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Entanglement in Condensates involving Strong Interactions

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We look at two well known examples of interacting systems relating to condensed matter in which we put the strong interacting parameters. At high quark chemical potentials and low temperatures we study the entropy arising from the excitation in the BCS model of superconductivity and the Bose-Einstein condensation (BEC) of colored quark pairs. We compare it with the ground state entropy for a system consisting of two colored quarks. In the BCS model we found that the entropy strongly depends on the energy gap. Both for the very small values of the momenta as well as those much greater than the characterizing Fermi momentum p_f, the ground state entropy is dominant. For the BEC case we suggest a phenomenological model to build up colored bosonic quark pairs. Here the entropy entirely depends upon the short ranged repulsive interactions between the quark pairs and vanishes for large momenta.

Keywords (BACS): Phenomenological quark models (12.39.-x), Entanglement and decoherence in Bose-Einstein condensates (03.75.Gg), Degenerate Fermi gases (03.75.Ss)

1 Introduction

At high quark chemical potentials and low temperatures the hadronic matter has been conjectured to dissolve into degenerate fermionic quarks. The matter of very cold dense quarks might exist in the interior of compact stellar objects. Due to the difficulties of performing lattice simulations with high chemical potentials, it is still not possible to simulate the physics of these phases by using the lattice QCD. Non-perturbative analysis at finite baryon density has been recently carried out on the lattice by using the Nambu-Jona-Lasinio model [1]. The degenerate quarks near to the Fermi surfaces are generally expected to interact according to quantum chromodynamics so that they can

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build up Cooper pairs. There are various mechanisms by which these quark pairs appear to be condensed, which depend on their total momenta. For example the Bardeen-Cooper-Schrieffer theory of superconductivity, which is often referred to as just BCS [2, 3], can be applied to the quarks in pairs with identical opposite momenta. Similarly, an extension of which is often called LOFF [4, 5] is applied to the pairs of quarks with different momenta. Far away from the Fermi surfaces the quarks are blocked according to the Pauli principle. Therefore, they behave like free particles. In the present work we are interested only in the physics near to the Fermi surfaces where the quarks are attractively interacting. If we take into account the colors as the effective degrees of freedom while we keep all the other quantum numbers relating to simple symmetrical forms, the attractive interaction clearly leads to a breaking of the color gauge symmetry and therefore, the Cooper pairs get color superconducting. This structure are analogous to $SU(2)_c$ baryons which are symmetric bosons of two colors. In the nuclear matter BCS pairings and BEC have been studied some years ago [6, 7].

In this work we shall apply our previous calculations for the ground state entropy [8–10] on these states of quarks under such extreme conditions. We shall compare the entropy arising from the excitations with that characterizing the ground state of colored quark pairs. Here obviously we are dealing with a large number of quark pairs. The other difference between this work and [8–10] places in the nature of couplings between the quarks. With the quantum entropy we mean the entropy arising from quantum fluctuations. The latter, which differs from the thermal fluctuations, can also exists at zero temperature. Hereafter we refer to quantum entropy as ground state entropy and *vice versa* whenever unambiguous. For the BCS condensate we can directly apply the models given in [8, 9]. But we simply consider the differences between the nature of quark-pairs and their interactions at the Fermi level in BCS and the structure and mixing of quarks in the colorless confined hadronic singlet and octet states. For mixed states consisting of two colored quarks, the quantum entropy is expected to be T independent and equal $\ln 4$ [8–10].

Furthermore, since we have bosonic states consisting of quark pairs we can study a particular case of Bose-Einstein condensation. This condensate entirely does not depend upon the Fermi level. However, it represents a stimulating tool for the understanding the extraneous condensates like BCS and LOFF, for instance. Moreover, it is a rich phenomenological example since we know the physics of BEC much better relative to the physics of the condensates at such high quark chemical potential and very low temperature. Here we shall not discuss the possible transition from color superconductivity of BCS at low density to BEC of Cooper pairs at high density. In these two phases we individually calculate the entropy arising from the excitation for different momenta and compare it with that for the ground state. The transition from BCS to BEC has been discussed in many works [6, 7, 11].

