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New Modular Relations for the Rogers-Ramanujan Type Functions of Order Thirteen with Applications to Partitions

Nasser Abdo Saeed Bulkhali[1] *and Ranganatha D*[2,*]

[1] Department of Mathematics, Faculty of Education-Zabid, Hodeidah University, Hodeidah, Yemen
[2] Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru -570006, India

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Abstract: In this paper, we establish some modular relations for the Rogers-Ramanujan type functions of order thirteen which are analogues to Ramanujan forty identities for Rogers-Ramanujan functions and as an application we extract some theorems in partitions.

Keywords: Rogers-Ramanujan functions, Modular relations, theta functions, partitions

1 Introduction

In the sequel, we assume that $|q| < 1$. For positive integer n, we use the standard notation

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$$
$$\text{and } (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also write

$$(a_1,a_2,a_3,\cdots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

In the theory of q-series, two of the most important results are the classical Rogers-Ramanujan identities which state that

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q^1;q^5)_\infty(q^4;q^5)_\infty}$$

and

$$H(q) := \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Ramanujan [18] Recorded forty modular relations involving the functions $G(q)$ and $H(q)$ including following two beautiful identities:

$$H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1+11q\{G(q)H(q)\}^6 \tag{1}$$

and

$$H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1. \tag{2}$$

For other details, proofs and further references, see [8, 11]. In view of the Ramanujan forty identities, many researchers studied Rogers–Ramanujan type functions and established several modular relations involving them and extracted some theorems in partitions. Two beautiful analogues of the Rogers-Ramanujan functions are the Göllnitz-Gordon identities, given by [13, 16]

$$S(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)}{(q^2;q^2)}q^{n^2} = \frac{1}{(q^1;q^8)_\infty(q^7;q^8)_\infty(q^4;q^8)_\infty}$$

and

$$T(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)}{(q^2;q^2)}q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty(q^5;q^8)_\infty(q^4;q^8)_\infty}.$$

Using the idea of Rogers, Watson [23] and Bressoud [12], Huang [16] and Chen and Huang [13] have established several modular relations for the Göllnitz-Gordan functions and Baruah et al. [10] have given alternative

* Corresponding author e-mail: ddranganatha@gmail.com

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proofs some of them by using Schröter's formulas and some simple theta functions identities of Ramanujan. These functions were studied by Xia and Yao [24]. Septic analogues of Rogers-Ramanujan type functions were studied by Hahn [15,17], Nonic analogues of Rogers-Ramanujan type functions were studied by Baruah and Bora [9] and sextodecic analogues of the Rogers-Ramanujan functions were studied by Gugg [14] and Adiga and Bulkhali [3]. Adiga et al. [4,5,6] have studied several Rogers-Ramanujan type functions of different orders. In [20], Srivastava and Chaudhary have established relationships between $q-$product identities, continued fraction identities and combinatorial partition identities. Recently, Srivastava et al. [21] have derived several results involving $q-$series and associated continued fractions.

For $|ab| < 1$, Ramanujan's general theta function is defined by [1]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{3}$$

The Jacobi triple product identity in Ramanujan's notation is given by [1, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{4}$$

The function $f(a,b)$ satisfies the following basic properties [1]:

$$f(a,b) = f(b,a), \tag{5}$$

$$f(1,a) = 2f(a,a^3), \tag{6}$$

$$f(-1,a) = 0. \tag{7}$$

Furthermore, if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{8}$$

Ramanujan defined the following three special cases of (3) [1, Entry 22]:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{9}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{10}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{11}$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty,$$

for a positive integer n. In this paper, we consider the following six functions of order thirteen which are analogues to the Rogers-Ramanujan functions:

$$U(q) := \frac{(q,q^{12},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q,-q^{12})}{f(-q)}, \tag{12}$$

$$V(q) := \frac{(q^2,q^{11},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q^2,-q^{11})}{f(-q)}, \tag{13}$$

$$W(q) := \frac{(q^3,q^{10},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q^3,-q^{10})}{f(-q)}, \tag{14}$$

$$X(q) := \frac{(q^4,q^{9},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q^4,-q^{9})}{f(-q)} \tag{15}$$

$$Y(q) := \frac{(q^5,q^{8},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q^5,-q^{8})}{f(-q)}, \tag{16}$$

and

$$Z(q) := \frac{(q^6,q^{7},q^{13};q^{13})_\infty}{(q;q)_\infty} = \frac{f(-q^6,-q^{7})}{f(-q)}. \tag{17}$$

A. V. Sills [22, Eqs (4.20) and (4.21)], established that

$$U(q) = \sum_{n,r\geq 0} \frac{q^{n^2+2r^2+2nr+2n+3r}(q;q)_{n+r+1}}{(q;q)_{2n+2r+2}(q;q)_n(q;q)_r}$$

and

$$Z(q) = \sum_{n,r\geqq 0} \frac{q^{n^2+2r^2+2nr}(q;q)_{n+r}}{(q;q)_{2n+2r}(q;q)_n(q;q)_r}.$$

In 1974, G. E. Andrews [7] obtained a generalization of the well-known Rogers-Ramanujan functions to odd moduli, namely for all $k \geq 2$, $1 \leq i \leq k$,

$$\sum_{n_1,n_2,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \frac{f(-q^i,-q^{2k+1-i})}{f(-q,-q^2)} \tag{18}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$. We observe that, functions defined in (12)–(17) can be obtained by setting $k = 6$ and $i = 1,2,3,4,5,6$ in the right-hand side of (18).

