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On the existence and Multiplicity of Nondecreasing Positive Solutions for Fractional Pantograph Type Equations

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Abstract: We investigate the existence and multiplicity of nondecreasing positive solutions for a fractional pantograph equation via fixed point theorems. To prove the multiplicity of solutions we use a fixed point theorem due to Leggett-Williams. We give some examples to exemplify our main results.

Keywords: Fractional pantograph equation, nondecreasing positive solution, fixed point theorem, multiplicity of solution.

1 Introduction

Fractional derivatives and fractional integrals are very useful tools in the modeling of many complex phenomena. To see some of the applications of fractional differential equations (FDEs) we refer the reader to [1,2,3,4]. In the book [5], Baleanu et al. studied the recent developments in nonlinear fractional dynamics, nonlinear vibration and control. In [6] there are some recent achievements in fractional dynamics. In [7,8,9,10,11] one can find applications of fractional differential equations in viscoelastic materials, biology, signal processing, heat conduction and thermal systems. In [12], Benson studied advection and dispersion of solutes in natural porous or fractured media by using fractional calculus. For the theory of fractional calculus one can see the monographs of Kilbas et al. [13], Podlubny [14] and Samko et al. [15]. Some recent existence results to FDEs can be found in articles [16,17,18,19,20,21,22,23]. Bhrawy et al. [24] used spectral methods to solve various types of FDEs.
In this work, we investigate the problem

$$\begin{cases} D^{\alpha}_{0+}\Big(w(s)-P(s)w(\beta s)\Big)(t)=f(t,w(t),w(\gamma t))), \ \ t\in(0,T], \\ w(0)=w'(0)=0, \end{cases} \tag{1}$$

where $1<\alpha\leq 2$, D^{α}_{0+} denotes the Riemann-Liouville fractional derivative of order α, $0<\beta,\gamma<1$, $f:J\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous where $\mathbb{R}^+:=[0,\infty)$, $J:=[0,T]$ and $P:J\to\mathbb{R}^+$ is of class C^1.

The pantograph type equations emerge in the modeling of many problems in sciences and engineering such as economy, electrodynamics, control and biology [25,26,27]. Recently, Doha et. al [28] utilized a collocation method to solve a class of FDEs of pantograph type. Balachandran et al. [29] established the existence of solutions for the pantograph problems of the form, namely

$$\begin{cases} D^{\alpha}_{0+}\Big(w(s)-h(s,w(\beta s))\Big)(t)=f(t,w(t),w(\beta t))), \ \ t\in(0,T], \\ w(0)=w_0, \end{cases} \tag{2}$$

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when $0<\alpha,\beta<1$, h is a Lipschitz continuous function and f is completely continuous on a Banach space X. They consider the above problem when $h=0$ with a nonlocal condition. Balachandran et al. make used of the Banach and the Krasnoselskii fixed point theorems to perform the existence of a solution for the mentioned problems.

The main purpose of this article is to discuss the existence and multiplicity of nondecreasing positive solutions to problem (1) with $1<\alpha\le 2$, utilizing the fixed point theorems.

We organize the manuscript as follows. In Section 2, some basic definitions and results concerning fractional calculus are shown. Also two required fixed point theorems are recalled in this section. In Section 3, we establish the existence and multiplicity of solutions for problem (1). In Section 4, three examples are given to exemplify the main reported results.

2 Preliminaries

Assume $-\infty<a<b<+\infty$, $\eta,\alpha,\beta\in\mathbb{C}$ and denote the real part of $z\in\mathbb{C}$ by $Re(z)$. The α^{th}-order Riemann-Liouville fractional derivative and integral are defined by

$$\begin{aligned}(D^{\alpha}_{a+}f)(t)&=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{f(s)}{(t-s)^{\alpha-n+1}}ds\\&=\frac{d^n}{dt^n}(I^{n-\alpha}_{a+}f)(t),\ \ t>a,\ \ Re(\alpha)\ge 0,\end{aligned}\tag{3}$$

and

$$(I^{\alpha}_{a+}f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\quad t>a,\ \ Re(\alpha)>0,\tag{4}$$

respectively, such that $n=[Re(\alpha)]+1$ when $\alpha\notin\mathbb{N}$ ($[x]$ means the integer part of $x\in\mathbb{R}$) [13].
The following Lemma shows the semigroup property of the operator I^{α}_{a+} and the composition relation of the operators I^{α}_{a+} and D^{β}_{a+}.

