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The Complementary Exponentiated BurrXII Poisson Distribution: model, properties and application

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Abstract: A new class of lifetime distribution called complementary exponentiated BurrXII Poisson (CEBXIIP) distribution is introduced. The distribution contain several lifetime models such as the BurrXII-zero truncated Poisson, complementary exponentiated log-logistic Poisson, complementary log-logistic Poisson, complementary exponentiated Lomax poissson and the Poisson-Lomax distributions. Several properties of the new distribution are investigated. Inferences are obtained via maximum likelihood procedure. An application of the new model to a real data set is presented for illustration purposes.

Keywords: Exponentiated BurrXII, Moments, Maximum likelihood estimates

1 Introduction

In probability modeling, several authors considered different approaches of mixing distributions in order to establish more flexible distributions, some distributions were generated by mixing two or more continuous distributions while some others by mixing continuous and discrete probability distributions. For example, in the case of mixing two continuous distributions, we recall the procedure of [14] for the beta generated families of distributions by mixing beta distribution and normal distribution, which was followed by [15] for beta-linear failure rate distribution and beta BurrXII distribution by [16] among others. In another way [17] proposed Lindley distribution by mixing the gamma$(2,\theta)$ and the exponential(θ) distribution, by this approach we have the gamma Lindley by [18], generalized Lindley by [19] and pseudo Lindley by [20] among others. To read more about compounding procedures in various approaches [21] provide an extensive review on the compounding distributions. In this work we consider the mixing continuous and discrete probability distributions, these families of distributions provides some flexibility in modeling data in practical applications, most of the distributions generated by this procedure are obtained by either compounding the minimum or maximum failure rate of the random variable distributed according to the continuous distribution under consideration, therefore, we may have two different distributions generated from the same continuous and discrete distributions and are not special cases of each other. For example, the exponential geometric (EG) distribution was proposed by [1] while [2] proposed complementary exponential geometry (CEG) distribution as a complementary to the (EG) distribution, [3] introduced the exponential Poisson (EP) distribution, [4] proposed Poisson-exponential (PE) distribution as a complementary to the (EP) distribution. Others include the complementary exponentiated exponential geometric [22], complementary Poisson-Lindley [23], Lindley-Poisson [24], Burr XII Poisson [9], Burr XII zero-truncated Poisson [6], complementary exponentiated inverted Weibull power series [25], Complementary Burr III Poisson [26], Poisson-half logistic [27], half logistic Poisson [28], generalized Gompertz power series [29], inverse burr negative binomial [32], Dagum-Poisson by [34], geometric-weibull Poisson (GWP) by [36], generalized Gompertz-power series (GGPS) by [31], bivariate Weibull-power series by [33] and generalized exponential power series distributions proposed by [30]. However, recently [5] proposed exponentiated BurrXII Poisson (Exp-BXIIP) distribution which was obtained through the mixing of the exponentiated BurrXII and Poisson distributions by considering the minimum failure rate of independent and identically random variables distributed exponentiated BurrXII distribution. Here, we proposed a new lifetime distribution called the complementary exponentiated BurrXII Poisson (CEBXIIP) distribution as a complementary to the Exp-BXIIP distribution by considering the maximum failure rate of independent and identically

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random variables distributed exponentiated BurrXII distribution. The CEBXIIP distribution exhibit decreasing and upside-down bathtub hazard rate function. The paper is arranged as follows. In section 2 we derive the CEBXIIP distribution and its basic properties. In section 3 we discuss the quantile function, r^{th} moment, moment generating function, skewness and kurtosis of the proposed distribution. In section 4, distribution of the order statistics and their moments are derived. In section 5, estimation of the unknown parameters by the method of maximum likelihood is presented. Real application is provided in section 6. Section 7 conclusions.

2 The CEBXIIP distribution

In this section, we briefly discuss about the BurrXII (BXII) and Exponentiated BurrXII (EBXII) distributions. Also we obtained the pdf of the proposed model, basic properties and the sub models of the new distribution. The exponentiated BurrXII (EBXII) distribution is a probability model which generalized the BurrXII (BXII) distribution and it is commonly used to work out various practical problems particularly in lifetime analysis. In this case, we consider the two parameter BurrXII distribution with the cdf and pdf given respectively by

$$G(y;\alpha,\beta)=1-\frac{1}{(1+y^{\alpha})^{\beta}},\quad y>0, \tag{1}$$

$$g(y,\alpha,\beta,)=\frac{\alpha\beta\, y^{\alpha-1}}{(1+y^{\alpha})^{\beta+1}}, \tag{2}$$

where $\alpha>0$, and $\beta>0$ are the shape parameters. The r^{th} moment of the BXII distribution is given by

$$E(X^{r})=\beta B\left(\beta-\frac{r}{\alpha},1+\frac{r}{\alpha}\right), \tag{3}$$

where $B(.,.)$ is a beta function defined as $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$. The cdf of the exponentiated BurrXII (EBXII) distribution is given by

$$G(y;\alpha,\beta,\theta)=\left(1-(1+y^{\alpha})^{-\beta}\right)^{\theta}, \tag{4}$$

while its corresponding pdf is

$$g(y;\alpha,\beta,\theta)=\alpha\beta\theta\, y^{\alpha-1}\,(1+y^{\alpha})^{-(\beta+1)}\,(1-(1+y^{\alpha})^{-\beta})^{\theta-1}, \tag{5}$$

where $\theta>0$, $\alpha>0$ and $\beta>0$ are the shape parameters.
Given $Z\in\mathbb{N}$, let $Y_1,Y_2,\cdots,Y_z$, be independent and identically distributed (iid) random variables with EBXII distribution. Let Z follow discrete random variable distributed Poisson truncated at zero with probability mass function given by

$$P(Z;\lambda)=\lambda^{z}((\exp(\lambda)-1)\;z!)^{-1},\qquad \lambda>0,\;z=1,2,3,4,\cdots. \tag{6}$$

In this case, we will consider the distribution of the maximum of Y_i' s, say $X=\max\{Y_i\},\ i=1,2,3,\cdots,z$. The conditional probability density function of X can be obtained from

$$f(x|z)=zg(x)[G(x)]^{z-1}, \tag{7}$$

where, $G(.)$ and $g(.)$ are the pdf and cdf of the EBXII distribution. The probability density function of X is obtained as follows.

Remark 2.1 *If $X=\min\{Y_i\}$, we have the Exponentiated BurrXII Poisson (Exp-BXIIP) distribution proposed by [5].*

See [35] for some compound class of lifetime model and poisson distribution.

Proposition 2.2 *Let $X=\max\{Y_i\}$, where $Y_i\sim EBXII(\alpha,\beta,\theta)$, then, according to (6) and (7), X is distributed according to complementary Exponentiated BurrXII Poisson (CEBXIIP) distribution given by*

$$f(x)=\frac{\alpha\beta\lambda\theta\, x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)}}{(\exp(\lambda)-1)}(1-(1+x^{\alpha})^{-\beta})^{\theta-1}\exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta}), \tag{8}$$

with parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\lambda > 0$.