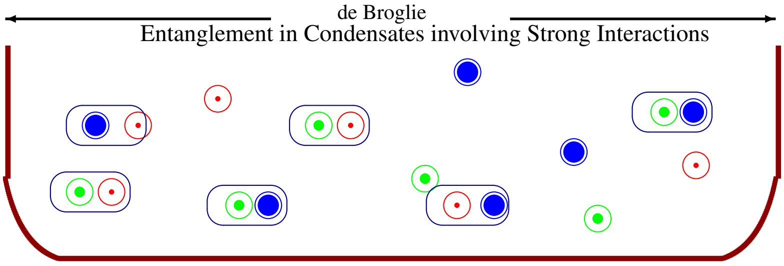

Figure 2.1: Configuration of our phenomenological model for the Bose-Einstein condensation of coupling two colors as quark-pairs. Within the de Broglie wavelength the enclosed system is colorless. The colored q q pairs are coherently distributed. Different quark colors are represented by different solid circles and the pairs by ovals.

In the present work we shall use the usual statistical properties for the interacting many-particle Fermi and Bose gases with very short-ranged interactions [12]. In this case one can utilize the second quantized formalism to rewrite down the effects of the interaction by means of the canonical Bogoliubov transformation as a new gas with the same statistical properties but with correspondingly modified energy spectrum.

This article is organized as follows: The next section develops the model for $SU(2)_c$ of symmetric bosons of two colors. In the two following sections we present, respectively, the formulation for the Bose-Einstein condensation and the BCS model of superconductivity. The next section is devoted to the presentation of the results followed by the discussion. Finally in the last section we state the conclusions.

2 Model for $SU(2)_c$

The $SU(2)$ color structure has been investigated for finite baryon number in a very special model involving the group characters [13]. The thermodynamics of $SU(2)$ gauge theory with staggered fermions has been studied at finite baryon density and zero temperature in the strong coupling limit [14]. A special property of the group structure of $SU(2)$ is that the action remains real, meanwhile the basic group structure for $SU(3)$ is complex. Analogous to $SU(2)_c$ symmetric colored bosons we suggest a phenomenological model for the Bose-Einstein condensation in a system consisting of tightly bound quark-pairs embedded within non-degenerate fermions. We take the scale of the order of the de Broglie wavelength $\lambda^2 = h^2/(2\pi m T)$. After having been accepted that the degenerate quarks build up subsystems of atom-like pairs, we assume that the pairs are coherently distributed. Between each of the two pairs there is a short-ranged repulsive interaction characterized by the interaction strength U_0 which is usually given in units of energy volume. Each quark is assumed to be coupled with its counterpart by a strongly attractive force mediated by a kind of soft gluonic matter that in nature might differ from the usual epoxy matter [15] which supposed to hold the quarks in the confined hadronic states. In the ground state of the system an arbitrary number of identical quark pairs is allowed to occupy the same state. This homogeneous system in turn can be considered as a large number of individual and inseparable but still distinguishable subsystems, so that for binary interacting systems the canonical measure would be the entropy arising from the interaction. In other words for this bosonic subsystems we can estimate the entropy from the reduced density matrix as done in [8, 9, 16]. This configuration might be illustrated as in Fig. 2.1. For the case where on this condensate an external potential is applied, the system becomes inhomogeneous. Nevertheless, we are still able to consider it as consisting of individual, inseparable and distinguishable subsystems. For this model we suppose that all quark pairs are located within the de Broglie scale and the quarks exhibit coherent. Therefore, each of them can be considered to be correlated with all others. And the correlations which are included into the quantum fluctuations are short as well as large ranged, so that they do not depend on the distance. For the simplicity we can only consider light quarks with binary correlations. Clearly the inclusion of many-body interactions between the quark pairs leads to modification of both ground and excited states and therefore their thermodynamics. In order to deal with this problem, we can apply the Hartree-Fock approximations in the second quantization formalism. The first is very familiar in the atomic physics, meanwhile the second is basically utilizing the many-body quantum field theory. The quark pairs are coupled with each other but their couplings in nature differ from those in the confined mesonic states. Nevertheless, their constituents are also supposed to be asymptotically free. Now we apply the Bose-Einstein statistics over these coupled states of two colors and symmetric flavor.