The following identity is an easy consequence of Entry 31 [1] when $n = 2$:

$$f(a,b) = f(a^3b,ab^3) + af(b/a,a^5b^3). \tag{19}$$

Setting $a = b = q$ in (19), we find that

$$\varphi(q^4) + 2q\psi(q^8) = \varphi(q). \tag{20}$$

Using (20), one can easily establish the following lemma:

Lemma 1.

$$\varphi(-q^a)\varphi(q^b) - \varphi(q^a)\varphi(-q^b) = 4q^b\left\{\varphi(q^{4a})\psi(q^{8b}) - q^{a-b}\varphi(q^{4b})\psi(q^{8a})\right\}. \tag{21}$$

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Lemma 2. *We have*

$$\varphi(q)=\frac{f_2^5}{f_1^2 f_4^2},\quad \psi(q)=\frac{f_2^2}{f_1},$$
$$\varphi(-q)=\frac{f_1^2}{f_2}\ \text{and}\ \ \psi(-q)=\frac{f_1 f_4}{f_2}.$$

This lemma is a consequence of (4) and Entry 24 of [1, p. 34].

Lemma 3. *Let* $m=\left[\frac{s}{s-r}\right]$, $l=m(s-r)-r$, $k=-m(s-r)+s$ *and* $h=mr-\frac{m(m-1)(s-r)}{2}$, $0\le r<s$. *Here* $[x]$ *denote the largest integer less than or equal to x. Then*

$$(i)\ \ f(q^{-r},q^{s})=\ q^{-h}f(q^{l},q^{k}),$$
$$(ii)\ \ f(-q^{-r},-q^{s})=\ (-1)^{m}q^{-h}f(-q^{l},-q^{k}).$$

For a proof of Lemma 3, see [2].
The main aim of this paper is to establish several modular relations involving the Rogers-Ramanujan type functions in (12)–(17) which are analogous to Ramanujan's forty identities and we extract some theorems in partitions from our main results.

2 Main Results

In this section, we present a list of modular relations involving the functions defined in (12)–(17). For simplicity, for a positive integer n, we set $U_n:=U(q^n)$, $V_n:=V(q^n)$, $W_n:=W(q^n)$, $X_n:=X(q^n)$, $Y_n:=Y(q^n)$ and $Z_n:=Z(q^n)$.
We prove our main results using ideas similar to those of Watson [23] and Bressoud [12].

Theorem 1. *If* $1\le r\le 6$, *then the following modular relation holds true:*

$$q^{15}U_{6+r}U_{7-r}+q^{10}V_{6+r}V_{7-r}+q^{6}W_{6+r}W_{7-r}+q^{3}X_{6+r}X_{7-r}$$
$$+qY_{6+r}Y_{7-r}+Z_{6+r}Z_{7-r}=\frac{f_{21-r(r-1)/2}^2 f_2^2}{f_{42-r(r-1)}f_{6+r}f_{7-r}f_1}. \tag{22}$$

Proof. Using (12)–(17) and Lemma 2, one may rewrite (22) in the form

$$\begin{aligned}
&q^{15}f\left(-q^{6+r},-q^{72+12r}\right)f\left(-q^{7-r},-q^{84-12r}\right)\\
&+q^{10}f\left(-q^{12+2r},-q^{66+11r}\right)f\left(-q^{14-2r},-q^{77-11r}\right)\\
&+q^{6}f\left(-q^{18+3r},-q^{60+10r}\right)f\left(-q^{21-3r},-q^{70-10r}\right)\\
&+q^{3}f\left(-q^{24+4r},-q^{54+9r}\right)f\left(-q^{28-4r},-q^{63-9r}\right)\\
&+qf\left(-q^{30+5r},-q^{48+8r}\right)f\left(-q^{35-5r},-q^{56-8r}\right)\\
&+f\left(-q^{36+6r},-q^{42+7r}\right)f\left(-q^{42-6r},-q^{49-7r}\right)\\
=&\ \varphi(-q^{21-r(r-1)/2})\psi(q).
\end{aligned}\tag{23}$$

Suppose that $1\le r\le 6$. Then, by using (6), (9) and (10), we have

$$\begin{aligned}
&2\varphi(-q^{21-r(r-1)/2})\psi(q)\\
&=f\left(-q^{21-r(r-1)/2},-q^{21-r(r-1)/2}\right)f(1,q)\\
&=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{(21-r(r-1)/2)m^2+(n^2+n)/2}.
\end{aligned}\tag{24}$$

In this representation, we make the change of indices by setting

$$(7-r)m+n=13M+a\qquad\text{and}\qquad -(6+r)m+n=13N+b$$

where a and b have values selected from the set

$$\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\}$$

Then

$$m=M-N+\frac{a-b}{13}$$

and

$$n=(6+r)M+(7-r)N+\frac{(6+r)a+(7-r)b}{13}.$$

It follows easily that $a=b$, and so $m=M-N$ and $n=(6+r)M+(7-r)N+a$, where $-6\le a\le 6$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-6\le a\le 6$. From (24), we find that

$$\begin{aligned}
&2\varphi(-q^{21-r(r-1)/2})\psi(q)\\
&=\sum_{a=-6}^{6}q^{(a^2+a)/2}\sum_{M=-\infty}^{\infty}(-1)^M q^{(6+r)\left(13M^2+(2a+1)M\right)/2}\\
&\quad\times\sum_{N=-\infty}^{\infty}(-1)^N q^{(7-r)\left(13N^2+(2a+1)N\right)/2}\\
&=\sum_{a=-6}^{6}q^{(a^2+a)/2}f\left(-q^{(6+r)(7+a)},-q^{(6+r)(6-a)}\right)\\
&\quad\times f\left(-q^{(7-r)(7+a)},-q^{(7-r)(6-a)}\right)\\
&=q^{15}f\left(-q^{6+r},-q^{72+12r}\right)f\left(-q^{7-r},-q^{84-12r}\right)\\
&\quad+q^{10}f\left(-q^{12+2r},-q^{66+11r}\right)f\left(-q^{14-2r},-q^{77-11r}\right)\\
&\quad+q^{6}f\left(-q^{18+3r},-q^{60+10r}\right)f\left(-q^{21-3r},-q^{70-10r}\right)\\
&\quad+q^{3}f\left(-q^{24+4r},-q^{54+9r}\right)f\left(-q^{28-4r},-q^{63-9r}\right)\\
&\quad+qf\left(-q^{30+5r},-q^{48+8r}\right)f\left(-q^{35-5r},-q^{56-8r}\right)\\
&\quad+f\left(-q^{36+6r},-q^{42+7r}\right)f\left(-q^{42-6r},-q^{49-7r}\right)\\
&\quad+f\left(-q^{42+7r},-q^{36+6r}\right)f\left(-q^{49-7r},-q^{42-6r}\right)\\
&\quad+qf\left(-q^{48+8r},-q^{30+5r}\right)f\left(-q^{56-8r},-q^{35-5r}\right)
\end{aligned}$$