Proof:
By considering (6) and (7) the pdf of the CEBXIIP$(\alpha, \beta, \lambda, \theta)$ can be computed by simplifying the unconditional probability of X

$$f(x) = \sum_{z=1}^{\infty} f(x|z)P(Z = z).$$

The cumulative distribution function of the CEBXIIP distribution is given by

$$F(x) = \frac{\exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta}) - 1}{\exp(\lambda) - 1}, \tag{9}$$

the survival and hazard rate function of the CEBXIIP distribution are, respectively given by

$$s(x) = \frac{\exp(\lambda) - \exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta})}{\exp(\lambda) - 1} \tag{10}$$

and

$$h(x) = \frac{\alpha\beta\lambda\theta\, x^{\alpha-1}\,(1+x^{\alpha})^{-(\beta+1)}\,(1-(1+x^{\alpha})^{-\beta})^{\theta-1}\exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta})}{\exp(\lambda) - \exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta})}. \tag{11}$$

Theorem 2.3 *The limiting distribution given by (9) when* $\lambda \to 0^{+}$ *is* $\lim_{\lambda \to 0^{+}} F(x;\alpha,\beta,\lambda,\theta) = (1-(1+x^{\alpha})^{-\beta})^{\theta}$, *which is the cdf of EBXII*(α,β,θ).

Corollary 2.4 *The probability density function and the hazard rate function of the CEBXIIP distribution can be express in terms of the cdf and pdf of the EBXII distributions as*

$$f(x) = \omega\, g(x)\, e^{\lambda G(x)} \quad \text{and} \quad h(x) = \frac{\lambda g(x)\, e^{\lambda G(x)}}{e^{\lambda} - e^{\lambda G(x)}},$$

respectively, where $\omega = \lambda(e^{\lambda} - 1)^{-1}$. $G(x)$ *and* $g(x)$ *are the cdf and pdf of the EBXII distribution given by (4) and (5) respectively.*
The shapes of the density and the hazard rate function of the CEBXIIP distribution can be analytically described as follows. The critical points of the probability density function $f(x)$ of the CEBXIIP distribution are the roots of the equation:

$$g'(x) + \lambda\, g^{2}(x) = 0, \tag{12}$$

equation (12) may have more than one root, let, $\vartheta(x) = d^{2}\log f(x)/dx^{2}$, we have

$$\vartheta(x) = g''(x) + 2\lambda\, g(x)g'(x). \tag{13}$$

Suppose that, $x = x_0$ is a root of (12), then, it corresponds to a local minimum if $\vartheta(x) < 0$ for all $x < x_0$ and $\vartheta(x) > 0$ for all $x > x_0$. It gives a local maximum if $\vartheta(x) > 0$ for all $x < x_0$ and $\vartheta(x) < 0$ for all $x > x_0$. And it refers to an inflexion point if either $\vartheta(x) > 0$ for all $x \neq x_0$ or $\vartheta(x) < 0$ for all $x \neq x_0$.
The critical points of the hazard rate function $h(x)$ of the CEBXIIP distribution are the roots of the equation:

$$g'(x) + \lambda\, g^{2}(x) + \frac{\lambda\, g^{2}(x)\; e^{\lambda G(x)}}{e^{\lambda} - e^{\lambda G(x)}} = 0, \tag{14}$$

equation (14) may have more than one root. Let, $\vartheta^{*}(x) = d^{2}\log h(x)/dx^{2}$, we have

$$\vartheta^{*}(x) = g''(x) + 2\lambda\, g(x)g'(x) + \frac{\lambda\, g(x)\, e^{\lambda G(x)}\,[2g'(x)\,e^{\lambda} - 2g'(x)\,e^{\lambda G(x)} + \lambda\, g^{2}(x)\,e^{\lambda}]}{(e^{\lambda} - e^{\lambda G(x)})^{2}}. \tag{15}$$

Suppose that, $x = x_0$ is a root of (14), then, $\vartheta^*(x)$ follows similar conditions to that of $\vartheta(x)$ given by (13). Figure 1 and 2 shows the plots of the probability density function f(x) and hazard rate function h(x) of the complementary exponentiated BurrXII poisson distribution (CEBXIIP) for some values of parameters respectively.

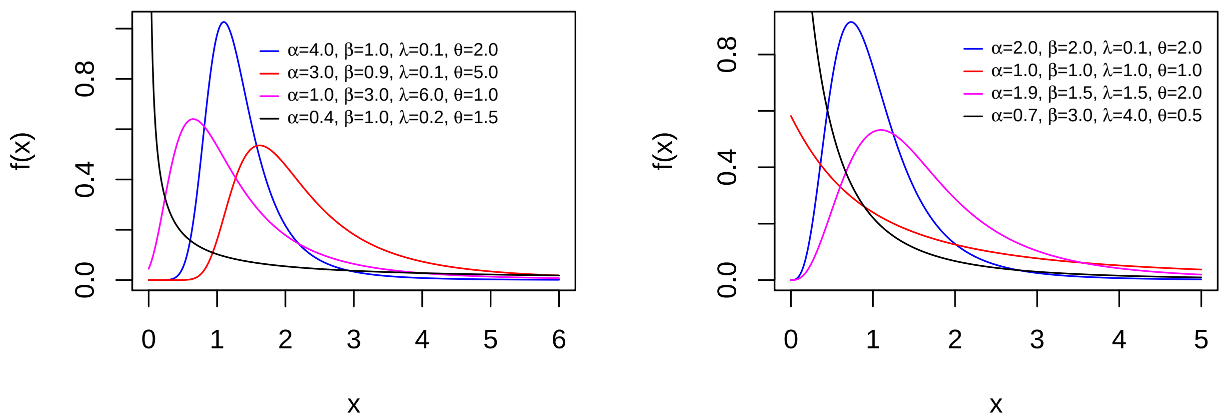

Fig. 1: Plots of probability density function of the complementary exponentiated BurrXII poisson distribution for different values of parameters.

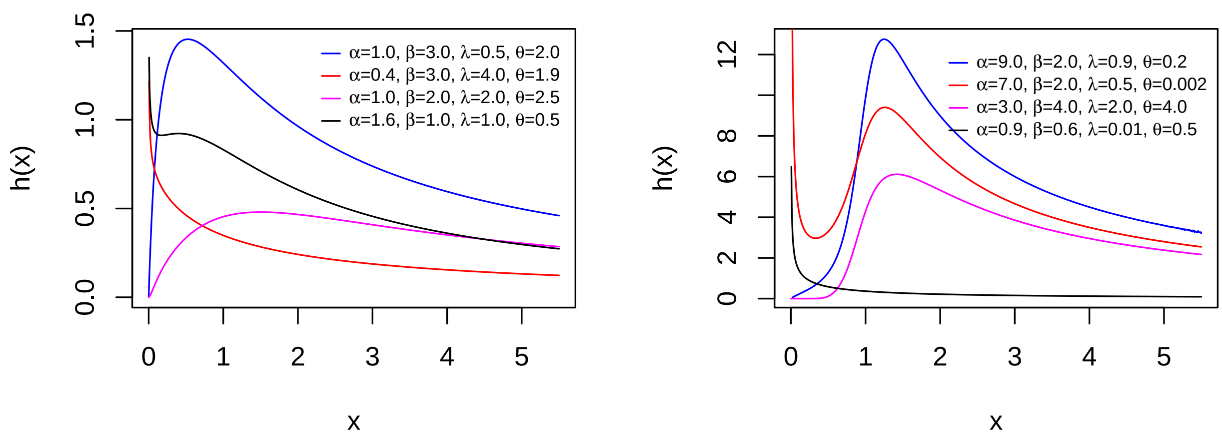

Fig. 2: Plots of hazard rate function of the complementary exponentiated BurrXII poisson distribution for different values of parameters.