3 Formulation for Entanglement in Bose-Einstein condensation

We look at the problem of entanglement in an interacting Bose gas consisting of pairs of light quarks by using the second quantization formalism [17]. The quarks effective degrees of freedom considered in this work are just the colors. The flavors and other quantum numbers are kept identical for both light quarks. For creation and annihilation operators, $\hat{\Psi}^\dagger$, $\hat{\Psi}$ and effective interaction, U_0 the Hamiltonian for interacting Bose gas can be written as

$$H = \int d\mathbf{r}\, \hat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m}\nabla^2\hat{\Psi} + V\hat{\Psi} + \frac{U_0}{2}\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\right] \tag{3.1}$$

$$= \sum_p \epsilon_p^0 a_p^\dagger a_p + \frac{U_0}{2V}\sum_{p_1,p_2,q} a^\dagger_{p_1+q} a^\dagger_{p_2-q}\, a_{p_2} a_{p_1} \tag{3.2}$$

$\hat{\Psi} = \Psi + \delta\hat{\Psi}$ gives the quantum fluctuations whereas Ψ is the single particle wavefunction. The first term in Eq. 3.1 represents the non-interacting part of the Hamiltonian. In Eq. 3.2 the Hamiltonian of the degenerate and nearly ideal Bose gas the operators are given in momentum space. Simultaneously, we have inserted two bosonic creation and annihi-

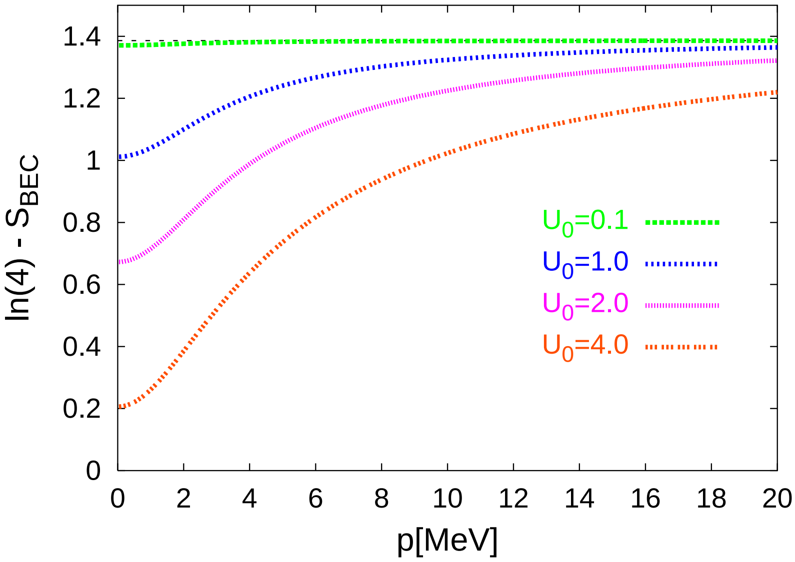

Figure 3.2: The difference between the ground state entropy, $\ln 4$, and the entropy S_{BEC}^{ent} of the excitation in BEC is given as a function of momentum p and for different values of the effective interaction, U_0. For $p \to 0$, S_{BEC}^{ent} reaches its maximum value and it decreases with increasing p. For large p the ground state entropy is dominant. For small U_0, S_{BEC}^{ent} vanishes and the entropy is just given by the ground state value $\ln 4$.

