right)^2 \text{ and } \overset{a}{\mathbf{B}} = \left(\overset{\alpha}{\mathbf{V}}\right)^2. \tag{74}$$

Moreover, the eigenvalues of $\overset{a}{\mathbf{U}}$ and $\overset{a}{\mathbf{V}}$ are the same, but the eigenvectors $\overset{a}{\mathbf{u}}$ of $\overset{a}{\mathbf{U}}$ and $\overset{a}{\mathbf{v}}$ of $\overset{a}{\mathbf{V}}$ are related by $\overset{\alpha}{\mathbf{v}} = \mathbf{R}\overset{\alpha}{\mathbf{u}}$. In fact $\overset{\alpha}{\mathbf{v}}$ is directed along a principal direction (eigenvector) of the strain tensor $\overset{\alpha}{\mathbf{V}}$ with $\overset{\alpha}{\mathbf{u}}$ been the eigenvector of $\overset{\alpha}{\mathbf{U}}$. In other words, principal directions refer to the vectors $\overset{\alpha}{\mathbf{u}} = u_A^a \mathbf{e}_A$ and $\overset{\alpha}{\mathbf{v}} = v_A^a \mathbf{e}_A$.

9 Deformation of Volume and Surface

Consider three non-coplanar line elements $d^\alpha \mathbf{X}^{(1)}, d^\alpha \mathbf{X}^{(2)}, d^\alpha \mathbf{X}^{(3)}$ at the point X in B so that:

$$\mathbf{d}^\alpha \mathbf{y}^{(i)} = \overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-1} d^\alpha \mathbf{X}^{(i)} \tag{75}$$

with $d^\alpha \mathbf{y}^i$ the corresponding fractional differential vectors in the current placement. Further, the volume dV is derived by

$$dV = d^\alpha \mathbf{X}^{(1)} \cdot (d^\alpha \mathbf{X}^{(2)} \wedge d^\alpha \mathbf{X}^{(3)}). \tag{76}$$

Alternatively

$$dV = \det(d^\alpha \mathbf{X}^{(1)}, d^\alpha \mathbf{X}^{(2)}, d^\alpha \mathbf{X}^{(3)}) \tag{77}$$

in which $d\mathbf{X}^{(1)}$ denotes a column vector (i=1,2,3). The corresponding volume dv in the deformed configuration is

$$dv = \det(d^\alpha \mathbf{y}^{(1)}, d^\alpha \mathbf{y}^{(2)}, d^\alpha \mathbf{y}^{(3)}) \tag{78}$$

and

$$dv = \det(\overset{a}{\mathbf{F}}) \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-3} dv \equiv \overset{a}{J} \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-3} dV \tag{79}$$

since

$$\overset{a}{J} = (\det \overset{a}{\mathbf{F}}) \text{ and } dV = d^\alpha \mathbf{X}^{(1)} \cdot d^\alpha \mathbf{X}^{(2)} \wedge d^\alpha \mathbf{X}^{(3)}. \tag{80}$$

Consider, further, an infinitesimal vector element of material surface dS in the neighborhood of the point X in B with $d^a\mathbf{S} = \mathbf{N} d^a S$ the surface vector corresponding to the normal vector N. Furthermore $d^\alpha \mathbf{X}$ is an arbitrary fiber cutting the edge $d^\alpha \mathbf{S}$ such that $d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S} > 0$. The volume of the cylinder with base $d^\alpha \mathbf{S}$ and generators $d^\alpha \mathbf{X}$ has volume $dV = d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S}$. If $d^\alpha \mathbf{x}$ and $d^\alpha \mathbf{s}$ are the deformed configurations of $d^\alpha \mathbf{X}$ and $d^\alpha \mathbf{S}$ respectively, with $d^\alpha \mathbf{s} = n d^\alpha \mathbf{s}$, where n is the normal vector to the deformed surface, the volume dV in the reference placement corresponds to the volume $dv = d^\alpha \mathbf{x} \cdot d^\alpha \mathbf{s}$ in the current configuration so that:

$$dv = d^\alpha \mathbf{y} \cdot d^\alpha \mathbf{s} = \overset{\alpha}{J} \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-3} d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S}. \tag{81}$$