2.1 Special cases

The special cases of the CEBXIIP distribution for selected values of the parameters are as follows:

1.When $\theta = 1$, we obtain BurrXII-zero truncated poisson distribution (BXIIZTP) by [6].
2.When $\beta = 1$, we obtain complementary exponentiated log-logistic poissson distribution (CELLP).
3.When $\theta = 1$ and $\beta = 1$, we have, complementary log-logistic poissson distribution (CLLP).
4.When $\alpha = 1$, we obtain complementary exponentiated Lomax poissson distribution (CELP).
5.When $\alpha = 1$ and $\theta = 1$, we obtain, poisson-Lomax distribution (PL) proposed by [7].

2.2 Series expansion of CEBXIIP distribution

We demonstrate that the pdf of the CEBXIIP distribution can be written as an infinite series of EBXII distribution or BXII distribution. Using the exponential expansion of $\exp(\lambda(1-(1+x^{\alpha})^{-\beta})^{\theta})$ in (1) and some algebraic manipulations, we can express the pdf of the CEBXIIP as

$$f(x) = \sum_{i=0}^{\infty} \phi_i \, g(x; \alpha, \beta, \theta(i+1)), \tag{16}$$

where

$$\phi_i = \frac{\lambda^{i+1}}{(\exp(\lambda)-1)\,(i+1)i!} \tag{17}$$

and $'g(x; \alpha, \beta, \theta(i+1))$ is an exponentiated BurrXII distribution with parameters α, β and $\theta(i+1)$. Also for $b > 0$ real and non-integer, $|\nu| < 1$,

$$(1-\nu)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\, j!} \nu^j. \tag{18}$$

For $\theta(i+1) > 0$ real and non-integers we can applying (18) in the expansion of the expression $(1-(1+x^{\alpha})^{-\beta})^{a(i+1)-1}$ in (16) and after some algebraic manipulations we obtain

$$f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varrho_{i,j} \, g(x; \alpha, \beta(j+1)), \tag{19}$$

where

$$\varrho_{i,j} = \frac{(-1)^j \, \theta \lambda^{i+1} \, \Gamma(\theta(i+1))}{(\exp(\lambda)-1)\,(j+1)\Gamma(\theta(i+1)-j) i!\, j!}$$

and $g(x; \alpha, \beta(j+1))$ is a BurrXII distribution with parameters α and $\beta(j+1)$.

3 Quantile and moments

The quantile function of the CEBXIIP distribution is computed directly by inverting (9) and can be used to generate a random data with generating a random data from uniform distribution. The quantile $\zeta(p)$ of the CEBXIIP distribution is given by

$$\zeta(p) = \left(\left(1 - \left(\frac{\log(p(e^{\lambda}-1)+1)}{\lambda} \right)^{\frac{1}{\theta}} \right)^{-\frac{1}{\beta}} - 1 \right)^{\frac{1}{\alpha}}, \qquad 0 < p < 1. \tag{20}$$

The r^{th} moment, moment generating function, skewness and kurtosis of a probabiliy distribution are very useful characteristics and one of the significant measures used in studying the features of a distribution. The r^{th} moments of the CEBXIIP distribution can be computed using

$$E(X^r) = \int_0^{\infty} x^r \, f(x) dx, \tag{21}$$

through (19) we have

$$E(X^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varrho_{i,j} \int_0^{\infty} x^r \, g(\alpha, \beta(j+1)) dx, \tag{22}$$

where the integral part is the r^{th} moment of BXII distribution with parameter α and $\beta(j+1)$ by considering (3) we have,

$$E(X^r) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varrho^*_{i,j}\, B(\beta(j+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}), \tag{23}$$

where

$$\varrho^*_{i,j} = \frac{(-1)^j\, \beta\theta\lambda^{i+1}\,\Gamma(\theta(i+1))}{(\exp(\lambda)-1)\,\Gamma(\theta(i+1)-j)i!\,j!},$$

and $B(.,.)$ is a complete beta function. The moment generating function of the CEBXIIP can be computed by substituting (23) in (24)

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r), \tag{24}$$

thus,

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\xi_{i,j,r}\, B((\beta(j+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}),$$

where

$$\xi_{i,j,r} = \frac{(-1)^j \beta\theta\, \lambda^{i+1}\Gamma(\theta(i+1))\, t^r}{(\exp(\lambda)-1)\,\Gamma(\theta(i+1)-j)\, i!\, j!\, r!}.$$

The skewness (γ_3) and kurtosis (γ_4) of the CEBXIIP distribution are respectively obtained from

$$\gamma_3 = \frac{1}{\sigma^3}\sum_{r=0}^{3}\binom{3}{r}(-1)^{r+1}\mu^{3-r}E(X^r) \tag{25}$$

and

$$\gamma_4 = \frac{1}{\sigma^4}\sum_{r=0}^{4}\binom{4}{r}(-1)^{r}\mu^{4-r}E(X^r) \tag{26}$$

as

$$\gamma_3 = \sum_{r=0}^{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{3}{r}\frac{(-1)^{j+r+1}\,\beta\theta\lambda^{i+1}\,\mu^{3-r}\Gamma(\theta(i+1))}{(\exp(\lambda)-1)\,\sigma^3\Gamma(\theta(i+1)-j)i!\,j!}B(\beta(j+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}) \tag{27}$$

and

$$\gamma_4 = \sum_{r=0}^{4}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{4}{r}\frac{(-1)^{j+r}\,\beta\theta\lambda^{i+1}\,\mu^{4-r}\Gamma(\theta(i+1))}{(\exp(\lambda)-1)\,\sigma^4\Gamma(\theta(i+1)-j)i!\,j!}B(\beta(j+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}), \tag{28}$$

where μ and σ are the mean and standard deviation of the CEBXIIP distribution. Furthermore the skewness and kurtosis of CEBXIIP can be analyzed using the quantile function via the Bowley skewness (B) and Moores kurtosis (M). The Bowley skewness and Moores kurtosis are defined respectively by

$$B = \frac{\zeta(3/4) + \zeta(1/4) - 2\,\zeta(2/4)}{\zeta(3/4) - \zeta(1/4)} \quad \text{and} \quad M = \frac{\zeta(3/8) - \zeta(1/8) + \zeta(7/8) - \zeta(5/8)}{\zeta(6/8) - \zeta(2/8)},$$

where $\zeta(.)$ is a quantile fuction given by (20). Figure 3 show the plots of the Bowley skewness and Moores kurtosis of CEBXIIP distribution.