Lemma 1([13]) *Assume $Re(\alpha),Re(\beta)>0$ and $f\in C[a,b]$. Then for any $t\in[a,b]$ we conclude that*

(*a*)
$$(I^{\alpha}_{a+}I^{\beta}_{a+}f)(t)=(I^{\alpha+\beta}_{a+}f)(t).\tag{5}$$

(*b*)
$$(D^{\alpha}_{a+}I^{\alpha}_{a+}f)(t)=f(t).\tag{6}$$

(*c*)*If $Re(\alpha)>Re(\beta)$, then*
$$(D^{\beta}_{a+}I^{\alpha}_{a+}f)(t)=(I^{\alpha-\beta}_{a+}f)(t).\tag{7}$$

(*d*)*Suppose $Re(\alpha)\notin\mathbb{N}$ and $n=[Re(\alpha)]+1$. Also assume $f_{n-\alpha}(t)=(I^{n-\alpha}_{a+}f)(t)\in C^n[a,b]$. Then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(t)=f(t)-\sum_{k=1}^{n}\frac{f^{(n-k)}_{n-\alpha}(a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k}.\tag{8}$$

We denote $C_\eta[a,b]$ to be the space of functions f defined on $(a,b]$ fulfilling $(t-a)^\eta f(t)\in C[a,b]$ with the norm

$$||f||_{C_\eta}=||(t-a)^\eta f(t)||_C.$$

We recall that we have $\eta=0$, $C_\eta[a,b]=C[a,b]$. The following lemma is about the continuity of the operator

$$I^{\alpha}_{a+}:C_\eta[a,b]\to C[a,b].$$

Lemma 2([13]) *Assume $Re(\alpha)>0$ and $0\le Re(\eta)\le 1$. If $Re(\eta)\le Re(\alpha)$, then the operator $I^{\alpha}_{a+}:C_\eta[a,b]\to C[a,b]$ is bounded,namely*

$$\begin{aligned}&||I^{\alpha}_{a+}f||_C\le l||f||_{C_\eta},\\&l=(b-a)^{R(\alpha-\eta)}\frac{\Gamma(R(\alpha))|\Gamma(1-R(\eta))|}{|\Gamma(\alpha)|\Gamma(1+R(\alpha-\eta))}.\end{aligned}$$

To establish the existence of solutions for (1), we recall the following theorems.

Theorem 3([30]) *Assume Y is a Banach space and $\mathscr{P} \subset Y$ is a cone in Y. Let Ω_1, Ω_2 be open subsets of Y with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and suppose $F : \mathscr{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathscr{P}$ is a completely continuous map such that either*

$(H_1)||Fw|| \geq ||w||$, $w \in \mathscr{P} \cap \partial\Omega_1$, *and* $||Fw|| \leq ||w||$, $w \in \mathscr{P} \cap \partial\Omega_2$, *or*
$(H_2)||Fw|| \leq ||w||$, $w \in \mathscr{P} \cap \partial\Omega_1$, *and* $||Fw|| \geq ||w||$, $w \in \mathscr{P} \cap \partial\Omega_2$.

Then F possess a fixed point in $\mathscr{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the sequel, we recall a fixed point theorem due to the Leggett-Williams [31].

Definition 1*Assume Y is a real Banach space and $\mathscr{P}$ is a cone in Y. We say a map $\theta : \mathscr{P} \to [0, \infty)$ is a nonnegative continuous concave functional on $\mathscr{P}$ provided θ is continuous and*

$$\theta(\lambda v + (1-\lambda)w) \geq \lambda\theta(v) + (1-\lambda)\theta(w),$$

for any $v, w \in \mathscr{P}$ and $\lambda \in [0, 1]$.