Since, $d\mathbf{y} = \overset{a}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-1} d^\alpha \mathbf{X}$, we obtain

$$\left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-T} \mathbf{F}^T d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{s} = \overset{a}{J} \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X}\right)^{-3} d^\alpha \mathbf{X} \cdot d^\alpha \mathbf{S} \tag{82}$$

removing the arbitrary $d^{\alpha}\mathbf{X}$. Therefore

$$d^{\alpha}\mathbf{s} = \overset{\alpha}{J} \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{\overset{\alpha}{-3}} (\overset{\alpha}{\mathbf{F}})^{-T} \left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right) \overset{\alpha}{\mathbf{dS}} \tag{83}$$

and

$$\mathbf{n}d^{\alpha}s = \overset{\alpha}{J} \det\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{\overset{\alpha}{-3}} (\overset{\alpha}{\mathbf{F}})^{-T} \left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right) \mathbf{N}\,\overset{\alpha}{\mathbf{d}}S. \tag{84}$$

The relation between the area elements corresponding to reference and current configurations is the well known Nansons formula for the fractional deformations. It is evident that the correspondence lies between $\mathbf{d}^{\alpha}\mathbf{S}$ and $\mathbf{d}^{\alpha}\mathbf{s}$

10 Examples of Deformations

(a)Homogeneous deformations

The most general homogeneous deformation of the body B from its reference configuration is expressed by:

$$\mathbf{x} = \mathbf{AX}. \tag{85}$$

Hence,

$$\overset{a}{\mathbf{F}} = \frac{\partial^{a}\mathbf{x}}{\partial \overset{a}{\mathbf{X}}} = \mathbf{A}\frac{\partial^{a}\mathbf{x}}{\partial \mathbf{X}^{a}}.$$

Recalling that the Caputo's fractional derivatives are given by:

$${}_{a}^{c}D_{t}^{a}(t-a)^{v} = \frac{\Gamma(v+1)}{\Gamma(-\alpha+v+1)}(t-a)^{v-a}$$

and

$${}_{a}^{c}D_{X}^{a}X = \frac{\Gamma(2)}{\Gamma(2-a)}X^{1-a} \tag{86}$$

the fractional deformation gradient is defined by

$$\overset{a}{\mathbf{F}} = \underset{ij}{\overset{a}{x}} = A_{ik}\frac{\partial^{a}X_{k}}{\partial X_{j}^{a}} = \frac{\Gamma(2)}{\Gamma(2-a)}A_{ik}\delta_{kj}X_{j}^{1-a}. \tag{87}$$

Hence, we have

$$\overset{a}{\mathbf{F}} = \frac{\Gamma(2)}{\Gamma(2-a)}\begin{bmatrix} A_{11}X_{1}^{1-a} & A_{12}X_{2}^{1-a} & A_{13}X_{3}^{1-a} \\ A_{21}X_{1}^{1-a} & A_{22}X_{2}^{1-a} & A_{23}X_{3}^{1-a} \\ A_{31}X_{1}^{1-a} & A_{32}X_{2}^{1-a} & A_{33}X_{3}^{1-a} \end{bmatrix}. \tag{88}$$

Furthermore, we have

$$\underset{\mathbf{X}}{\overset{\alpha}{\nabla}}\mathbf{X} = \frac{\Gamma(2)}{\Gamma(2-a)}\begin{bmatrix} X_{1}^{1-a} & 0 & 0 \\ 0 & X_{2}^{1-a} & 0 \\ 0 & 0 & X_{3}^{1=a} \end{bmatrix} \tag{89}$$

thus we conclude that

$$\overset{a}{\mathbf{C}} = \left(\overset{\alpha}{\mathbf{F}}\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{-1}\right)^{T}\overset{a}{\mathbf{F}}\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{-1} \text{ and } \overset{a}{\mathbf{B}} = \overset{a}{\mathbf{F}}\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{-1}\left(\overset{\alpha}{\mathbf{F}}\left(\overset{\alpha}{\nabla}_{\mathbf{X}}\mathbf{X}\right)^{-1}\right)^{T}. \tag{90}$$