lation operators, $a_p^\dagger$ and a_p, respectively for the usual spatially dependent operators in the previous equation. U_0 is the interaction strength from the collision of two particles with momenta, p_1 and p_2, which produce two new particles with momenta, $p_1 + q$ and $p_2 - q$, respectively. Here we have assumed that the interaction does not depend on the momentum change q [18]. With the Bogoliubov canonical transformations we get a Hamiltonian of non-interacting system. The latter is solvable and therefore we can estimate its eigenvalues. The second quantized operators in p direction, $a, a^\dagger$ and in p direction, $b, b^\dagger$ are commutative, so that we can replace a with b in H, which remains invariant. In doing the Bogoliubov transformations, we introduce two variables, α and β, where

$$\alpha = u(p)a + v(p)b^\dagger \quad (3.3)$$

$$\beta = u(p)b + v(p)a^\dagger \quad (3.4)$$

and the p-dependent variables $u(p)$ and $v(p)$ are just the coefficients, which will be defined in Eq. 3.8, 3.9. α and β, and their counterparts in $-p$ direction are the operators, which create and destroy the elementary excitations (quasi-particles). We choose the phases of real $u(p)$ and $v(p)$ so that the interacting parts in H are entirely removed. In doing this, we diagonalize H and therefore we can set $a_p = u(p)\alpha - v(p)\alpha_{-p}^\dagger$ and

$a_{-p} = u(p)\alpha_{-p} - v(p)\alpha_p^\dagger$.

$$H = \frac{N^2 U_0}{2V} + \sum_{p\neq 0} \epsilon_p^0 \alpha_p^\dagger \alpha_p - \frac{1}{2}\sum_{p\neq 0}\left(\epsilon_p^0 - n_0 U_0 - \epsilon(p)\right) \tag{3.5}$$

ϵ_p^0 is the single-particle energy and $\epsilon(p)$ will be given in Eq. 3.11. As given in [17] the ground state in such entangled collection of decoupled Boson pairs in p and $-p$ directions is given as

$$|\psi_0 > \;\; = \;\; g \prod_{p\neq 0} \frac{1}{u(p)} \sum_{i=0}^{\infty} \left(-\frac{v(p)}{u(p)}\right)^i \; |n_{-p} = i; n_p = i > \tag{3.6}$$

and the entropy from the excitation of two bosons with equal and oppositely directed momenta p is given by

$$S_{BEC}^{ent} \;\; = \;\; g \sum_{p\neq 0} \left\{ \frac{\ln\left(\frac{u(p)}{v(p)}\right)^2}{\left(\frac{u(p)}{v(p)}\right)^2 - 1} \; - \; \ln\left(1 - \left(\frac{v(p)}{u(p)}\right)^2\right) \right\} \tag{3.7}$$

where g is the degeneracy factor and the two coefficients of the Bogoliubov canonical transformation are defined as

$$u(p) \;\; = \;\; \left[\frac{1}{2}\left(\frac{\zeta(p)}{\epsilon(p)} + 1\right)\right]^{1/2} \tag{3.8}$$

$$v(p) \;\; = \;\; \left[\frac{1}{2}\left(\frac{\zeta(p)}{\epsilon(p)} - 1\right)\right]^{1/2} \tag{3.9}$$

From Eq. 3.8, 3.9, we get $u(p)^2 - v(p)^2 = 1$ which assures that the energy over the space of the amplitudes for unoccupation u and occupation v is minimum.