Assume $a, b, c > 0$ are constants. We define

$$\mathscr{P}_c = \{w \in \mathscr{P} : ||w|| < c\}, \tag{9}$$

$$\overline{\mathscr{P}}_c = \{w \in \mathscr{P} : ||w|| \leq c\}, \tag{10}$$

$$\mathscr{P}(\theta, a, b) = \{w \in \mathscr{P} : \theta(w) \geq a, ||w|| \leq b\}. \tag{11}$$

Theorem 4([31]) *Assume that Y is a real Banach space, $\mathscr{P}$ is a cone in Y and $c > 0$. Let θ be a concave nonnegative continuous functional on $\mathscr{P}$ with $\theta(w) \leq ||w||$ for any $w \in \overline{\mathscr{P}}_c$. Suppose $F : \overline{\mathscr{P}}_c \to \overline{\mathscr{P}}_c$ denotes a completely continuous map. Assume that there exist constants $0 < a < b < d \leq c$ such that*

$(h_1)\{w \in \mathscr{P}(\theta, b, d) : \theta(w) > b\} \neq \emptyset$ *and* $\theta(Fw) > b$ *for* $w \in \mathscr{P}(\theta, b, d)$;
$(h_2)||Fw|| < a$ *for* $||w|| \leq a$;
$(h_3)\theta(Fw) > b$ *for* $w \in \mathscr{P}(\theta, b, c)$ *with* $||Fw|| > d$.

Thus, F admits at least three fixed point w_1,w_2 and w_3 in $\overline{\mathscr{P}}_c$ fulfilling

$$||w_1|| < a, \;\; b < \theta(w_2), \;\; a < ||w_3|| \text{ with } \theta(w_3) < b.$$

3 Existence of Solutions

Suppose $[a, b] \subset \mathbb{R}$ is an interval. We denote by $C^1[a, b]$ the space of continuously differentiable functions on $[a, b]$ equipped with the norm

$$||w|| = ||w||_C + ||w'||_C,$$

where $||w||_C := \sup\limits_{t\in[a,b]} |w(t)|$.

Using the Lemmas 1 and 2, it can be easily show that $w \in C^1(J)$ is a solution of problem (1) iff $w \in C^1(J)$ is a solution of the integral equation

$$w(t) = P(t)w(\beta t) + \left(I_{0+}^{\alpha} f(s, w(s), w(\gamma s))\right)(t), \quad t \in J. \tag{12}$$

So, we study the existence of solutions for integral equation (12). We consider (12) under the following conditions.

$(i)P : J \to \mathbb{R}^+$ is a function in the space $C^1(J)$ with $(2+\beta)||P|| < 1$ and $P'(t) \geq 0$.
$(ii) f : J \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which is nondecreasing with respect to its variables. Moreover there exist constants $c > b > 0$ such that

$$\left(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\right) f(T, c, c) \leq c, \tag{13}$$

$$\frac{f(\eta, \frac{b}{2}, 0)(T-\eta)^{\alpha}}{\Gamma(\alpha+1)} \geq b, \tag{14}$$

where $\eta = \max\{0, T - \frac{1}{2}\}$.

Let $B_r \subset C^1(J)$ be an open ball centered at zero with radius r and define the operator F on $C^1(J)$ as follows:

$$(Fw)(t) := P(t)w(\beta t) + \left(I^{\alpha}_{0+} f(s, w(s), w(\gamma s))\right)(t). \tag{15}$$

Next, define the cone

$$\mathscr{P} := \{w \in C^1(J) : w(t) \geq 0, w'(t) \geq 0\}. \tag{16}$$

By the definition (15) fixed points of F are solutions of integral equation (12).