Specializing the homogeneous deformations with the example of simple shear we discuss the deformation

$$\begin{aligned} x_1 &= X_1 + \gamma X_2, \\ x_2 &= X_2, \\ x_3 &= X_3. \end{aligned} \tag{91}$$

Hence the deformation gradient:

$$\overset{a}{\mathbf{F}} = \underset{\mathbf{X}}{\overset{a}{\nabla}} \mathbf{x} = \frac{\Gamma(2)}{\Gamma(2-\alpha)} \begin{vmatrix} X_1^{1-a} & \gamma X_2^{1-a} & 0 \\ 0 & X_2^{1-a} & 0 \\ 0 & 0 & X_3^{1-a} \end{vmatrix}. \tag{92}$$

Therefore, the Cauchy-Green deformation tensors $\overset{a}{\mathbf{C}}$ and $\overset{a}{\mathbf{B}}$ become:

$$\overset{a}{\mathbf{C}} = \left(\overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \right)^{T} \overset{a}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} = \begin{vmatrix} 1 & \gamma & 0 \\ \gamma & (\gamma^2+1) & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{93}$$

and

$$\overset{a}{\mathbf{B}} = \overset{a}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \left(\overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \right)^{T} = \begin{vmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{94}$$

The Green (Langrange) Fractional strain tensor is:

$$\overset{a}{\mathbf{E}} = \frac{1}{2}(\overset{\alpha}{\mathbf{C}} - \mathbf{I}) = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & \gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{95}$$

and the Euler-Almansi strain tensor is given by

$$\overset{\alpha}{\mathbf{A}} = \frac{1}{2}(\mathbf{I} - (\overset{\alpha}{\mathbf{B}})^{-1}) = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & -\gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{96}$$

It would seem strange that the results are exactly the same as the ones of the conventional elasticity. However, there is mathematical explanation for the homogeneous deformations. Just taking into consideration the definition of the fractional derivative and differential, the differential for a linear function of the form

$$f(x) = Ax \tag{97}$$

the fractional differential is given by

$$d^{\alpha} f(x) = A d^{\alpha} x. \tag{98}$$

The fractional differential of the function has almost the same form as the conventional one. Nevertheless, for the nonlinear function,

$$f(x) = x^4. \tag{99}$$

The fractional differential

$$d^{\alpha} f(x) = {}_{a}^{c}D_x^{\alpha} x^4 \cdot ({}_{a}^{c}D_x^{\alpha} x)^{-1} d^{\alpha} x \tag{100}$$

is equal to

$$d^{\alpha} f(x) = \frac{\Gamma(5)\,\Gamma(2-\alpha)}{\Gamma(5-\alpha)} x^3 d^{\alpha} x \tag{101}$$

with coefficient depending upon the α -fractional dimension. That makes the difference in non-homogeneous deformations discussed in the next section.

(b)The non-homogeneous deformations The non-homogeneous deformation is defined by the equations,

$$\begin{aligned} x_1 &= X_1 + \gamma X_2^4, \\ x_2 &= X_2, \\ x_3 &= X_3. \end{aligned} \tag{102}$$

Therefore, the deformation gradient is given by

$$\overset{a}{\mathbf{F}} = \overset{a}{\underset{\mathbf{X}}{\nabla}}\mathbf{x} = \begin{vmatrix} \frac{\Gamma(2)}{\Gamma(2-\alpha)} X_1^{1-a} & \gamma \frac{\Gamma(5)}{\Gamma(5-\alpha)} X_2^{1-a} & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} X_2^{1-a} & 0 \\ 0 & 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} X_3^{1-a} \end{vmatrix}. \tag{103}$$