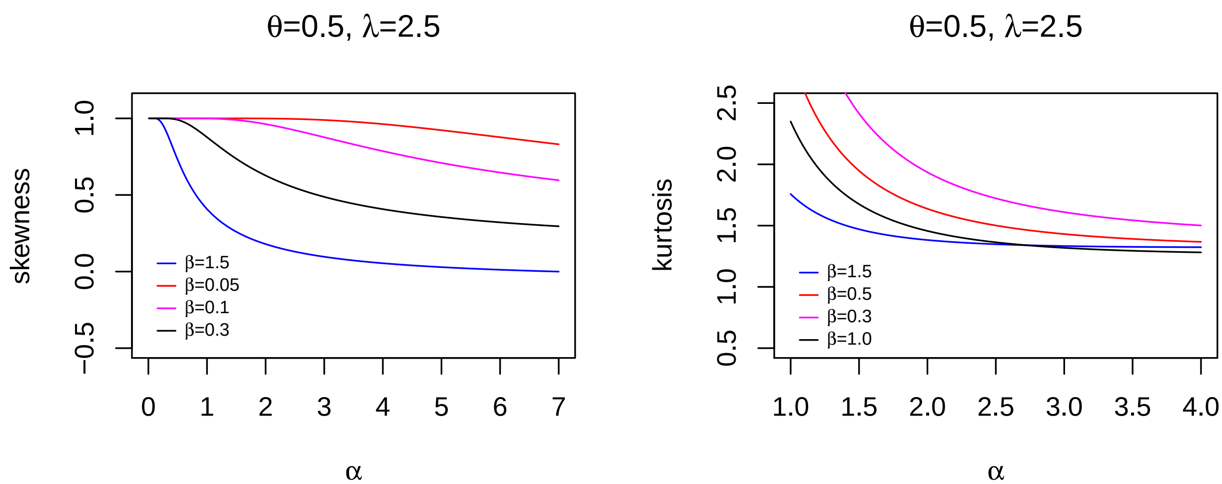

Fig. 3: Plots of B-skewness and M-kurtosis of the complementary exponentiated BurrXII poisson distribution for different values of parameter β.

4 Order statistics

Let $X_1, X_2, X_3, \cdots, X_n$ be a simple random sample from CEBXIIP distribution, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, be the order statistics obtained from this random sample, then for $j = 1, 2, 3, \cdots, n$, the corresponding pdf, say $f_{j:n}(x)$ is obtained as follows. The pdf of the j^{th} order statistics can be computed from

$$f_{x_j:n}(x;\alpha,\beta,\lambda,\theta) = \frac{n!}{(j-1)!(n-j)!} f(x)(F(x))^{j-1}(1-F(x))^{n-j},$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the CEBXIIP distribution.

$$f_{x_j:n}(x;\alpha,\beta,\lambda,\theta) = \sum_{l=0}^{n-j} \frac{n!\,(-1)^l}{(j-1)!(n-j-l)!\,l!} f(x)(F(x))^{j+l-1},$$

using the binomial expansion of F^{j+l-1} we have,

$$f_{x_j:n}(x) = \sum_{l=0}^{n-j}\sum_{k=0}^{j+l-1} \frac{(-1)^{j+k+2l-1}\,n!\,(j+l-1)!\exp(\lambda\,k(1-(1+x^{\alpha})^{-\beta})^{\theta})}{(\exp(\lambda)-1)^{j+l-1}(j+l-k-1)!\,(n-j-l)!\,(j-1)!k!l!}\; f(x), \tag{29}$$

after some algebraic manipulations, we obtained the pdf as

$$f_{x_j:n}(x;\alpha,\beta,\lambda,\theta) = \sum_{l=0}^{n-j}\sum_{k=0}^{j+l-1} \tau_{k,l}\, f(x;\alpha,\beta,\lambda(k+1),\theta), \tag{30}$$

where

$$\tau_{k,l} = \frac{n!(-1)^{j+k+2l-1}\,(j+l-1)!\,(\exp(\lambda(k+1))-1)}{(\exp(\lambda)-1)^{j+l}(j+l-k-1)!\,(n-j-l)!\,(j-1)!\,(k+1)\,k!l!}, \tag{31}$$

and $f(x;\alpha,\beta,\lambda(k+1),\theta)$ is the pdf of the CEBXIIP distribution with parameters $\alpha, \beta, \lambda(k+1)$ and θ. The r^{th} moment of the j^{th} order statistics is given by

$$E(X_{j:n}^r) = \int_0^\infty x^r\, f_{x_j:n}(x)dx \tag{32}$$

$$E(X_{j:n}^r) = \sum_{l=0}^{n-j}\sum_{k=0}^{j+l-1} \tau_{k,l} \int_0^\infty x^r\, f(x;\alpha,\beta,\lambda(k+1),\theta)dx, \tag{33}$$

thus, by considering (3)

$$E(X_{j:n}^r) = \sum_{l=0}^{n-j}\sum_{k=0}^{j+l-1}\sum_{i=0}^{\infty}\sum_{w=0}^{\infty} \tau_{k,l}^{*} B(\beta(w+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}) \tag{34}$$

where

$$\tau_{k,l}^{*} = \frac{n!(-1)^{j+k+2l+w-1}\,\beta\,\theta\,\lambda^{i+1}\,(k+1)^i\,(j+l-1)!\Gamma(\theta(i+1))}{(j+l-k-1)!\,(n-j-l)!\,(j-1)!\,\Gamma(\theta(i+1)-w)\,i!\,k!\,l!\,w!}. \tag{35}$$

5 Estimation and inference

In this section, we discuss the estimation of the parameters of the CEBXIIP distribution by the method of maximum likelihood. Let $X_1, X_2, \cdots, X_n$ be a random sample with observation values at $x_1, x_2, \cdots, x_n$ from CEBXIIP distribution with parameters α, β, λ and θ. Let Θ be the vector of the parameters $\Theta = (\alpha, \beta, \lambda, \theta)^T$, then, the total log-likelihood function is obtained as

$$\begin{aligned}\log \ell(\Theta) &= n\log\theta + n\log\alpha + n\log\beta + n\log\lambda - n\log(\exp(\lambda)-1)\\ &\quad + (\alpha-1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log(1+x_i^{\alpha})\\ &\quad + (\theta-1)\sum_{i=1}^{n}\log\left(1-((1+x_i^{\alpha})^{-\beta})\right) + \lambda\sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}.\end{aligned}$$

The associate score function is given by $U(\Theta) = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\theta})^T$, where