Alternatively, we can obtain these results, if we directly diagonalize the Hartree-Fock Hamiltonian together with the self-consistency equation of the order parameter Δ. For the interacting Bose gas of quark-pairs with particle density $n = N_0/V_0 > 0$, we apply the following definitions:

$$\zeta(p) \;\; = \;\; \epsilon_0(p) + nU_0 \tag{3.10}$$

$$\epsilon(p) \;\; = \;\; \left(\epsilon_0(p)^2 + 2\epsilon_0(p) n U_0\right)^{1/2} \tag{3.11}$$

$\epsilon_0(p) = (p^2 + m^2)^{1/2}$ - as given above - is the energy of single-particle excitation and m is the reduced mass of the quark-pair. The validity of above definitions in Eq. 3.10, 3.11 is guaranteed by $nU_0 \geq \epsilon_0(p)$. After plugging into Eq. 3.7 we find that

$$S_{BEC}^{ent}(p) \;\; = \;\; g \sum_{p\neq 0} \left\{ \frac{\zeta(p) - \epsilon(q)}{2\,\epsilon(p)} \; \ln\left(\frac{\zeta(p)+\epsilon(p)}{\zeta(p)-\epsilon(p)}\right) \right.$$
$$\left. + \ln\left(\frac{\zeta(p)+\epsilon(p)}{2\,\epsilon(p)}\right) \right\} \tag{3.12}$$

the sum of the excitation entropy for each boson pair with the momentum p.

3.1 Ground state entropy of colored quark-pairs

In the limit of low temperature T and small distances R between the two static quarks, the free energy for the $q\,q$ system is presently being studied in $SU(3)$ pure gauge theory on the lattice [19]. By taking the colors N_c as the effective degrees of freedom, the quantum entropy in the ground state of a system consisting of two colored quarks is given as [8–10]

$$S_{qq} = \ln N_c^2 = \ln 4 \tag{3.13}$$

This value is clearly temperature independent. Moreover, we should mention that for $T = [0, m]$ it is all-dominant against the thermal entropy and as discussed in [8–10], for this reason it is used to be abstracted away for high temperatures. But if we look at a system of massive quarks [20, 21], the region of temperatures where the ground state entropy remains significant will be correspondingly large. The effects of the quark mass on the color superconductivity [22] and on the stability of hybrid stars [23] are recently reported. Thus we think that the ground state entropy is an essential physical observation in understanding the compact stellar objects and the coled quark matter with very high chemical potential.

To compare this constant value with the entropy arising from the excitation in the BEC condensate, we define

$$S_{BEC}(p) = S_{qq} - S^{ent}_{BEC}(p) = \ln 4 - S^{ent}_{BEC}(p) \tag{3.14}$$

The results are given in Fig. 3.2. $S^{ent}_{BEC}(p)$ is maximum when $p \to 0$, i.e. $m >> p$ in ϵ^0_p and correspondingly, $u(p) \sim v(p)$. For high momenta, $S^{ent}_{BEC}(p) \to 0$, and $u(p) \to 1$ meanwhile $v(p) \to 0$. But as we will see, this behavior is further strongly depending on the interaction strength, U_0.

For $U_0 \to 0$, the entropy $S^{ent}_{BEC}(p) \to 0$. Thus for constant momentum the controlling parameter over S^{ent}_{BEC} is the U_0 which is defined as the interaction strength in the units of energy volume. For the non-interacting BEC of an ideal gas consisting the coupled quark pairs $S^{ent}_{BEC} = 0$. Clearly in this case the excitation entropy, S^{ent}_{BEC}, is a phenomenon which does not depend on the Fermi surface.

4 Formulation for Entanglement in the BCS Model

As introduced in section 1 the overlap between BCS theory and the phenomenon of BEC will not be taken directly into consideration in this work. For the entropy in the paired phase we look at the ground state entropy which arises from the correlations in order to determine the full extent of the entanglement of the ground state. The expectation