$$+q^3 f\left(-q^{54+9r},-q^{24+4r}\right) f\left(-q^{63-9r},-q^{28-4r}\right)$$
$$+q^6 f\left(-q^{60+10r},-q^{18+3r}\right) f\left(-q^{70-10r},-q^{21-3r}\right)$$
$$+q^{10} f\left(-q^{66+11r},-q^{12+2r}\right) f\left(-q^{77-11r},-q^{14-2r}\right)$$
$$+q^{15} f\left(-q^{72+12r},-q^{6+r}\right) f\left(-q^{84-12r},-q^{7-r}\right)$$
$$+q^{21} f\left(-q^{13r+78},-1\right) f\left(-q^{-13r+91},-1\right),$$

which is equivalent to (23) as the last term equal to zero by (7).

Theorem 2.*If $1 \le r \le 6$, then the following modular relation holds true:*

$$q^{30}U_{27-2r}U_{2r-1}+q^{20}V_{27-2r}V_{2r-1}+q^{12}W_{27-2r}W_{2r-1}$$
$$+q^6X_{27-2r}X_{2r-1}+q^2Y_{27-2r}Y_{2r-1}+Z_{27-2r}Z_{2r-1}$$
$$=\frac{1}{f_{27-2r}f_{2r-1}}\left(\frac{f^5_{2(27-2r)(2r-1)}f_4^2}{f^2_{(27-2r)(2r-1)}f^2_{4(27-2r)(2r-1)}f_2}\right.$$
$$\left.-q^{42-(r-7)^2}\frac{f_2^5 f^2_{4(27-2r)(2r-1)}}{f_1^2 f_4^2 f_{2(27-2r)(2r-1)}}\right). \tag{25}$$

*Proof.*Using (12)–(17) and Lemma 2, we see that (25) is equivalent to

$$q^{30} f\left(-q^{27-2r},-q^{324-24r}\right) f\left(-q^{2r-1},-q^{24r-12}\right)$$
$$+q^{20} f\left(-q^{54-4r},-q^{297-22r}\right) f\left(-q^{4r-2},-q^{22r-11}\right)$$
$$+q^{12} f\left(-q^{81-6r},-q^{270-20r}\right) f\left(-q^{6r-3},-q^{20r-10}\right)$$
$$+q^{6} f\left(-q^{108-8r},-q^{243-18r}\right) f\left(-q^{8r-4},-q^{18r-9}\right)$$
$$+q^{2} f\left(-q^{135-10r},-q^{216-16r}\right) f\left(-q^{10r-5},-q^{16r-8}\right)$$
$$+ f\left(-q^{162-12r},-q^{189-14r}\right) f\left(-q^{12r-6},-q^{14r-7}\right)$$
$$=\varphi\left(q^{(2r-1)(27-2r)}\right)\psi\left(q^2\right)$$
$$-q^{42-(r-7)^2}\varphi(q)\,\psi\left(q^{2(2r-1)(27-2r)}\right). \tag{26}$$

Now changing q to q^4 in (26), and then applying Lemma 1 in the resulting identity, we may rewrite (26) in the form

$$\frac{1}{4q}\left\{\varphi\left(-q^{(2r-1)(27-2r)}\right)\varphi(q)-\varphi\left(q^{(2r-1)(27-2r)}\right)\varphi(-q)\right\}$$
$$=q^{120} f\left(-q^{108-8r},-q^{1296-96r}\right) f\left(-q^{8r-4},-q^{96r-48}\right)$$
$$+q^{80} f\left(-q^{216-16r},-q^{1188-88r}\right) f\left(-q^{16r-8},-q^{88r-44}\right)$$
$$+q^{48} f\left(-q^{324-24r},-q^{1080-80r}\right) f\left(-q^{24r-12},-q^{80r-40}\right)$$
$$+q^{24} f\left(-q^{432-32r},-q^{972-72r}\right) f\left(-q^{32r-16},-q^{72r-36}\right)$$
$$+q^{8} f\left(-q^{540-40r},-q^{864-64r}\right) f\left(-q^{40r-20},-q^{64r-32}\right)$$
$$+ f\left(-q^{648-48r},-q^{756-56r}\right) f\left(-q^{48r-24},-q^{56r-28}\right). \tag{27}$$

Thus we need only to establish (27). We have

$$\varphi\left(-q^{(2r-1)(27-2r)}\right)\varphi(q)$$
$$=f\left(-q^{(2r-1)(27-2r)},-q^{(2r-1)(27-2r)}\right)f(q,q)$$
$$=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{(2r-1)(27-2r)m^2+n^2}. \tag{28}$$

In the above representation, we make the following change of indices:

$$(2r-1)m+n=26M+a \quad\text{and}\quad -(27-2r)m+n=26N+b,$$

where a and b have values selected from the set

$$\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm7,\pm8,\pm9,\pm10,\pm11,\pm12,13\}.$$

Then

$$m=M-N+\frac{a-b}{26}$$

and

$$n=(27-2r)M+(2r-1)N+\frac{(27-2r)a+(2r-1)b}{26}.$$

It follows easily that $a=b$, and so $m=M-N$ and $n=(27-2r)M+(2r-1)N+a$, where $-12\le a\le 13$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-12\le a\le 13$. From (28), we find that

$$\varphi\left(-q^{(2r-1)(27-2r)}\right)\varphi(q)=\sum_{a=-12}^{13}q^{a^2}$$
$$\times\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{(27-2r)\left(26M^2+2aM\right)+(2r-1)\left(26N^2+2aN\right)}$$
$$=\sum_{a=-12}^{13}q^{a^2} f\left(-q^{(27-2r)(26+2a)},-q^{(27-2r)(26-2a)}\right)$$
$$\times f\left(-q^{(2r-1)(26+2a)},-q^{(2r-1)(26-2a)}\right).$$

Changing q to $-q$ in the above identity and then subtracting the resulting identity from the above identity and after some simplifications, we obtain (27).