Taking into consideration Eqs.(75, 76) we get the fractional Cauchy-Green deformation tensors expressed by

$$\overset{a}{\mathbf{C}} = \left(\overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \right)^T \overset{a}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} = \begin{vmatrix} 1 & \frac{24\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{24\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & 1 + \frac{576\gamma^2 \Gamma(2-\alpha)^2 X_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{104}$$

and

$$\overset{a}{\mathbf{B}} = \overset{a}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \left(\overset{\alpha}{\mathbf{F}} \left(\overset{\alpha}{\nabla}_{\mathbf{X}} \mathbf{X} \right)^{-1} \right)^T = \begin{vmatrix} 1 + \frac{576\gamma^2 \Gamma(2-\alpha)^2 X_2^6}{\Gamma(5-\alpha)^2} & \frac{24\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{24\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{105}$$

Hence, Green-Lagrange strain tensor is expressed by

$$\overset{a}{\mathbf{E}} = \begin{vmatrix} 0 & \frac{12\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{12\gamma \Gamma(2-\alpha) X_2^3}{\Gamma(5-\alpha)} & \frac{288\gamma^2 \Gamma(2-\alpha)^2 X_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{106}$$

and Euler-Almansi strain tensor is defined as

$$\overset{a}{\mathbf{A}} = \begin{vmatrix} 0 & \frac{12\gamma \Gamma(2-\alpha) x_2^3}{\Gamma(5-\alpha)} & 0 \\ \frac{12\gamma \Gamma(2-\alpha) x_2^3}{\Gamma(5-\alpha)} & -\frac{288\gamma^2 \Gamma(2-\alpha)^2 x_2^6}{\Gamma(5-\alpha)^2} & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{107}$$

It is evident that the strain tensor strongly depends upon the fractional dimension α for the present non-homogeneous deformation.

11 The Infinitesimal Deformations

Since there has been introduced, in the literature, as quite evident to consider fractional strain tensors, in infinitesimal deformations, by simply substituting the common derivatives to fractional ones, it would be wise to study whether that idea is valid or not. Unfortunately it is proven a mistake. Fractional strain with simple substitution of derivatives does not have any physical meaning. Considering the fractional Green-Lagrange strain tensor, Eq.(65), where u is the small displacement vector with $|\mathbf{u}| \ll 1$ and $\left| \overset{a}{\nabla}_{\mathbf{X}} \mathbf{u} \right| \ll 1$, we restrict into the linear deformation analysis, and the infinitesimal (linear) fractional Euler-Lagrange strain tensor $\underset{lin}{\overset{\alpha}{\mathbf{E}}}$ becomes

$$\underset{lin}{\overset{\alpha}{\mathbf{E}}} = \frac{1}{2} \left[(\overset{a}{\nabla}_{\mathbf{X}} \mathbf{X})^{-T} \cdot (\overset{a}{\nabla}_{\mathbf{X}} \mathbf{u})^T + \overset{a}{\nabla}_{\mathbf{X}} \mathbf{u} \cdot (\overset{a}{\nabla}_{\mathbf{X}} \mathbf{X})^{-1} \right]. \tag{108}$$

Further recalling that the current placement $\mathbf{x} = \mathbf{X} + \mathbf{u}$ where u is the displacement vector the fractional left Cauchy-Green deformation tensor $\overset{\alpha}{\mathbf{B}}$ becomes

$$\overset{\alpha}{\mathbf{B}} = \mathbf{I} + (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^T + \overset{a}{\nabla}_{\mathbf{X}}\mathbf{u} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1} + \overset{a}{\nabla}_{\mathbf{X}}\mathbf{u} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1} (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^T. \tag{109}$$