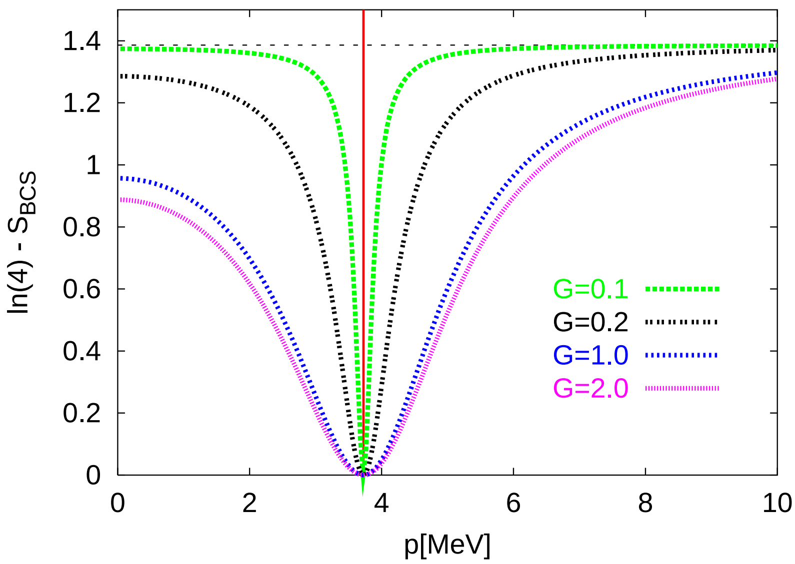

Figure 4.3: The difference between $\ln 4$ (quantum entropy for a system of two colored quarks) and the entropy S_{BCS}^{ent} which arises from the excitation is given for different momenta p and couplings G. We note that S_{BCS}^{ent} strongly depends on the momentum space and the energy gap. At the Fermi level p_f (vertical line), S_{BCS}^{ent} reaches a maximum value, $\ln 4$. This is valid for all couplings and the correspondingly energy gaps. Obviously, for small couplings it vanishes both below as well above p_f.

is that the extent of the entanglement is determined by the mixing as we have determined in [8–10] and the excitations in the ground state of BCS, which analogously arises from the electron-phonon interaction. As a theoretical replacement for this coupling we have the octet structure which couples through the gluons to the particular colors. This is true for all the gluons building colored pairs except for the coupling generated from the Gell-Mann matrix λ_8. In the second quantized formalism the interactions can be related to the entanglement [17]. We use the description of superconductors [18, 24], which can be treated in a similar way to the interaction causing a Bose-Einstein condensation of the Cooper pairs. For finite couplings nonzero bound state energies exist. There is another basic feature of BCS which involves oppositely directed momenta and spins above and below the Fermi level. As we have discussed in the previous section, we carry out the Bogoliubov canonical transformation for the field amplitudes deoccupation and occupation of the states, respectively,

$$u(p) = \left[\frac{1}{2}\left(1+\frac{\zeta(p)}{\epsilon(p)}\right)\right]^{1/2}, \tag{4.1}$$

$$v(p) = \left[\frac{1}{2}\left(1-\frac{\zeta(p)}{\epsilon(p)}\right)\right]^{1/2}, \tag{4.2}$$

and therefore, $u(p)^2 + v(p)^2 = 1$. We find by taking into account the proper statistics

$$S_{BCS}^{ent}(p) \quad = \quad g \sum_{p\neq 0} \left\{ \frac{\ln\left(\frac{u(p)}{v(p)}\right)^2}{\left(\frac{u(p)}{v(p)}\right)^2 + 1} + \ln\left(1 + \left(\frac{v(p)}{u(p)}\right)^2\right) \right\} \tag{4.3}$$

where

$$\zeta(p) \quad = \quad \epsilon_0(p) - \mu, \tag{4.4}$$

$$\epsilon(p) \quad = \quad \left(\zeta(p)^2 + \Delta^2\right)^{1/2} \tag{4.5}$$

and μ is the quark chemical potential. We call to mind that the change of the minus signs in Eq. 3.7 to the plus signs here arises on account of the Fermi statistics in the present case.