Theorem 3.*We have*

$$-q^{45}U_{40}V_3+q^{28}V_{40}X_3-q^{15}W_{40}Z_3+q^6X_{40}Y_3-qY_{40}W_3$$
$$+Z_{40}U_3=\frac{f_1f_4f_{30}f_{120}}{f_2f_3f_{40}f_{60}}, \tag{29}$$
$$q^{28}U_{24}V_7-q^{15}V_{24}X_7+q^6W_{24}Z_7-qX_{24}Y_7+Y_{24}W_7$$
$$-q^3Z_{24}U_7=\frac{f_1f_4f_{42}f_{168}}{f_2f_7f_{24}f_{84}}, \tag{30}$$

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$$q^6V_8X_{11}-q^{15}U_8V_{11}-qW_8Z_{11}+X_8Y_{11}-q^3Y_8W_{11}+q^{10}Z_8U_{11}=\frac{f_1f_4f_{22}f_{88}}{f_2f_8f_{11}f_{44}}, \quad (31)$$

$$-q^3W_{16}Z_9+X_{16}Y_9-qY_{16}W_9+q^6Z_{16}U_9+q^{10}V_{16}X_9-q^{21}U_{16}V_9=\frac{f_1f_4f_{36}f_{144}}{f_2f_9f_{16}f_{72}}, \quad (32)$$

$$-q^{21}V_{32}X_5+q^{10}W_{32}Z_5-q^3X_{32}Y_5+Y_{32}W_5-qZ_{32}U_5+q^{36}U_{32}V_5=\frac{f_1f_4f_{40}f_{160}}{f_2f_5f_{32}f_{80}}, \quad (33)$$

$$-q^{55}U_{48}V_1+q^{36}V_{48}X_1-q^{21}W_{48}Z_1+q^{10}X_{48}Y_1-q^3Y_{48}W_1+Z_{48}U_1=\frac{f_4f_{12}}{f_2f_{24}}. \quad (34)$$

*Proof.*Using (10), We have

$$\psi(-q^{30})\psi(-q)=f(-q^{30},-q^{90})f(-q,-q^3)=\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{30m(2m-1)+n(2n+1)}. \quad (35)$$

In the above representation, we make the following change of indices:

$$3m+n=13M+a \quad \text{and} \quad -10m+n=13N+b,$$

where a and b have values selected from the set $\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\}$. Then

$$m=M-N+\tfrac{a-b}{13} \quad \text{and} \quad n=10M+3N+\tfrac{10a+3b}{13}.$$

It follows easily that $a=b$, and so $m=M-N$ and $n=10M+3N+a$, where $-6\le a\le 6$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-6\le a\le 6$. From (35), we find that

$$\begin{aligned}\psi(-q^{30})\psi(-q)&=\sum_{a=-6}^{6}(-1)^a q^{2a^2+a}\times\sum_{M,N=-\infty}^{\infty}(-1)^M q^{260M^2+20(-1+2a)M+78N^2+3(11+4a)N}\\&=\sum_{a=-6}^{6}(-1)^a q^{2a^2+a}f\left(-q^{240+40a},-q^{280-40a}\right)\times f\left(q^{111+12a},q^{45-12a}\right)\\&=q^{66}f\left(-1,-q^{520}\right)f\left(q^{39},q^{117}\right)-q^{45}f\left(-q^{40},-q^{480}\right)\\&\times f\left(q^{51},q^{105}\right)+q^{28}f\left(-q^{80},-q^{440}\right)f\left(q^{63},q^{93}\right)\\&-q^{15}f\left(-q^{120},-q^{400}\right)f\left(q^{75},q^{81}\right)+q^6f\left(-q^{160},-q^{360}\right)\\&\times f\left(q^{87},q^{69}\right)-qf\left(-q^{200},-q^{320}\right)f\left(q^{99},q^{57}\right)\\&+f\left(-q^{240},-q^{280}\right)f\left(q^{111},q^{45}\right)-q^3f\left(-q^{280},-q^{240}\right)\\&\times f\left(q^{123},q^{33}\right)+q^{10}f\left(-q^{320},-q^{200}\right)f\left(q^{135},q^{21}\right)\\&-q^{21}f\left(-q^{360},-q^{160}\right)f\left(q^{147},q^9\right)+q^{36}f\left(-q^{400},-q^{120}\right)\\&\times f\left(q^{159},q^{-3}\right)-q^{55}f\left(-q^{440},-q^{80}\right)f\left(q^{171},q^{-15}\right)\\&+q^{78}f\left(-q^{480},-q^{40}\right)f\left(q^{183},q^{-27}\right).\end{aligned} \quad (36)$$

Using (19) in (36) and then after some simplification, we obtain

$$\begin{aligned}&-q^{45}f\left(-q^{40},-q^{480}\right)f\left(-q^6,-q^{33}\right)+q^{28}f\left(-q^{80},-q^{440}\right)\\&\times f\left(-q^{12},-q^{27}\right)-q^{15}f\left(-q^{120},-q^{400}\right)f\left(-q^{18},-q^{21}\right)\\&+q^6f\left(-q^{160},-q^{360}\right)f\left(-q^{15},-q^{24}\right)-qf\left(-q^{200},-q^{320}\right)\\&\times f\left(-q^9,-q^{30}\right)+f\left(-q^{240},-q^{280}\right)f\left(-q^3,-q^{36}\right)\\&=\psi\left(-q^{30}\right)\psi(-q).\end{aligned} \quad (37)$$

Now if we employ (12)–(17) and Lemma 2 in the above identity, we obtain (29). The proofs of (30)–(34) follow in a similar way.