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{(1+x_i^{\alpha})} + (\theta-1)\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{(1-(1+x_i^{\alpha})^{-\beta})(1+x_i^{\alpha})^{\beta+1}}\\ &\quad + \beta\lambda\theta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\,(1-(1+x_i^{\alpha})^{-\beta})^{\theta-1}}{(1+x_i^{\alpha})^{\beta+1}}.\end{aligned} \tag{36}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \lambda\theta\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})(1-(1+x_i^{\alpha})^{-\beta})^{\theta-1}}{(1+x_i^{\alpha})^{\beta}} + (\theta-1)\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})}{(1-(1+x_i^{\alpha})^{-\beta})(1+x_i^{\alpha})^{\beta}}. \tag{37}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \frac{n}{(1-\exp(-\lambda))} + \sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}. \tag{38}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(1-(1+x_i^{\alpha})^{-\beta}\right)\\ &\quad + \lambda\sum_{i=1}^{n}((1-(1+x_i^{\alpha})^{-\beta})^{\theta})\log\left(1-(1+x_i^{\alpha})^{-\beta}\right).\end{aligned} \tag{39}$$

The MLEs of Θ i.e $\hat{\Theta}$, can be obtained by solving the non-linear system $U(\Theta) = 0$, the solution of the system $U(\Theta) = 0$ can be obtained by using mathematical or statistical software. For the interval estimation and hypothesis tests on the parameters, we required $I_n(\Theta)$. The fisher information matrix is given by $I_n(\Theta) = -\left[I_{ij}\right]_{ij=1}^{4}$. For a very large sample, the approximate of the MLEs of Θ say $\hat{\Theta}$ can be approximated as $N_4(0, J_n(\Theta))$, where the $J_n(\Theta) = E[I_n(\Theta)]$ Under the usual condition that are fulfilled for the parameter space but not on the boundary. The asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is the $N_4(0, J_n(\Theta))$ where $J_n(\Theta) = \lim_{x \to\infty} n^{-1} I_n(\Theta)$ is the unit information matrix which can be used to construct the approximate confidence interval for each of the parameter. A $100(1-\epsilon)\%$ asymptotic confidence interval for each parameter Θ_r is given by $ACI_r = (\hat{\Theta}_r - Z_{\frac{\epsilon}{2}}\sqrt{\hat{I}^{rr}}, \hat{\Theta}_r + Z_{\frac{\epsilon}{2}}\sqrt{\hat{I}^{rr}})$, where I^{rr} is the (r,r) diagonal element of $I_n(\Theta)^{-1}$ for $r = 1,2,3,4$ and $Z_{\frac{\epsilon}{2}}$ is the quantile $(1-\frac{\epsilon}{2})$ of the standard normal distribution. The elements of $J_n(\Theta)$ are given in appendix A. Besides that, we can consider the test of the null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ and this is equivalent to compare the BXIIZTP distribution with the CEBXIIP distribution and the LR statistic reduces to $w = 2[\ell(\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \ell(1, \breve{\alpha}, \breve{\beta}, \breve{\lambda})]$ where $\hat{\Theta}$ and $\breve{\Theta}$ are the unrestricted and restricted MLEs of Θ respectively. Under the null hypothesis w is asymptotically distributed as χ_1^2. For a given level ξ the LR test rejects H_0 if w exceeds the $(1-\xi)$-quantile of the χ_1^2 distribution. Theorems 5.1, 5.2, 5.3 and 5.4 established the existence and uniqueness of the MLE, under some certain conditions when the other parameters are known.

Theorem 5.1 *Let $\delta_1(\alpha; \beta, \lambda, \theta, x_i)$ be the function on the right hand side of equation (36), where β, λ and θ are the true values of the parameters, then, the root of $\delta_1(\alpha; \beta, \lambda, \theta, x_i) = 0$ can take one of the followings.*
(1) For $\theta \geq 1$ and $max\{X_i\} < 1$
(2) For $\theta \geq 1$ and $min\{X_i\} > 1$

Theorem 5.2 *Let $\delta_2(\beta; \alpha, \lambda, \theta, x_i)$ be the function of the right hand side of equation (37) where α, λ and θ are the true values of the parameters, then, the root of $\delta_2(\beta; \alpha, \lambda, \theta, x_i) = 0$ say $\hat{\beta}$ can take one of the following forms.*
(1) *For a given $\lambda \in (0,1)$ and $\theta = 1$ the root lies in the interval $\left(n(\sum_{i=1}^{n} \log(1+x_i^{\alpha}))^{-1}, n(\sum_{i=1}^{n}(1-\lambda)\log(1+x_i^{\alpha}))^{-1}\right)$ and is unique.*
(2) *For a given $\lambda > 1$ and $\theta = 1$ the root lies in the interval $\left(0, n(\sum_{i=1}^{n} \log(1+x_i^{\alpha}))^{-1}\right)$ and is unique.*
(3) *For $\theta \neq 1$ the equation $\delta_2(\beta; \alpha, \lambda, \theta, x_i) = 0$, has at least one root.*

Theorem 5.3 *Let $\delta_3(\lambda; \alpha, \beta, \theta, x_i)$ denote the function on the right hand side of equation (38), where α, β and θ are the true values of the parameters, then, $\delta_3(\lambda; \alpha, \beta, \theta, x_i) = 0$, has at least one root for $n > \sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}$.*

Theorem 5.4 *Let $\delta_4(\theta; \alpha, \beta, \lambda, x_i)$ be the function on the right hand side of the equation (39), where α, β and λ are the true values of the parameters, then, the root of $\delta_4(\theta; \alpha, \beta, \lambda, x_i) = 0$ lies in the interval*
$\left(\frac{-n}{(1+\lambda)\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}, \frac{-n}{\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}\right)$.
For the proofs of theorem **5.1**, **5.2**, **5.3** and **5.4** see appendix B.

5.1 Simulation

In this subsetion, we assess the performance of the maximum likelihood estimates by conducting a simulation studies. We generate ten thousand samples from the CEBXIIP distribution each of sample size 200, 400 and 600. The sample size (n), actual values, estimated values and standard deviations for various values of parameters are listed in Table 1 below. The result of the simulation shows that the maximum likelihood method performed consistently and the standard deviations of the MLEs decrease as the sample size increases.

Table 1: MLEs and standard deviations for various values of parameter.