This dispersion relation is to be modified by the existence of coherent effects (pairings) at $T = 0$. $\epsilon(p)$ is equivalent to the Bogoliubov quasiparticle energy, which according to μ characterizes the minimum of the energy gap. Δ is an interval in which no eigenenergies are allowed in the one-particle energy spectrum. It is usually called the energy gap and plays the role of order parameter.

$$S_{BCS}^{ent}(p) \quad = \quad g \sum_{p\neq 0} \left\{ \frac{\epsilon(p) - \zeta(p)}{2\,\epsilon(p)} \ln\left(\frac{\epsilon(p) + \zeta(p)}{\epsilon(p) - \zeta(p)}\right) \ln\left(\frac{\epsilon(p) + \zeta(p)}{2\epsilon(p)}\right) \right\} \tag{4.6}$$

In Eq. 3.12, 4.6 we can replace $\sum_{p\neq 0}$ by $V/(2\pi^2) \int p^2\, dp$ and set the degeneracy factor $g = 2$.

Eq. 4.6 gives the sum of the entanglement entropy of BCS pairs with momenta $]0, p]$. In order to evaluate this entropy arsing from the entanglement in the ground state, we must know more about the system. In the BCS limit, we set the chemical potential $\mu \equiv \epsilon_f = (\pi^2\, n^{2/3} + m^2)^{1/2}$, the Fermi level at vanishing temperature. The gap in the spectrum of single-particle excitation which controls the sign of chemical potential, μ, can be found by solving the following integral equation [18, 25]:

$$1 \quad = \quad \frac{3U_0 V}{2\pi^2} \int_0^\infty p^2 dp \left[\frac{\epsilon(p) - \zeta(p)}{\epsilon(p)\zeta(p)}\right]. \tag{4.7}$$

We note that $\zeta(p)$ is the kinetic energy from the Fermi level and clearly depends upon the chemical potential. Therefore, for the excitation energy in the ground state Δ can be parametrized as follows:

$$\Delta \quad = \quad \epsilon_f \, \left(\frac{2}{e}\right)^{7/3} \exp\left(\frac{-1}{4\, G}\right) \tag{4.8}$$

where ϵ_f is the Fermi energy, which per definition at $T = 0$ is equivalent to the ground state quark chemical potential μ_f. G is the coupling strength, which represents the controlling parameter in the BCS model. For vanishing G we have $\Delta = 0$ and then $S^{ent}_{BCS} = 0$. As in Eq. 3.14 we define

$$S_{BCS}(p) \quad = \quad \ln 4 - S^{ent}_{BCS}(p) \tag{4.9}$$

This entropy difference is given in Fig. 4.3 for different couplings and momenta. So far we can conclude that the part of the entanglement entropy is entirely arising from the interactions/excitations between the quark pairs with opposite momenta at the surface of Fermi sea. We can also conclude that at the Fermi surface the maximum excitation entropy equals to the entropy for the mixing of two colored quarks (quantum entropy). The latter is T and p independent and all-dominant for $T = [0, m]$.

5 Results and discussion

In Fig. 3.2 the difference between the entropy of the excitation in BEC of two colored bosonic quark-pairs (Eq. 3.12), which are characterized by the phenomenological model given in section 2, and the ground state entropy (Eq. 3.13) is depicted in dependence upon the momenta p and for different interaction lengths U_0. Here Eq. 3.12 is numerically solved for equal successive momentum intervals p. We note that S^{ent}_{BEC} starts from a maximum value at $p \to 0$ and decays with increasing p. For large p it radically vanishes indicating no contribution to the entropy from the excitation, since for large momentum the scaling exceeds the de Broglie wavelength and therefore the correlation between the bosonic quark pairs entirely disappears. We also notice how the entropy difference depends on the interaction strength U_0. Larger U_0, smaller is the difference at $p \to 0$ and slower is the increasing towards the asymptotic value, $\ln 4$. For $U_0 \to 0$, we have from Eq. 3.12 that $S^{ent}_{BEC} \to 0$. Thus we can conclude that the interaction strength U_0 fully determines the properties of the S^{ent}_{BEC} as the entropy arising from the entanglement. Therefore, it can be taken as the controlling parameter. On the other hand, the asymptotically vanishing value of S^{ent}_{BEC} in the condensate of quark pairs disappears for large momenta since there is no remaining interaction between the quark pairs.