Considering the infinitesimal deformations, the fractional left Cauchy-Green strain tensor reduces to

$$\overset{\alpha}{\mathbf{B}} = \mathbf{I} + (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^T + \overset{a}{\nabla}_{\mathbf{X}}\mathbf{u} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1} \tag{110}$$

and the infinitesimal fractional Euler-Almansi strain tensor becomes:

$$\underset{lin}{\overset{\alpha}{\mathbf{A}}} = \frac{1}{2}(\mathbf{I} - \left(\overset{a}{\mathbf{B}}\right)^{-1}) = \frac{1}{2}((\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-T} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{u})^T + \overset{a}{\nabla}_{\mathbf{X}}\mathbf{u} \cdot (\overset{a}{\nabla}_{\mathbf{X}}\mathbf{X})^{-1}). \tag{111}$$

Both fractional strain tensors Eqs.(107,110) coincide in infinitesimal deformation analysis, however they do not coincide with the widely used strain definition in the fractional mechanics literature.

12 Fractional Stresses

Pointing out that the fractional tangent space of a surface has different orientation of the conventional one, the fractional normal vector $\mathbf{n}^{\alpha}$ does not coincide with the conventional normal vector n. Hence we should expect the stresses and consequently the stress tensor to differ from the conventional ones not only in the values. If $d^{\alpha}\mathbf{P}$ is a contact force acting on the deformed area $d^{\alpha}\mathbf{a} = \mathbf{n}^{\alpha} d^{\alpha}$a lying on the fractional tangent plane where $\mathbf{n}^{\alpha}$ is the unit outer normal to the element of area $d^{\alpha}\mathbf{a}$ then the α-fractional stress vector is defined by

$$\mathbf{t}^{\alpha} = \lim_{d^{\alpha}\mathrm{a}\text{ -¿ }0} \frac{d^{\alpha}\mathbf{P}}{d^{\alpha}\mathrm{a}}. \tag{112}$$

However, the α-fractional stress vector does not have any connection with the conventional one

$$\mathbf{t} = \lim_{d\mathrm{a}\text{ -¿ }0} \frac{d\mathbf{P}}{d\mathrm{a}}. \tag{113}$$

Since the conventional tangent plane has different orientation from the -fractional tangent plane and the corresponding normal vectors too. Following similar procedures as the conventional ones we may establish Cauchy fundamental theorem, see Truesdell [5].

If $\mathbf{t}^{\alpha}(\cdot, \mathbf{n}^{\alpha})$ is a continuous function of the transplacement vector y, there is an α-fractional Cauchy stress tensor field

$$\mathbf{T}^{\alpha} = \left[\sigma_{ij}^{\alpha}\right]. \tag{114}$$

13 The Balance Principles

Almost all balance principles are based upon Reynolds transport theorem. Hence the modification of that theorem, just to conform to fractional analysis is presented. The conventional Reynolds transport theorem is expressed by:

$$\frac{d}{dt}\int_W A dV = \int_W \frac{dA}{dt} dV + \int_{\partial W} A\mathbf{v}_{\mathbf{n}} dS. \tag{115}$$

For a vector field A applied upon region W with boundary ∂W and v_n is the normal velocity of the boundary ∂W.

(a)Material derivatives of volume, surface and line integrals.

For any scalar, vector or tensor property that may be represented by:

$$P_{ij}(t) = \int_V P^*_{ij}(x,t) dV, \tag{116}$$

where V is the volume of the current placement in the conventional calculus. The material time derivative of $P^*_{ij}(x,t)$ is expressed by

$$\frac{dP^*_{ij}}{dt} = \frac{\partial P^*_{ij}}{\partial t} + v\frac{\partial P^*_{ij}}{\partial x_j} = \frac{\partial P^*_{ij}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} P^*_{ij}. \tag{117}$$