Sample size	Actual values				Estimated values				Standard deviations			
n	α	β	λ	θ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$sd(\alpha)$	$sd(\beta)$	$sd(\lambda)$	$sd(\theta)$
200	1.0	0.1	0.5	0.5	1.0789	0.1011	0.8556	0.4770	0.2821	0.0325	1.1390	0.1547
	0.1	0.1	0.2	1.2	0.1313	0.0920	0.7577	1.0418	0.0750	0.0314	1.1092	0.2680
	1.0	0.2	1.0	1.0	1.0530	0.2016	1.1073	1.0339	0.3477	0.0885	1.5863	0.5219
	1.2	0.8	0.3	0.5	1.1874	0.8108	0.3204	0.5592	0.3599	0.6101	0.8193	0.4125
	1.2	0.2	1.2	0.5	1.0094	0.2713	0.8981	0.6961	0.2903	0.0978	1.3413	0.2830
	1.3	0.3	1.3	0.4	1.2379	0.3896	1.3115	0.6002	0.3553	0.3788	1.1770	2.4320
	0.2	0.2	1.0	1.2	0.2331	0.2013	1.2317	1.2032	0.1121	0.0884	1.3247	0.4514
	1.9	0.2	0.2	0.9	1.7115	0.2525	0.3254	0.9999	0.4621	0.0706	0.8735	0.2407
	1.2	0.2	1.2	0.5	1.0094	0.2713	0.8981	0.6961	0.2903	0.0978	1.3413	0.2823
	0.6	0.9	1.0	0.4	0.6157	1.1164	1.3219	0.4664	0.2481	0.6929	1.1225	0.2806
	1.2	1.3	0.4	0.5	1.2398	1.3082	0.5622	0.5216	0.3953	0.5317	1.0449	0.1975
	0.5	0.6	0.9	0.3	0.5552	0.8700	1.1614	0.4099	0.3098	0.5487	0.9617	0.2589
	3.0	4.0	1.0	2.8	3.3120	3.7373	0.8468	1.9631	0.7774	0.8136	1.0693	1.1281
	2.5	2.0	1.5	1.1	2.8455	2.0391	1.6470	1.3136	1.2772	0.9040	1.3536	1.1053
400	1.0	0.1	0.5	0.5	1.0578	0.0993	0.7511	0.4799	0.2236	0.0232	0.9223	0.1307
	0.1	0.1	0.2	1.2	0.1154	0.0948	0.5584	1.0487	0.0388	0.0214	0.8733	0.2047
	1.0	0.2	1.0	1.0	1.0429	0.1984	1.1261	1.0090	0.2766	0.0577	1.3323	0.3298
	1.2	0.8	0.3	0.5	1.1977	0.7434	0.2600	0.5060	0.2811	0.5058	0.6792	0.2280
	1.2	0.2	1.2	0.5	1.0442	0.2522	0.9020	0.6542	0.2618	0.0706	1.1099	0.2378
	1.3	0.3	1.3	0.4	1.2654	0.3362	1.2834	0.4715	0.2640	0.1684	0.8598	1.2717
	0.2	0.2	1.0	1.2	0.2222	0.1972	1.2281	1.1789	0.0781	0.0638	1.2626	0.3964
	1.9	0.2	0.2	0.9	1.6554	0.2536	0.1894	1.0207	0.3622	0.0600	0.6042	0.1908
	1.2	0.2	1.2	0.5	1.0442	0.2522	0.9020	0.6542	0.2618	0.0706	1.1099	0.2378
	0.6	0.9	1.0	0.4	0.6254	1.0185	1.1931	0.4254	0.2233	0.6190	0.9377	0.2044
	1.2	1.3	0.4	0.5	1.2200	1.2912	0.4857	0.5131	0.3116	0.4851	0.9056	0.1348
	0.5	0.6	0.9	0.3	0.5844	0.7675	1.0467	0.3764	0.3087	0.3806	0.6899	0.1857
	3.0	4.0	1.0	2.0	3.2392	3.7629	0.8420	1.9291	0.6014	0.7494	0.9690	0.6878
	2.5	2.0	1.5	1.1	2.7681	2.8843	1.6530	1.1668	1.0061	0.7884	1.2332	0.6449
600	1.0	0.1	0.5	0.5	1.0443	0.0993	0.6709	0.4862	0.1976	0.0197	0.7662	0.1178
	0.1	0.1	0.2	1.2	0.1089	0.0970	0.4334	1.0517	0.0263	0.0162	0.6788	0.1655
	1.0	0.2	1.0	1.0	1.0371	0.1986	1.1144	1.0044	0.2562	0.0410	1.2070	0.3034
	1.2	0.8	0.3	0.5	1.1999	0.6445	0.1807	0.4910	0.2296	0.4250	0.5413	0.1840
	1.2	0.2	1.2	0.5	1.0657	0.2431	0.9164	0.6317	0.2452	0.0622	0.9618	0.2192
	1.3	0.3	1.3	0.4	1.2714	0.3226	1.2851	0.4339	0.2140	0.0809	0.7068	0.1338
	0.2	0.2	1.0	1.2	0.2205	0.1970	1.2331	1.1678	0.0738	0.0607	1.1858	0.3740
	1.9	0.2	0.2	0.9	1.6271	0.2553	0.1397	1.0317	0.3158	0.0561	0.4917	0.1715
	1.2	0.2	1.2	0.5	1.0692	0.2416	0.9325	0.6268	0.2419	0.0612	0.9648	0.2169
	0.6	0.9	1.0	0.4	0.6418	0.9414	1.0903	0.4008	0.2148	0.5903	0.8267	0.1851
	1.2	1.3	0.4	0.5	1.2100	1.2820	0.4360	0.5147	0.2723	0.4634	0.8125	0.1159
	0.5	0.6	0.9	0.3	0.5850	0.8727	0.9765	0.3646	0.2930	0.3072	0.5106	0.1669
	3.0	4.0	1.0	2.0	3.2190	3.7673	0.8136	1.9281	0.5493	0.7255	0.8855	0.5999
	2.5	2.0	1.5	1.1	2.7458	1.9858	1.6036	1.1268	0.8957	0.7422	1.1059	0.5100

6 Applications

In this section, we provide an application of the CEBXIIP to a real data set for illustrative purpose and compare its fit with some other existing models based on the Akaike information criterion (AIC), Bayesian information criteria (BIC) and Kolmogorov Smirnov (KS) test statistic. The distribution with the smallest value of these measures fit the data better than the other distributions. The competing models includes the Generalize BurrXII-poisson (GBXIIP) by [8], BurrXII zero truncated poisson (BXIIZTP) by [6], BurrXII- poisson (BXIIP) by [9], BurrXII (BXII) by [10] and the exponenciated BurrXII (EBXII) distributions. The data set were revealed in [11,12] and recently studied by [13]. It is the observe time to failure of Kevlar 49/Epoxy strands tested at various stress level: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89. The results of the data set are summarized as follows. For the LR test, we test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ i.e we compare the CEBXIIP distribution with the BXIIZTP distribution. We obtain $w = 5.40(p - value = 0.020) > 3.842 = \chi^2_1(at\, 0.05)$, hence, in this case CEBXIIP distribution is superior than the BXIIZTP distribution. Table 2 provides the MLEs and the log likelihood $\ell(\Theta)$ which we obtained using *nmlinb in R-software*. Table 3 shows the numerical vales of the AIC, BIC, KS and p-value of the competing distributions.

Table 2: Maximum likelihood estimates (MLEs) and $\ell(\Theta)$ for the data set

Model	α	β	λ	θ	a	$\ell(\Theta)$
CEBXIIP(α, β, λ, θ)	**2.8447**	**0.6553**	**1.6097**	**0.2236**	**-**	**-103.50**
GBXIIP(α, β, λ, a)	2.3411	0.6829	4.98e-7	-	0.3902	-105.99
BXIIZTP(α, β, λ)	0.9856	2.3856	1.7333	-	-	-106.20
BXIIP(α, β, λ)	1.1737	1.6327	1.98e-8	-	-	-108.55
EBXII(α, β, θ)	0.3215	2.7890	-	0.5385	-	-105.86
BXII(α, β)	1.1736	1.6327	-	-	-	-108.55

Table 3: AIC, BIC and KS and its p-value for the data set

Model	AIC	BIC	K-S	p-value
CEBXIIP(α, β, λ, θ)	**214.99**	**225.45**	**0.0851**	**0.4579**
GBXIIP(α, β, λ, a)	219.97	230.43	0.1269	0.0772
BXIIZTP(α, β, λ)	218.40	226.25	0.1058	0.2080
BXIIP(α, β, λ)	223.10	230.94	0.1349	0.0506
EBXII(α, β, θ)	217.72	225.57	0.1251	0.0849
BXII(α, β)	223.10	230.94	0.1353	0.0494

Table 3 shows that the CEBXIIP has the least value of the AIC, BIC and K-S, thus, CEBXIIP fit the data better than the other models. Figure 4 below illustrate the fitted pdfs and cdfs of the competing distribution for the given data set.