Lemma 5*Suppose $P : J \to \mathbb{R}^+$ is a function in the space $C^1(J)$ with $P'(t) \geq 0$ and $f : J \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing with respect to its variables. Then, $F : \mathscr{P} \cap (\overline{B}_c \setminus B_b) \to \mathscr{P}$ is continuous and compact.*

*Proof.*We prove this lemma in the following steps.
1. $F(\mathscr{P}) \subset \mathscr{P}$.
Let $w \in \mathscr{P}$. Since f and P are nonnegative $(Fw)(t) \geq 0$ and by the assumption $1 < \alpha \leq 2$, Lemmas 1 and 2 we have

$$\frac{d}{dt}\left(I^{\alpha}_{0+} f(s, w(s), w(\gamma s))\right)(t) = \left(I^{\alpha-1}_{0+} f(s, w(s), w(\gamma s))\right)(t) \geq 0, \ \ t \in J.$$

Then by the assumption $P'(t) \geq 0$ we have $(Fw)'(t) \geq 0$. This implies that $Fw \in \mathscr{P}$.
2. $F : \mathscr{P} \cap (\overline{B}_c \setminus B_b) \to \mathscr{P}$ is continuous.
Fix $\varepsilon > 0$ and take arbitrarily $w, v \in \mathscr{P} \cap (B_c \setminus B_b)$ with $||w - v|| \leq \varepsilon$. For $t \in J$ we obtain

$$\begin{aligned}|(Fw)(t) - (Fv)(t)| &\leq |P(t)(w(\beta t) - v(\beta t))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T \frac{|f(s, w(s), w(\gamma s)) - f(s, v(s), v(\gamma s))|}{(T-s)^{1-\alpha}} ds \\ &\leq ||P||\,||w - v|| + \frac{\omega_c(f, \varepsilon) T^{\alpha}}{\Gamma(\alpha+1)},\end{aligned} \tag{17}$$

where

$$\omega_c(f, \varepsilon) = \sup\{|f(t, w_1, v_1) - f(t, w_2, v_2)| : t \in J,\ w_i, v_i \in [0, c],\ |w_i - v_i| \leq \varepsilon,\ i = 1, 2\}. \tag{18}$$

Also we discover

$$\begin{aligned}|(Fw)'(t) - (Fv)'(t)| &\leq |P'(t)(w(\beta t) - v(\beta t))| + \beta |P(t)(w'(t) - v'(t))| \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^T \frac{|f(s, w(s), w(\gamma s)) - f(s, v(s), v(\gamma s))|}{(T-s)^{2-\alpha}} ds \\ &\leq (1+\beta)||P||\,||w - v|| + \frac{\omega_c(f, \varepsilon) T^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned} \tag{19}$$

Inequalities (17) and (19) yield that

$$||F(w) - F(v)|| \leq (2+\beta)||P||\,||w - v|| + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)\omega_c(f, \varepsilon). \tag{20}$$

Since f is uniformly continuous on bounded subsets of $J \times \mathbb{R}^+ \times \mathbb{R}^+$ and letting $\varepsilon \to 0$, we infer that $\omega_c(f, \varepsilon) \to 0$. Thus inequality (20) implies that $F : \mathscr{P} \cap (\overline{B}_c \setminus B_b) \to \mathscr{P}$ is continuous.
3. $F : \mathscr{P} \cap (\overline{B}_c \setminus B_b) \to \mathscr{P}$ is compact.
Let $B \subset \mathscr{P} \cap (\overline{B}_c \setminus B_b)$ and put

$$M_0 = \sup\{f(t, w, v) : t \in J, w, v \in [0, c]\}.$$

Then for any $w \in B$ we have

$$\begin{aligned}||Fw|| = ||Fw||_C + ||(Fw)'||_C &\leq (2+\beta)||P||\,||w|| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T \frac{|f(s, w(s), w(\gamma s)) - f(s, v(s), v(\gamma s))|}{(T-s)^{1-\alpha}} ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^T \frac{|f(s, w(s), w(\gamma s)) - f(s, v(s), v(\gamma s))|}{(T-s)^{2-\alpha}} ds \\ &\leq (2+\beta)c||P|| + 2\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)M_0.\end{aligned}$$