The proof of our modular relations in Theorem 6 is strongly depends upon the results of Rogers [19] and Bressoud [12]. We adopt Bressoud's notation, except that we use $q^{\frac{n}{24}}f(-q^n)$ instead of P_n, and the variable q instead of x. Let $g_\alpha^{(p,n)}$ and $\Phi_{\alpha,\beta,m,p}$ be defined as follows:

$$g_\alpha^{(p,n)}:=g_\alpha^{(p,n)}(q)=q^{\alpha\left(\frac{12n^2-12n+3-p}{24p}\right)}\times\prod_{r=0}^{\infty}\frac{\left(1-(q^\alpha)^{pr+\frac{p-2n+1}{2}}\right)\left(1-(q^\alpha)^{pr+\frac{p+2n-1}{2}}\right)}{\prod_{k=1}^{p-1}\left(1-(q^\alpha)^{pr+k}\right)}, \quad (38)$$

for any positive odd integer p, integer n, and natural number α, and

$$\Phi_{\alpha,\beta,m,p}:=\Phi_{\alpha,\beta,m,p}(q)=\sum_{n=1}^{p}\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{\frac12\left\{p\alpha\left(r+m\frac{2n-1}{2p}\right)^2+p\beta\left(s+\frac{2n-1}{2p}\right)^2\right\}}, \quad (39)$$

where α,β and p are natural numbers, and m is an odd positive integer. Then we can easily obtain the following propositions.

Proposition 1.*We have*

$$g_\alpha^{(13,6)}=q^{\frac{175}{156}\alpha}U_\alpha,\quad g_\alpha^{(13,5)}=q^{\frac{115}{156}\alpha}V_\alpha,\quad g_\alpha^{(13,4)}=q^{\frac{67}{156}\alpha}W_\alpha,$$
$$g_\alpha^{(13,3)}=q^{\frac{31}{156}\alpha}X_\alpha,\quad g_\alpha^{(13,2)}=q^{\frac{7}{156}\alpha}Y_\alpha,\quad g_\alpha^{(13,1)}=q^{-\frac{5}{156}\alpha}Z_\alpha.$$

Theorem 4.*[12, Proposition 5.4]. For odd $p>1$,*

$$\Phi_{\alpha,\beta,m,p}=2q^{\frac{\alpha+\beta}{24}}f(-q^\alpha)f(-q^\beta)\left(\sum_{n=1}^{(p-1)/2}g_\beta^{(p,n)}g_\alpha^{(p,(2mn-m+1)/2)}\right).$$

Using Theorem 4, Proposition 1 and Lemma 3, we obtain the following proposition:

Proposition 2. *We have*

$$\begin{aligned}\Phi_{\alpha,\beta,3,13} =&2q^{\frac{\alpha+\beta}{24}} f_\alpha f_\beta \Bigg(q^{\frac{7\alpha-5\beta}{156}} Y_\alpha Z_\beta + q^{\frac{115\alpha+7\beta}{156}} V_\alpha Y_\beta \\ &- q^{\frac{175\alpha+31\beta}{156}} U_\alpha X_\beta - q^{\frac{31\alpha+67\beta}{156}} X_\alpha W_\beta \\ &- q^{\frac{-5\alpha+115\beta}{156}} Z_\alpha V_\beta + q^{\frac{67\alpha+175\beta}{156}} W_\alpha U_\beta \Bigg),\\ \Phi_{\alpha,\beta,7,13} =&2q^{\frac{\alpha+\beta}{24}} f_\alpha f_\beta \Bigg(q^{\frac{67\alpha-5\beta}{156}} W_\alpha Z_\beta - q^{\frac{31\alpha+7\beta}{156}} X_\alpha Y_\beta \\ &- q^{\frac{115\alpha+31\beta}{156}} V_\alpha X_\beta + q^{\frac{7\alpha+67\beta}{156}} Y_\alpha W_\beta \\ &+ q^{\frac{175\alpha+115\beta}{156}} U_\alpha V_\beta - q^{\frac{-5\alpha+175\beta}{156}} Z_\alpha U_\beta \Bigg),\\ \Phi_{\alpha,\beta,9,13} =&2q^{\frac{\alpha+\beta}{24}} f_\alpha f_\beta \Bigg(q^{\frac{115\alpha-5\beta}{156}} V_\alpha Z_\beta - q^{\frac{-5\alpha+7\beta}{156}} Z_\alpha Y_\beta \\ &+ q^{\frac{67\alpha+31\beta}{156}} W_\alpha X_\beta + q^{\frac{175\alpha+67\beta}{156}} U_\alpha W_\beta \\ &- q^{\frac{7\alpha+115\beta}{156}} Y_\alpha V_\beta + q^{\frac{31\alpha+175\beta}{156}} X_\alpha U_\beta \Bigg),\\ \Phi_{\alpha,\beta,11,13} =&2q^{\frac{\alpha+\beta}{24}} f_\alpha f_\beta \Bigg(q^{\frac{175\alpha-5\beta}{156}} U_\alpha Z_\beta - q^{\frac{67\alpha+7\beta}{156}} W_\alpha Y_\beta \\ &+ q^{\frac{7\alpha+31\beta}{156}} Y_\alpha X_\beta - q^{\frac{-5\alpha+67\beta}{156}} Z_\alpha W_\beta \\ &+ q^{\frac{31\alpha+115\beta}{156}} X_\alpha V_\beta - q^{\frac{115\alpha+175\beta}{156}} V_\alpha U_\beta \Bigg).\end{aligned}$$

Corollary 1. *[12, Corollary 5.5 and 5.6]. If $\Phi_{\alpha,\beta,m,p}$ is defined by (39), then*

$$\Phi_{\alpha,\beta,m,1} =0, \tag{40}$$

$$\Phi_{\alpha,\beta,1,3} =2q^{\frac{\alpha+\beta}{24}} f(-q^\alpha)f(-q^\beta). \tag{41}$$