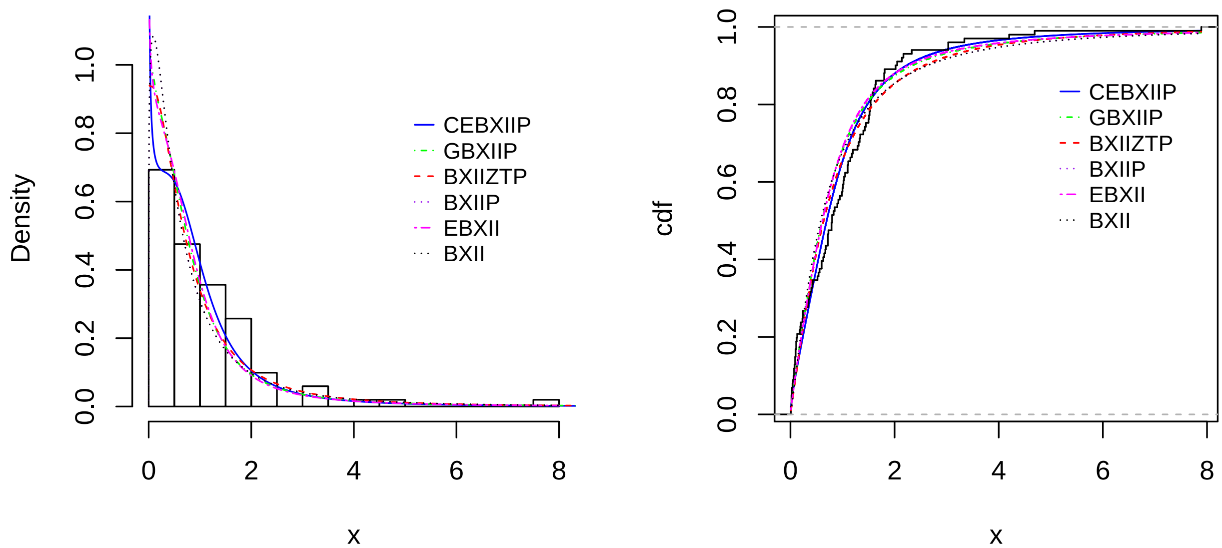

Fig. 4: Fitted pdfs and cdf for the data set.

7 Conclusion

We introduce and study a new four parameter lifetime distribution called the complementary exponentiated BurrXII Poisson distribution (CEBXIIP). We provide several mathematical properties of the CEBXIIP distribution which includes the Shape characteristic of its pdf and hazard function. We also present an explicit expressions of the r^{th} moment, moment generating function, skewness, kurtosis, the density of the order statistics and the r^{th} moment of the order statistics. Estimation of the four unknown parameters by maximum likelihood is considered and the observed Fisher information matrix is obtained. The existence and uniqueness of the MLEs of CEBXIIP are studied under some certain conditions. An application of the CEBXIIP distribution to a real data is demonstrated to express the usefulness of the proposed distribution.

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Appendix A

the elements of $J(\Theta)$

$$
\begin{aligned}
I_{11} = & -\frac{n}{\alpha^2} - (\beta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{u_i} + (\beta+1)\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2}{u_i} \\
& + (\theta-1)\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1-u_i^{-\beta})u_i^{\beta+1}} - (\theta-1)(\beta+1)\beta\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2}{(1-u_i^{-\beta})u_i^{\beta+2}} \\
& - (\theta-1)\beta^2\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2}{(1-u_i^{-\beta})^2\, u_i^{2(\beta+1)}} + \theta\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}} \\
& + \theta(\theta-1)\beta^2\lambda\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2(1-u_i^{-\beta})^{\theta-2}}{u_i^{2\beta+1}} - \theta\beta^2\lambda\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta+1}},
\end{aligned}
$$

$$I_{22} = -\frac{n}{\beta^2} - (\theta-1)\sum_{i=1}^{n}\frac{(\log u_i)^2}{(1-u_i^{-\beta})^2\, u_i^{2\beta}} - (\theta-1)\sum_{i=1}^{n}\frac{(\log u_i)^2}{(1-u_i^{-\beta})^2\, u_i^{\beta}}$$
$$+\, \theta(\theta-1)\lambda\sum_{i=1}^{n}\frac{(\log u_i)^2\,(1-u_i^{-\beta})^{\theta-2}}{u_i^{2\beta}} + \theta\lambda\sum_{i=1}^{n}\frac{(\log u_i)^2\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}},$$

$$I_{33} = -\frac{n}{\lambda^2} + \frac{n\exp(-\lambda)}{(1-\exp(-\lambda))^2},$$

$$I_{44} = -\frac{n}{\theta^2} + \lambda\sum_{i=1}^{n}(1-u_i^{-\beta})^{\theta}\,(\log(1-u_i^{-\beta}))^2,$$

$$I_{12} = -\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\,\log x_i}{u_i} + (\theta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1-u_i^{-\beta})\,u_i^{\beta+1}}$$
$$-\,(\theta-1)\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\log u_i}{(1-u_i^{-\beta})^2\,u_i^{2\beta+1}} - (\theta-1)\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\log u_i}{(1-u_i^{-\beta})\,u_i^{\beta+1}}$$
$$+\,\theta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}} + \theta(\theta-1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\ \log u_i,(1-u_i^{-\beta})^{\theta-2}}{u_i^{2\beta}}$$
$$-\,\theta\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\ \log u_i\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}},$$

$$I_{13} = \theta\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}},$$

$$I_{14} = \beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{(1-u_i^{-\beta})\,u_i^{\beta+1}} + \beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}}$$
$$+\,\theta\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i\,(\log(1-u_i^{-\beta}))\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}},$$

$$I_{23} = \theta\sum_{i=1}^{n}\frac{(1-u_i^{-\beta})^{\theta-1}\ \log u_i}{u_i^{\beta}},$$

$$I_{24} = \sum_{i=1}^{n}\frac{\log u_i}{(1-u_i^{-\beta})\,u_i^{\beta}} + \lambda\sum_{i=1}^{n}\frac{(1-u_i^{-\beta})^{\theta-1}\ \log u_i}{u_i^{\beta}}$$
$$+\,\theta\lambda\sum_{i=1}^{n}\frac{\log u_i\,(\log(1-u_i^{-\beta}))\,(1-u_i^{-\beta})^{\theta-1}}{u_i^{\beta}},$$

$$I_{34} = \sum_{i=1}^{n}(1-u_i^{-\beta})^{\theta}\log(1-u_i^{-\beta}),$$

where $u_i = (1 + x_i^{\alpha})$.