Then $F(B)$ is bounded. Put

$$M_1 := \sup\{F(w) : w \in B\}.$$

Now, we prove that $F(B)$ is an equicontinuous subset of $\mathscr{P}$. Let $t_1, t_2 \in J$ and $t_1 < t_2$. For any $w \in B$, by mean value theorem and using boundedness of B we conclude

$$\begin{aligned}
|(Fw)(t_1) - (Fw)(t_2)| &\leq |P(t_1)||w(\beta t_1) - w(\beta t_2)| + |P(t_1) - P(t_2)||w(\beta t_2)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} f(s, w(s), w(\gamma s))\Big(\frac{1}{(t_1 - s)^{1-\alpha}} - \frac{1}{(t_2 - s)^{1-\alpha}}\Big) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{f(s, w(s), w(\gamma s))}{(t_2 - s)^{1-\alpha}} ds \\
&\leq c\beta ||P||\, |t_1 - t_2| + c|P(t_1) - P(t_2)| \\
&+ \frac{M_0}{\Gamma(\alpha + 1)}\big(2(t_2 - t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha}\big) \to 0 \quad \text{as } t_1 \to t_2. \qquad (21)
\end{aligned}$$

Then $F(B)$ is equicontinuous.
By using the steps 1-3 and the Arzela-Ascoli theorem, we find $F : \mathscr{P} \cap (\overline{B}_c \setminus B_b) \to \mathscr{P}$ is continuous and compact. □

Theorem 6*Under assumptions (i) and (ii), problem (1) has a nondecreasing positive solution in $C^1(J)$.*

*Proof.*It is enough to prove that integral equation (12) has a nondecreasing positive solution in $C^1(J)$. To do this we prove that the equation $Fx = x$ has a nonzero solution in $\mathscr{P}$. Now we show that F satisfies the following inequalities

1)$||Fw|| \leq ||w||$, $w \in \mathscr{P} \cap \{w \in C^1(J) : ||w|| = c\}$;
2)$||Fw|| \geq ||w||$, $w \in \mathscr{P} \cap \{w \in C^1(J) : ||w|| = b\}$,

where constants b and c come from assumption (ii). Let $w \in \mathscr{P} \cap \{w \in C^1(J) : ||w|| = c\}$. Since $Fw \in \mathscr{P}$, and using condition (ii), we have

$$\begin{aligned}
||Fw|| &= ||Fw||_C + ||(Fw)'||_C \\
&\leq (2 + \beta)||P||\, ||w|| + \frac{1}{\Gamma(\alpha)} \int_0^T \frac{f(s, w(s), w(\gamma s))}{(T - s)^{1-\alpha}} ds \\
&+ \frac{1}{\Gamma(\alpha - 1)} \int_0^T \frac{f(s, w(s), w(\gamma s))}{(T - s)^{2-\alpha}} ds \\
&\leq (2 + \beta)||P||c + \frac{f(T, c, c)T^{\alpha}}{\Gamma(\alpha + 1)} + \frac{f(T, c, c)T^{\alpha-1}}{\Gamma(\alpha)} \\
&\leq c = ||w||. \qquad (22)
\end{aligned}$$

Then 1) it is satisfied. Now, let $w \in \mathscr{P} \cap \{w \in C^1(J) : ||w|| = b\}$. Since $w \in \mathscr{P}$, $b = ||w|| = w(T)$. Put $\eta = \max\{T - \frac{1}{2}, 0\}$. Using mean value theorem for any $t \in [\eta, T]$, we have $b - w(t) = w(T) - w(t) \leq b(T - t) \leq b(T - \eta)$. Thus,

$$\frac{b}{2} \leq b(1 - (T - \eta)) \leq w(t), \quad t \in [\eta, T]. \qquad (23)$$

By assumptions (i), (ii) and inequality (23), we get

$$\begin{aligned}
||Fw|| &= ||Fw||_C + ||(Fw)'||_C \geq ||Fw||_C \\
&\geq \frac{1}{\Gamma(\alpha)} \int_{\eta}^T \frac{f(s, w(s), w(\gamma s))}{(T - s)^{1-\alpha}} ds \\
&\geq \frac{f(\eta, \frac{b}{2}, 0)(T - \eta)^{\alpha}}{\Gamma(\alpha + 1)} \geq b = ||w||,
\end{aligned}$$