Recalling that we consider constant material points, during time derivation. Nevertheless, in fractional calculus, the fractional order for space is different from the fractional order for time, the material time derivative is given, for any tensor field P_{ij}, by:

$${}^{L}_{0}D^{\beta}_{t} P^*_{ij} = {}^{L}_{0}\partial^{\beta}_{t} P^*_{ij} + \left({}^{L}_{\mathrm{a}}\partial^{\alpha}_{k} P^*_{ij}\right)\left({}^{L}_{\mathrm{a}}\partial^{\alpha}_{k} x_l\right)\left({}^{L}_{0}D^{\beta}_{t} x_m\right)\delta_{km}, \tag{118}$$

where, the symbol ∂ denotes the partial Leibnitz derivative corresponding to D .
Since the fractional velocity is expressed by,

$$v_k = \left({}^{L}_{\mathrm{a}}\partial^{\alpha}_{k} x_l\right)\left({}^{L}_{o}D^{\beta}_{t} x_m\right)\delta_{km}. \tag{119}$$

The material derivative, into the context of fractional calculus, is expressed by

$${}^{L}_{o}D^{\beta}_{t}(\dots) = {}^{L}_{o}\partial^{\beta}_{t}\,() + \mathbf{v}\cdot\overset{\alpha}{\nabla}_{\mathbf{x}}\,(). \tag{120}$$

Hence, the acceleration is defined by

$$\mathbf{a} = {}^{c}_{o}D^{\beta}_{t}\mathbf{v} = {}^{c}_{o}\partial^{\beta}_{t}\mathbf{v} + \mathbf{v}\cdot\overset{\alpha}{\nabla}_{\mathbf{x}}\mathbf{v}. \tag{121}$$

Furthermore the material time derivative of $P_{ij}(t)$ is expressed in conventional analysis by:

$$\frac{d}{dt}[P_{ij}(t)] = \frac{d}{dt}\int_V P^*_{ij}(x,t)dV, \tag{122}$$

where b is the current placement of the region. It is well known that
$(d\dot{V}) = JdV$, and the Eq.(121) yields:

$$\frac{d}{dt}\int_V P^*_{ij}(x,t)dV = \int_V \left[\frac{\partial P^*_{ij}(x,t)}{dt} + P^*_{ij}(x,t)\frac{\partial v_p}{\partial x_p}\right]dV. \tag{123}$$

Recalling the material derivative operator Eq.(116), Eq.(122) yields,

$$\frac{d}{dt}\int_V P^*_{ij}(x,t)dV = \int_b \left[\frac{\partial P^*_{ij}(x,t)}{\partial t} + \frac{\partial(u_p P^*_{ij}(x,t))}{\partial x_p}\right]dV. \tag{124}$$

Yielding Reynolds Transport theorem:

$$\frac{d}{dt}\int_v P^*_{ij}(x,t)dV = \int_V \frac{\partial P^*_{ij}(x,t)}{\partial t}dV + \int_S v_p[P^*_{ij}(x,t)]dS_p. \tag{125}$$

Expressing Reynolds Transport Theorem in fractional form we get:

$${}^{L}_{o}D^{\beta}_{t}\left({}_{\omega}\int_V^{(\alpha)} P^*_{ij}(x,t)d^{\alpha}V\right) = {}_{\omega}\int_V^{(\alpha)} {}^{L}_{o}\partial^{\beta}_{t} P^*_{ij}(x,t)\, d^{\alpha}V + {}_{\omega}\int_S^{(\alpha)} v_p[P^*_{ij}(x,t)]d^{\alpha}S_p, \tag{126}$$

where $d^{\alpha}V$ and $d^{\alpha}S$ are the infinitesimal fractional volume and surface, respectively.
The volume integral of the material time derivative of $P_{ij}(t)$ may also be expressed by:

$${}^{L}_{o}D^{\beta}_{t}\left({}_{\omega}\int_V^{(\alpha)} P^*_{ij}(\mathbf{x},t)d^{\alpha}V\right) = {}_{\omega}\int_V^{(\alpha)} \left({}^{L}_{o}\partial^{\beta}_{t} P^*_{ij}(x,t) + div^{\alpha}[\mathbf{v}P^*_{ij}]\right) d^{\alpha}V. \tag{127}$$