Theorem 5. *[12, Corollary 7.3]. Let α_i, β_i, m_i, p_i where $i=1,2$, be positive integers with m_1 and m_2 both odd. If $\lambda_1 := (\alpha_1 m_1^2+\beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2+\beta_2)/p_2$, and the conditions*

$$\lambda_1 = \lambda_2,$$
$$\alpha_1\beta_1 = \alpha_2\beta_2,$$
$$\alpha_1 m_1 \equiv \alpha_2 m_2 \ (mod \lambda_1) \quad or \quad \alpha_1 m_1 \equiv -\alpha_2 m_2 \ (mod \lambda_1)$$

hold, then

$$\Phi_{\alpha_1,\beta_1,m_1,p_1} = \Phi_{\alpha_2,\beta_2,m_2,p_2}.$$

Theorem 6. *We have*

$$qX_3V_1 - q^3V_3U_1 - Z_3W_1 + Y_3X_1 - qW_3Y_1 + q^3U_3Z_1 = 0, \tag{42}$$

$$q^6V_2U_5 - q^3X_2V_5 + qZ_2W_5 - Y_2X_5 + W_2Y_5 - qU_2Z_5 = 0, \tag{43}$$

$$U_1Z_9 - q^{10}V_1U_9 + q^6X_1V_9 - q^3Z_1W_9 + qY_1X_9 - W_1Y_9 = 0, \tag{44}$$

$$U_1Z_{35} - q^2W_1Y_{35} + q^7Y_1X_{35} - q^{15}Z_1W_{35} + q^{26}X_1V_{35} - q^{40}V_1U_{35} = 1, \tag{45}$$

$$V_1Z_{23} - qZ_1Y_{23} + q^5W_1X_{23} + q^{11}U_1W_{23} - q^{17}Y_1V_{23} + q^{26}X_1U_{23} = 1, \tag{46}$$

$$Y_5Z_7 + q^4V_5Y_7 - q^7U_5X_7 - q^4X_5W_7 - q^5Z_5V_7 + q^{10}W_5U_7 = 1. \tag{47}$$

Proof. The proof follows using Corollary 1 and Theorem 5 in such away that α_i, β_i, m_i and p_i $(i=1,2)$ are selected, respectively, as in the following table:

α_1	β_1	m_1	p_1	α_2	β_2	m_2	p_2
1	3	7	13	1	3	1	1
2	5	11	13	1	10	3	1
1	9	11	13	1	9	1	1
1	35	11	13	1	35	1	3
1	23	9	13	1	23	1	3
5	7	3	13	5	7	1	3

3 Applications to the theory of partitions

For simplicity, we define

$$(q^{r_1\pm}; q^s)_\infty := (q^{r_1}, q^{s-r_1}; q^s)_\infty$$

and

$$(q^{r_1\pm, r_2\pm, r_3\pm, \cdots, r_k\pm}; q^s) = (q^{r_1\pm}; q^s)_\infty (q^{r_2\pm}; q^s)_\infty \cdots (q^{r_k\pm}; q^s)_\infty$$

where r_i, $1 \le i \le k$ and s are positive integers and $r_i < s$. In this section, we present a partition theoretic interpretations of (42) and (43). First, we need the notation of colored partitions.

Definition 1. *A positive integer n has l colors if there are l copies of n available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called "colored partitions".*

For example, if 1 is allowed to have two colors, say b (black), and g (green), then all the colored partitions of 4 are
4, $3+1_b$, $3+1_g$, $2+2$, $2+1_g+1_g$, $2+1_b+1_b$, $2+1_g+1_b$, $1_g+1_g+1_g+1_g$, $1_b+1_b+1_b+1_b$, $1_g+1_b+1_b+1_b$, $1_g+1_g+1_b+1_b$, $1_g+1_g+1_g+1_b$.
An important fact is that

$$\frac{1}{(q^r;\ q^s)_\infty^l}$$

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is the generating function for the number of partitions of n, where all the parts are congruent to $r \pmod{s}$ and have l colors.

Theorem 7. *Let $P_1(n)$ denote the number of partitions of n into parts not congruent to $\pm 2, \pm 11, \pm 13 \pmod{39}$ with parts congruent to $\pm 3, \pm 6, \pm 9$ and $\pm 18 \pmod{39}$ having two colors. Let $P_2(n)$ denote the number of partitions of n into parts not congruent to $\pm 1, \pm 13, \pm 14 \pmod{39}$ with parts congruent to $\pm 3, \pm 9, \pm 15$ and $\pm 18 \pmod{39}$ having two colors. Let $P_3(n)$ denote the number of partitions of n into parts not congruent to $\pm 10, \pm 13, \pm 16 \pmod{39}$ with parts congruent to $\pm 6, \pm 9, \pm 12$ and $\pm 15 \pmod{39}$ having two colors. Let $P_4(n)$ denote the number of partitions of n into parts not congruent to $\pm 13, \pm 14, \pm 17 \pmod{39}$ with parts congruent to $\pm 3, \pm 6, \pm 12$ and $\pm 18 \pmod{39}$ having two colors. Let $P_5(n)$ denote the number of partitions of n into parts not congruent to $\pm 5, \pm 8, \pm 13 \pmod{39}$ with parts congruent to $\pm 3, \pm 6, \pm 12$ and $\pm 15 \pmod{39}$ having two colors. Let $P_6(n)$ denote the number of partitions of n into parts not congruent to $\pm 7, \pm 13, \pm 19 \pmod{39}$ with parts congruent to $\pm 9, \pm 12, \pm 15$ and $\pm 18 \pmod{39}$ having two colors. Then, for any positive integer $n \geq 3$, we have*

$$P_1(n-1) - P_2(n-3) - P_3(n) + P_4(n) - P_5(n-1) + P_6(n-3) = 0.$$

Proof. With the help of (12)–(17), (42) can be written as follows:

$$\begin{aligned}
&\frac{q}{(q^{1\pm,4\pm,5\pm,7\pm,8\pm,10\pm,12\pm,14\pm,15\pm,16\pm,17\pm,19\pm};q^{39})_\infty} \times \frac{1}{(q^{3\pm,6\pm,9\pm,18\pm};q^{39})_\infty^2} \\
&- \frac{q^3}{(q^{2\pm,4\pm,5\pm,6\pm,7\pm,8\pm,10\pm,11\pm,12\pm,16\pm,17\pm,19\pm};q^{39})_\infty} \times \frac{1}{(q^{3\pm,9\pm,15\pm,18\pm};q^{39})_\infty^2} \\
&- \frac{1}{(q^{1\pm,2\pm,3\pm,4\pm,5\pm,7\pm,8\pm,11\pm,14\pm,17\pm,18\pm,19\pm};q^{39})_\infty} \times \frac{1}{(q^{6\pm,9\pm,12\pm,15\pm};q^{39})_\infty^2} \\
&+ \frac{1}{(q^{1\pm,2\pm,4\pm,5\pm,7\pm,8\pm,9\pm,10\pm,11\pm,15\pm,16\pm,19\pm};q^{39})_\infty} \times \frac{1}{(q^{3\pm,6\pm,12\pm,18\pm};q^{39})_\infty^2} \\
&- \frac{q}{(q^{1\pm,2\pm,4\pm,7\pm,9\pm,10\pm,11\pm,14\pm,16\pm,17\pm,18\pm,19\pm};q^{39})_\infty} \times \frac{1}{(q^{3\pm,6\pm,12\pm,15\pm};q^{39})_\infty^2} \\
&+ \frac{q^3}{(q^{1\pm,2\pm,3\pm,4\pm,5\pm,6\pm,8\pm,10\pm,11\pm,14\pm,16\pm,17\pm};q^{39})_\infty} \times \frac{1}{(q^{9\pm,12\pm,15\pm,18\pm};q^{39})_\infty^2} \\
&= 0.
\end{aligned}$$

Note that the six quotients in the left side of the above identity represent the generating functions for $P_k(n)$, $1 \leq k \leq 6$ respectively. Hence, it is equivalent to

$$q\sum_{n=0}^{\infty} P_1(n)q^n - q^3\sum_{n=0}^{\infty} P_2(n)q^n - \sum_{n=0}^{\infty} P_3(n)q^n + \sum_{n=0}^{\infty} P_4(n)q^n - q\sum_{n=0}^{\infty} P_5(n)q^n + q^3\sum_{n=0}^{\infty} P_6(n)q^n = 0,$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = P_5(0) = P_6(0) = 1$. Equating the coefficients of q^n $(n \geq 3)$ on both sides yields the desired result.

Example 1. The following table illustrates the case $n = 7$ in the Theorem 7.

$P_1(6) = 10$	$6_g, 6_r, 5+1, 4+1+1, 3_g+3_g, 3_g+3_r, 3_r+3_r, 3_g+1+1+1, 3_r+1+1+1, 1+1+1+1+1+1$
$P_2(4) = 2$	$4, 2+2$
$P_3(7) = 16$	$7, 6_g+1, 6_r+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$
$P_4(7) = 18$	$7, 6_g+1, 6_r+1, 5+2, 5+1+1, 3_g+3_g+1, 3_g+3_r+1, 3_r+3_r+1, 3_r+2+2, 3_g+2+2, 3_r+2+1+1, 3_g+2+1+1, 3_g+1+1+1+1, 3_r+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$
$P_5(6) = 15$	$6_g, 6_r, 4+1+1, 4+2, 3_g+3_r, 3_g+3_g, 3_r+3_r, 3_g+2+1, 3_r+2+1, 3_g+1+1+1, 3_r+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1$
$P_6(4) = 5$	$4, 3+1, 2+2, 2+1+1, 1+1+1+1$

Theorem 8. *Let $P_1(n)$ denote the number of partitions of n, where each part is a multiple of 2 or 5 and not congruent to $\pm 4, \pm 5, \pm 22, \pm 26, \pm 48, \pm 52, \pm 56, 65 \pmod{130}$ with parts congruent to $\pm 10, \pm 20, \pm 40, \pm 50 \pmod{130}$ having two colors. Let $P_2(n)$ denote the number of partitions of n, where each part is a multiple of 2 or 5 and not congruent to $\pm 8, \pm 18, \pm 26, \pm 34, \pm 44, \pm 52, \pm 55, 65 \pmod{130}$ with parts congruent to $\pm 20, \pm 30, \pm 40, \pm 50 \pmod{130}$ having two colors. Let $P_3(n)$ denote the number of partitions of n, where each part is a multiple of 2 or 5 and not congruent to $\pm 12, \pm 14, \pm 15, \pm 26, \pm 38, \pm 52, \pm 64, 65 \pmod{130}$ with parts congruent to $\pm 10, \pm 20, \pm 30, \pm 60 \pmod{130}$ having two colors. Let $P_4(n)$ denote the number of partitions of n, where each part is a multiple of 2 or 5 and not congruent*

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to $\pm16, \pm26, \pm36, \pm42, \pm45, \pm52, \pm62, 65 \pmod{130}$ *with parts congruent to* $\pm30, \pm40, \pm50, \pm60 \pmod{130}$ *having two colors. Let* $P_5(n)$ *denote the number of partitions of n, where each part is a multiple of* 2 *or* 5 *and not congruent to* $\pm6, \pm25, \pm26, \pm28, \pm32, \pm46, \pm52, \pm58, 65 \pmod{130}$ *with parts congruent to* $\pm10, \pm30, \pm50, \pm60 \pmod{130}$ *having two colors. Let* $P_6(n)$ *denote the number of partitions of n, where each part is a multiple of* 2 *or* 5 *and not congruent to* $\pm2, \pm24, \pm26, \pm28, \pm52, \pm54, \pm55, 65 \pmod{130}$ *with parts congruent to* $\pm10, \pm20, \pm40, \pm60 \pmod{130}$ *having two colors. Then, for any positive integer* $n \geq 6$, *we have*