Then 2) it is satisfied. By Lemma 5, (1) and (2), all conditions of the Theorem 3 hold. Then F has a fixed point in $\mathscr{P} \cap (\overline{B}_c \setminus B_b)$. □

In the sequel, using the Leggett-Williams fixed point theorem, we prove the existence of three nondecreasing positive solutions to problem (1). To do this we need the following assumption:

$(ii)' f : J\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which is nondecreasing with respect to its variables. Moreover there exist constants $c > b > a > 0$ such that

$$\Big(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\Big) f(T,c,c)\le c, \tag{24}$$

$$\frac{f(\eta,\frac{b}{2},0)(T-\eta)^{\alpha}}{\Gamma(\alpha+1)} > b, \tag{25}$$

$$\Big(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\Big) f(T,a,a) < a, \tag{26}$$

where $\eta=\max\{0,T-\frac{1}{2}\}$.

Theorem 7*Assume (i) and $(ii)'$ hold. Then problem (1) has three nondecreasing positive solutions w_1,w_2 and w_3 such that*

$$||w_1|| < a,\;\; b < w_2(T),\;\; a < ||w_3|| \textit{ with } w_3(T) < b.$$

*Proof.*Assume the operator F and the set $\mathscr{P}$ are defined by (15) and (16). Using (i) and (ii), for any $w\in\mathscr{P}_c=\{w\in\mathscr{P} : ||w|| < c$, we find $Fw\in\mathscr{P}$ and

$$\begin{aligned}||Fw|| &= ||Fw||_C+||(Fw)'||_C\\ &\le (2+\beta)||P||\,||w||+\frac{1}{\Gamma(\alpha)}\int_0^T\frac{f(s,w(s),u(\gamma s))}{(T-s)^{1-\alpha}}ds\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^T\frac{f(s,w(s),w(\gamma s))}{(T-s)^{2-\alpha}}ds\\ &< (2+\beta)c||P||+\frac{f(T,c,c)T^{\alpha}}{\Gamma(\alpha+1)}+\frac{f(T,c,c)T^{\alpha-1}}{\Gamma(\alpha)}\le c.\end{aligned} \tag{27}$$

Then $Fw\in\mathscr{P}_c$. Similar to the proof of Lemma 5, we can prove that $F:\overline{\mathscr{P}}_c\to\overline{\mathscr{P}}_c$ is completely continuous. Now, define the functional $\theta:\mathscr{P}\to\mathbb{R}^+$ as

$$\theta(w)=w(T).$$

Then θ is a nonnegative continuous concave functional on $\mathscr{P}$. Consider a,b,c given in assumption $(ii)'$. We define $w(t)=\frac{b+c}{2}$. Thus $w\in\mathscr{P}_c$ and $c>||w||=\theta(w)=\frac{b+c}{2}>b$. Therefore, $\{w\in\mathscr{P}(\theta,b,c):\theta(w)>b\}\neq\emptyset$. Let $w\in\mathscr{P}(\theta,b,c)$. Similar to the inequity (23), we observe $\frac{b}{2}\le w(t)$ for $t\in[\eta,T]$, where $\eta=\max\{T-\frac{1}{2},0\}$. By assumptions (i) and $(ii)'$ we obtain

$$\begin{aligned}\theta(Fw) &= (Fw)(T)\ge\big(I^{\alpha}_{0+}f(s,w(s),w(\gamma s))\big)(T)\\ &\ge\frac{1}{\Gamma(\alpha)}\int_{\eta}^T\frac{f(s,w(s),w(\gamma s))}{(T-s)^{1-\alpha}}ds\\ &\ge\frac{f(\eta,\frac{b}{2},0)(T-\eta)^{\alpha}}{\Gamma(\alpha+1)}>b.\end{aligned}$$