Appendix B

Proof of theorem 5.1 :

The $\lim_{\alpha\to 0}\delta_1(\alpha;\beta,\lambda,\theta,x_i) = \infty$, and
for $\theta \geq 1$, $\lim_{\alpha\to\infty}\delta_1(\alpha;\beta,\lambda,\theta,x_i) = \theta\sum_{x_i<1}^{n}\log x_i - \beta\sum_{x_i>1}^{n}\log x_i$.
To show that $\delta_1 < 0$ as $\alpha \to \infty$, we consider this cases.
(a) If $Max\{X_i\} < 1$, then, $\lim_{\alpha\to\infty}\delta_1(\alpha;\beta,\lambda,\theta,x_i) = a\sum_{x_i<1}^{n}\log x_i < 0$.
(b) If $Min\{X_i\} > 1$, then, $\lim_{\alpha\to\infty}\delta_1(\alpha;\beta,\lambda,\theta,x_i) = -\beta\sum_{x_i>1}^{n}\log x_i < 0$.
(c) If $Min\{X_i\} > 1$ and $Max\{X_i\} < 1$, then, $\lim_{\alpha\to\infty}\delta_1(\alpha;\beta,\lambda,\theta,x_i) = a\sum_{x_i<1}^{n}\log x_i - \beta\sum_{x_i>1}^{n}\log x_i < 0$. Thus, $\delta_1 < 0$ in all the cases, if and only if $x_i \neq 1$, for some $i = 1, 2, 3, \cdots, n$. Since δ_1 is contineous function which decreases

monotonically from positive to negetive values, therefore, there exist at least one root for $\delta_1(\alpha;\beta,\lambda,\theta,x_i)=0$.

Proof of theorem 5.2:

For $\theta=1$

Let $q_2(\beta;\alpha,\lambda,\theta,x_i) = \lambda\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})}{(1+x_i^{\alpha})^{\beta}}$. It is clear that $q_2(\beta;\alpha,\lambda,\theta,x_i)$ is strictly decreasing in β, and $\lim_{\beta\to\infty} q_2(\beta;\alpha,\lambda,\theta,x_i)=0$. It follows that,

$$\delta_2(\beta;\alpha,\lambda,\theta,x_i) > \frac{n}{\beta}-\sum_{i=1}^{n}\log(1+x_i^{\alpha})+\lim_{\beta\to\infty}\lambda\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})}{(1+x_i^{\alpha})^{\beta}} = \frac{n}{\beta}-\sum_{i=1}^{n}\log(1+x_i^{\alpha}).$$

and $\delta_2(\beta;\alpha,\lambda,\theta,x_i)>0$ when $\beta<n(\sum_{i=1}^{n}\log(1+x_i^{\alpha}))^{-1}$. On the other hand,

$\lim_{\beta\to 0}\delta_2(\beta;\alpha,\lambda,\theta,x_i)=\lambda\sum_{i=1}^{n}\log(1+x_i^{\alpha})$ so that,

$$\delta_2(\beta;\alpha,\lambda,\theta,x_i) < \frac{n}{\beta}-\sum_{i=1}^{n}\log(1+x_i^{\alpha})+\lim_{\beta\to 0}\lambda\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})}{(1+x_i^{\alpha})^{\beta}} = \frac{n}{\beta}-(1-\lambda)\sum_{i=1}^{n}\log(1+x_i^{\alpha})$$

Hence, $\delta_2(\beta;\alpha,\lambda,\theta,x_i)<0$ provided that $\beta>n(\sum_{i=1}^{n}(1-\lambda)\log(1+x_i^{\alpha}))^{-1}$.

(1) For $\lambda\in(0,1)$, there is at least one root of $\delta_2(\beta;\alpha,\lambda,\theta,x_i)=0$ in the interval $\left(n(\sum_{i=1}^{n}\log(1+x_i^{\alpha}))^{-1}, n(\sum_{i=1}^{n}(1-\lambda)\log(1+x_i^{\alpha}))^{-1}\right)$.

To prove that, the root is unique, it is enough to show that, the first derivative of δ_2 (J_2') is negetive.

$$\delta_2' = -\frac{n}{\beta^2}-\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^{\alpha}))^2}{(1+x_i^{\alpha})^{\beta}}. \tag{40}$$

Its clear that $\delta_2'<0$, hence, δ_2 is strictly decreasing in β.

(2) For $\lambda>1$, there is at least one root of of $\delta_2(\beta;\alpha,\lambda,\theta,x_i)=0$ in the interval $\left(0, n(\sum_{i=1}^{n}\log(1+x_i^{\alpha}))^{-1}\right)$. Where (40) prove the uniqueness.

(3) For $\theta\neq 1$

$\lim_{\beta\to 0}\delta_2=\infty$, then, we show that $\lim_{\beta\to\infty}\delta_2<0$.

$\lim_{\beta\to\infty}\delta_2=-\sum_{i=1}^{n}\log(1+x_i^{\alpha})<0$, therefore δ_2 is contineous monotone function which decreases from positive to negetive values, thus, $\delta_2(\beta;\alpha,\lambda,\theta,x_i)=0$ has at least one root.

Proof of theorem 5.3:

$\lim_{\lambda\to 0}\delta_3=\infty$, therefore, we show that $\lim_{\lambda\to\infty}\delta_3<0$.

$\lim_{\lambda\to\infty}\delta_3=-n+\sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}$, therefore, $\delta_3<0$ if $n>\sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}$,

thus, δ_3 is a contineous function which decreases monotonically from positive to negetive values, hence, $\delta_3(\lambda;\alpha,\beta,\theta,x_i)=0$ has at least one root.

Proof of theorem 5.4:

Let $q_4=\lambda\sum_{i=1}^{n}(1-(1+x_i^{\alpha})^{-\beta})^{\theta}\log(1-(1+x_i^{\alpha})^{-\beta})$, q_4 is strictly decreasing in θ, and $\lim_{\theta\to 0}q_4=\lambda\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})$, we have,

$\delta_4>\frac{n}{\theta}+\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})+\lim_{\theta\to 0}q_4 = \frac{n}{\theta}+(1+\lambda)\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})$, thus, $\delta_4>0$ when $\theta>\frac{-n}{(1+\lambda)\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}$. On the other hand, $\lim_{\theta\to\infty}q_4=0$, so,

$\delta_4<\frac{n}{\theta}+\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})+\lim_{\theta\to\infty}q_4=\frac{n}{\theta}+\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})$, thus, $\delta_4<0$ whenever $\theta<\frac{-n}{\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}$, hence, the root of $\delta_4(\theta;\alpha,\beta,\lambda,x_i)=0$ lies in the interval $\left(\frac{-n}{(1+\lambda)\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}, \frac{-n}{\sum_{i=1}^{n}\log(1-(1+x_i^{\alpha})^{-\beta})}\right)$.

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