$$P_1(n-6) - P_2(n-3) + P_3(n-1) - P_4(n) + P_5(n) - P_6(n-1) = 0.$$

*Proof.*Using (12)–(17) in (43) and simplifying the resulting identity, we obtain

$$\frac{q^6}{(q^{2\pm,6\pm,8\pm,12\pm,14\pm,15\pm,16\pm,18\pm,24\pm,25\pm,28\pm,30\pm,32\pm};q^{130})_\infty} \times \frac{1}{(q^{34\pm,35\pm,36\pm,38\pm,42\pm,44\pm,45\pm,46\pm,54\pm};q^{130})_\infty} \times \frac{1}{(q^{55\pm,58\pm,60\pm,62\pm,64\pm};q^{130})_\infty (q^{10\pm,20\pm,40\pm,50\pm};q^{130})^2_\infty}$$
$$-\frac{q^3}{(q^{2\pm,4\pm,5\pm,6\pm,10\pm,12\pm,14\pm,15\pm,16\pm,22\pm,24\pm,25\pm,28\pm};q^{130})_\infty} \times \frac{1}{(q^{32\pm,35\pm,36\pm,38\pm,42\pm,45\pm,46\pm,48\pm,54\pm};q^{130})_\infty} \times \frac{1}{(q^{56\pm,58\pm,60\pm,62\pm,64\pm};q^{130})_\infty (q^{20\pm,30\pm,40\pm,50\pm};q^{130})^2_\infty}$$
$$+\frac{q}{(q^{2\pm,4\pm,5\pm,6\pm,8\pm,16\pm,18\pm,22\pm,24\pm,25\pm,28\pm,32\pm,34\pm};q^{130})_\infty} \times \frac{1}{(q^{35\pm,36\pm,40\pm,42\pm,44\pm,45\pm,46\pm,48\pm,50\pm};q^{130})_\infty} \times \frac{1}{(q^{54\pm,55\pm,56\pm,58\pm,62\pm};q^{130})_\infty (q^{10\pm,20\pm,30\pm,60\pm};q^{130})^2_\infty}$$
$$-\frac{1}{(q^{2\pm,4\pm,5\pm,6\pm,8\pm,10\pm,12\pm,14\pm,15\pm,18\pm,20\pm,22\pm,24\pm};q^{130})_\infty} \times \frac{1}{(q^{25\pm,28\pm,32\pm,34\pm,35\pm,38\pm,44\pm,46\pm,48\pm};q^{130})_\infty} \times \frac{1}{(q^{54\pm,55\pm,56\pm,58\pm,64\pm};q^{130})_\infty (q^{30\pm,40\pm,50\pm,60\pm};q^{130})^2_\infty}$$
$$+\frac{1}{(q^{2\pm,4\pm,5\pm,8\pm,12\pm,14\pm,15\pm,16\pm,18\pm,20\pm,22\pm,24\pm,34\pm};q^{130})_\infty} \times \frac{1}{(q^{35\pm,36\pm,36\pm,38\pm,40\pm,42\pm,44\pm,45\pm,48\pm};q^{130})_\infty} \times \frac{1}{(q^{54\pm,55\pm,56\pm,62\pm,64\pm};q^{130})_\infty (q^{10\pm,30\pm,50\pm,60\pm};q^{130})^2_\infty}$$
$$-\frac{q}{(q^{4\pm,5\pm,6\pm,8\pm,12\pm14\pm,15\pm,16\pm,18\pm,22\pm,25\pm,30\pm,32\pm};q^{130})_\infty} \times \frac{1}{(q^{34\pm,35\pm,36\pm,38\pm,42\pm,44\pm,45\pm,46\pm,48\pm};q^{130})_\infty} \times \frac{1}{(q^{50\pm,56\pm,58\pm,62\pm,64\pm};q^{130})_\infty (q^{10\pm,20\pm,40\pm,60\pm};q^{130})^2_\infty} = 0.$$

Note that the six quotients in the left side of the above identity represent the generating functions for $P_k(n)$ where $1 \leq k \leq 6$ respectively. Hence, it is equivalent to

$$q^6 \sum_{n=0}^{\infty} P_1(n)q^n - q^3 \sum_{n=0}^{\infty} P_2(n)q^n + q \sum_{n=0}^{\infty} P_3(n)q^n - \sum_{n=0}^{\infty} P_4(n)q^n + \sum_{n=0}^{\infty} P_5(n)q^n - q \sum_{n=0}^{\infty} P_6(n)q^n = 0,$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = P_5(0) = P_6(0) = 1$. Equating the coefficients of q^n $(n \geq 6)$ on both sides yields the desired result.

*Example 2.*The following table illustrates the case $n = 15$ in the Theorem 8.

$P_1(8) = 3$
$P_2(11) = 3$
$P_3(13) = 4$
$P_4(14) = 14$
$P_5(14) = 12$
$P_6(13) = 2$

4 Conclusions

In this paper, we have used the Watson's method and Bresssoud method to establish several modular relations for the Rogers-Ramanujan type functions of order thirteen which are analogues to Ramanujan's forty identities for Rogers- Ramanujan functions.Almost all of our modular relations yield theorems in the theory of partitions.There is a need to establish a systematic way to establish modular relations for Rogers-Ramanujan type functions of different orders.

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Nasser Abdo Saeed Bulkhali is an Assistant Professor at Department of Mathematics Faculty of Education-Zabid, Hodeidah University, Hodeidah, Yemen. He received his Ph.D. degree in pure Mathematics(Number Theory) from University of Mysore. His research interests are in the areas of Special Functions, Number Theory, Partition Theory and Discrete Mathematics. He has published research articles in reputed international journals.

Ranganatha D is a Junior Research Fellow (UGC-BSR) in the Department of Studies in Mathematics, University of Mysore, Mysuru. He is doing Research in Number theory under the guidance of Prof.Chandrashekar Adiga. His research interests are in Number Theory, Partition Theory, Discrete Mathematics and Graph Theory. He has published research articles in reputed international journals.

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