Therefore, condition (h_1) of Theorem 4 holds if we put $d=c$. Now consider $w\in\mathscr{P}$ with $||w||\le a$. Similar to inequality (22) and using (26) we deduce

$$||Fw||\le(2+\beta)a||P||+\frac{f(T,a,a)T^{\alpha}}{\Gamma(\alpha+1)}+\frac{f(T,a,a)T^{\alpha-1}}{\Gamma(\alpha)}<a$$

The above inequality shows that assumption (h_2) of Theorem 4 holds.
Finally, we consider the assumption (h_3) of Theorem 4. By inequality (27) for every $w\in\mathscr{P}(\theta,b,c)$, $||Fw||\le c$. Now if we consider $d=c$, then (h_3) holds.
Consequently F satisfies the hypothesis of Theorem 4. Then problem (1) has three positive solutions w_1,w_2 and w_3 which are nondecreasing and

$$||w_1||<a,\;\; b<w_2(T),\;\; a<||w_3|| \text{ with } w_3(T)<b. \qquad\square$$

4 Examples

We now give three examples to demonstrate the efficiency of our results.

Example 1 *We consider the fractional pantograph equation with initial conditions*

$$\begin{cases} D^{1.75}\left(w(s)-\frac{2s-s^2}{22}w(\frac{s}{5})\right)(t)=\frac{w^2(t)+(t+1)\ln(|w(t)w(\frac{t}{2})|+1)+w^2(\frac{t}{2})+1}{24}, & t\in(0,1], \\ w(0)=w'(0)=0. \end{cases} \tag{28}$$

Set the followings

$$f(t,x,y)=\frac{x^2+(t+1)\ln(|xy|+1)+y^2+1}{24},\quad P(t)=\frac{2t-t^2}{22},$$
$$T=1,\ \alpha=1.75,\ \ \beta=\frac{1}{5},\ \ \gamma=\frac{1}{2}.$$

One can easily see that f is nondecreasing with respect to its variables and continuous on $[0,1]\times\mathbb{R}^+\times\mathbb{R}^+$ and $P(t)$ is continuously differentiable on $[0,1]$ with $P'(t)\geq 0$. Now, we consider conditions (i) and (ii) for problem (28). By calculation we observe that

$$||P||=\frac{3}{22},\quad \frac{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}<3,$$
$$\eta=\max\{0,T-\frac{1}{2}\}=\frac{1}{2},\quad \frac{(T-\eta)^\alpha}{\Gamma(\alpha+1)}>\frac{18}{100}.$$

For $c=1+\sqrt{3}$ and $b=\frac{3}{400}$, we find

$$\left(\frac{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\right)f(T,c,c)<3f(T,c,c)\approx 2.525\leq 1+\sqrt{3}=c$$
$$\frac{f(\eta,\frac{b}{2},0)(T-\eta)^\alpha}{\Gamma(\alpha+1)}>\frac{18}{100}f(\eta,\frac{b}{2},0)\approx 0.0077\geq\frac{3}{400}=b,$$

According to the above calculations, all assumptions of the Theorem 7 hold. Then, the problem (28) has a nondecreasing positive solution in the space $C^1[0,1]$.

Example 2 *Let us consider the fractional initial value problem*

$$\begin{cases} D^{1.5}\left(w(s)-\frac{s+1}{88}w(\frac{3s}{4})\right)(t)=f(t,w(t),w(\frac{t}{3})), & t\in(0,2], \\ w(0)=w'(0)=0. \end{cases} \tag{29}$$

where

$$f(t,x,y)=\begin{cases} \frac{x^2+\tanh(xy)}{10}+\frac{t+2}{128}, & t\in[0,2]\ x\in[0,1], y\in[0,\infty), \\ \frac{1+\tanh(y)}{10}+(x-1)^3+\frac{(t+2)x}{128}, & t\in[0,2]\ x\in[1,10], y\in[0,\infty), \\ \frac{\tanh(y)}{10}+\frac{(t+2)\sqrt{10x}}{128}+\frac{\sqrt{x}}{10\sqrt{10}}+729, & t\in[0,2]\ x\in[10,\infty), y\in[0,\infty). \end{cases}$$