(b)The balance of mass
The conventional balance of mass, expressing the mass preservation is expressed by:

$$\frac{d}{dt}\int_{W} \rho dV = 0. \tag{128}$$

In the fractional form it is given by:

$${}_{o}^{L}D_{t}^{\beta}\,{}_{\omega}\overset{(\alpha)}{\int_{W}} \rho\, d^{\alpha}V = 0, \tag{129}$$

where ${}_{o}^{c}D_{t}^{\beta}$ is the total time derivative.
Recalling the fractional Reynolds Transport Theorem, we get:

$${}_{o}^{L}D_{t}^{\beta}\,{}_{\omega}\overset{(\alpha)}{\int_{V}} \rho\, d^{\alpha}V = {}_{\omega}\overset{(\alpha)}{\int_{V}} \left({}_{o}^{L}\partial_{t}^{\beta}\rho(x,t) + div^{\alpha}[\mathbf{v}\rho]\right) d^{\alpha}V. \tag{130}$$

Since Eq. (129) is valid for any volume V, the continuity equation is:
${}_{o}^{c}\partial_{t}^{\beta}\rho + div^{a}[\mathbf{v}\rho] = 0,$
where, div^{α} has already been defined by Eq.(27). That is the continuity equation expressed in fractional form.
(c)It is reminded that the conventional balance of linear momentum is expressed in continuum mechanics by:

$${}_{o}^{L}\partial_{t}^{\beta}\rho + div^{a}[\mathbf{v}\rho] = 0, \tag{131}$$

where v is the velocity, $\mathbf{t}^{(n)}$ is the traction on the boundary and b is the body force per unit mass. Likewise that principle in fractional form is expressed by:

$${}_{o}^{c}D_{t}^{\beta}\,{}_{\omega}\overset{(\alpha)}{\int_{\Omega}} \rho\mathbf{v} d^{(\alpha)}V = {}_{\omega}\overset{(\alpha)}{\int_{\Omega}} [\rho\mathbf{b} + div^{a}(\mathbf{T}^{\alpha})]\, d^{(\alpha)}V. \tag{132}$$

Hence the equation of linear motion, expressing the balance of linear momentum is defined by,

$$div^{a}[\mathbf{T}^{\alpha}] + \rho\mathbf{b} - \rho\overset{\bullet}{\mathbf{v}} = 0. \tag{133}$$

It should be pointed out that div^{α} has already been defined by Eq.(27) and is different from the conventional definition of the divergence. Following similar steps as in the conventional case, the balance of rotational momentum yields the symmetry of Cauchy stress tensor.
(d)Balance of rotational momentum principle
Following similar procedure as in the conventional case, we may end up to the symmetry property of the fractional stress tensor, i.e.

$$\mathbf{T}^{\alpha} = (\mathbf{T}^{\alpha})^{T}. \tag{134}$$

14 Conclusion

Correcting, in the present work, the geometric icon of the fractional differential, the fractional tangent spaces have been established. The basic concepts of kinematics were studied for establishing the Fractional Continuum Mechanics principles. The various concepts of deformation and strain of the conventional Continuum Mechanics were modified, just to conform to the fractional differential law that is a homogeneous non-linear function of the variable differential. Since Continuum Mechanics is based upon the differential concept, that peculiarity introduces some important differences that make the simple transferring of the concepts and ideas from the conventional classical Continuum Mechanics to the Fractional Continuum Mechanics impossible. There is a need for the modification of the deformation and strain concepts that inevitably yield different transformation laws. Further, the, commonly used in the literature, linear fractional strain should be revised. The present analysis may be useful for solving updated problems in Mechanics and especially for lately proposed theories such as peridynamic theory, Silling [42], Lazopoulos & Lazopoulos [43].

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