Let us put

$$P(t)=\frac{t+1}{88},$$
$$T=2,\ \alpha=1.5,\ \ \beta=\frac{3}{4},\ \ \gamma=\frac{1}{3}.$$

Easily, we find

$$||P||=\frac{1}{22},\quad \frac{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\leq 4.26,$$
$$\eta=\frac{3}{2},\quad \frac{(T-\eta)^\alpha}{\Gamma(\alpha+1)}\geq 0.26.$$

For $a=1$, $b=10$ *and* $c=10^4$, *we deduce*

$$\Big(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\Big)f(T,c,c)\leq 4.26f(T,c,c)\approx 3161.43\leq 10^4=c,$$
$$\frac{f(\eta,\frac{b}{2},0)(T-\eta)^{\alpha}}{\Gamma(\alpha+1)}\geq 0.26f(\eta,\frac{b}{2},0)\approx 16.70>10=b,$$
$$\Big(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\Big)f(T,a,a)\leq 4.26f(T,a,a)\approx 0.89<1=a.$$

Then, using the Theorem 7, the problem (29) has three positive solutions w_1, w_2 *and* w_3 *in* $C^1[0,2]$ *which are nondecreasing and*

$$||w_1||<1,\;\; 10<w_2(2),\;\; 1<||w_3||\text{ with } w_3(2)<10.$$

Example 3*Finally, we consider the fractional pantograph equation with initial conditions*

$$\begin{cases} D^{1.9}\big(w(s)-\frac{\sin(\frac{s\pi}{2})}{15}w(\frac{s}{10})\big)(t)=(tw(t))^3+\sin^3(\frac{100\pi w(t)}{14})+tw^3(\frac{t}{8}),\;\; t\in(0,\frac{1}{2}],\\ w(0)=w'(0)=0. \end{cases} \tag{30}$$

Now, we set

$$f(t,x,y)=t^3x^3+\sin^3(\frac{100\pi x}{14})+ty^3,\;\; P(t)=\frac{\sin(\frac{t\pi}{2})}{15},$$
$$T=\frac{1}{2},\;\alpha=1.9,\;\;\beta=\frac{1}{10},\;\;\gamma=\frac{1}{8}.$$

Observe that $f(t,x,y)$ *is continuous and nondecreasing with respect to* t, x *and* y *on* $[0,\frac{1}{2}]\times\mathbb{R}^+\times\mathbb{R}^+$. *Also* $P(t)$ *is continuously differentiable on* $[0,\frac{1}{2}]$ *with* $P'(t)\geq 0$. *To verify the hypotheses (i) and (ii) of Theorem 7, we need the following estimates:*

$$||P||=\frac{\sqrt{2}}{30},\quad \frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}<0.79,$$
$$\eta=\max\{0,T-\frac{1}{2}\}=0,\quad \frac{(T-\eta)^{\alpha}}{\Gamma(\alpha+1)}>0.14.$$

For $c=0.28$ *and* $b=0.14$, *we observe that*

$$\Big(\frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}}{1-(2+\beta)||P||}\Big)f(T,c,c)<0.79f(T,c,c)=0.01372\leq c$$
$$\frac{f(\eta,\frac{b}{2},0)(T-\eta)^{\alpha}}{\Gamma(\alpha+1)}>0.14f(\eta,\frac{b}{2},0)=0.14=b.$$

In view of the above calculations, all assumptions of the Theorem 6 hold. Hence, the problem (30) has a nondecreasing positive solution in the space $C^1[0,\frac{1}{2}]$.

5 Conclusions

Motivated by the applications of fractional differential equations, we established the existence of solutions for the problem (1) including a fractional pantograph equation with initial conditions. We showed that when the nonlinear part of the equation is positive and monotonically nondecreasing with respect to each of its variable and satisfies some geometrical conditions, the problem (1) has nondecreasing positive solutions.

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