The BCS calculations are given in Fig. 4.3. The difference between the ground state entropy that arises from the entanglement in the BCS (Eq. 4.6) is depicted as a function of momenta p for different values of the coupling G. We should notice again that S^{ent}_{BCS} is highly structured. There are two regions where S^{ent}_{BCS} vanishes: one is well below while the other is well above the Fermi momentum p_f, which is shown as the entropy difference from the ground state value $\ln 4$. The peak, which is located around the Fermi surface, is shown by the sharp dip. When the value for S^{ent}_{BCS} starts to move above its vanishing value for the momenta higher in the Fermi sea. Near the Fermi surface the excitation entropy rapidly increases. This behavior reflects the appearance of correlations between the quarks below and their counterparts above p_f. The asymptotic value is again just $\ln 4$, the value of the quantum entropy for the mixing in a system of two colored quarks (Eq. 3.13). As discussed above this behavior strongly depends on the coupling G and thereby on the gap parameter Δ. Nevertheless, the general structure remains much the same over many values. For larger Δ, the values of S_{BCS}^{ent} get larger even at quite small and rather large p. Thus the asymptotic region will be moved towards much larger momenta. Nevertheless, the maximum value of S_{BCS}^{ent} at p_f does not itself depend on G. Therefore, this value characterizes the excitation at the Fermi surface. The entropy of excitation at p_f is the same as the value $\ln 4$ of the mixing of the paired colored quarks at $T = 0$, which is the size of the dip. Therefore, we can accordingly conclude that the condensate of BCS pairings can be a measure of the density of pair states around the Fermi surfaces. Thus in this model we would expect that as soon the gluonic interactions reach the Debye cutoff momenta that then S_{BCS}^{ent} also vanishes as does the excitation entropy.

The quark pairs in this BCS model are characterized by asymmetric color degrees of freedom and equal opposite momenta and spins. All the other degrees of freedom are kept identical. One quark is located just below the Fermi level, while its counterpart is found just above it. The coupling G and therefore the energy gap Δ are likewise model dependent [26]. Hence we are left with parameterizing these quantities. Eq. 4.8 gives the parametrization of Δ in relation to G for symmetric flavors, momenta and spins and asymmetric colors. This relationship signifies that for vanishing G, $\Delta \to 0$. With increasing G the energy gap Δ increases too. Δ can be illustrated as region around the Fermi level, in which S_{BCS}^{ent} is finite. Only those quarks whose momenta fit into this region are considered. In other words large Δ leads to large correlations with correspondingly large excitations between the quark pairs around p_f. Therefore, there are more pairs which are contributing to the value of S_{BCS}^{ent}. Vanishing values of Δ, on the other hand, lead to an absence of interacting quarks at the Fermi surface. Hence, the theoretical system of deconfined *free* quarks at these high quark chemical potentials and low temperatures can properly be considered as a closed pure state without any mixing structure [8–10]. This kind of system has according to Nernst's heat theorem zero entropy in the low temperature limit.

Another worthwhile result appearing in this figure is the shift of the asymptotically zero value of S_{BCS}^{ent} towards higher momenta for larger G. This situation happens because the larger values of Δ lead to inclusion a larger number of quarks deep within the Fermi surfaces and simultaneously more quarks with momenta greater than p_f. Then for much larger Δ we can freely move to higher momenta. Therefore, we notice for these cases that the region of nonzero values of the excitation entropy S_{BCS}^{ent} becomes correspondingly wider.

6 Conclusion

An important consideration in relation to this work is the existence of the excitations at $T = 0$. The only quark interactions allowed at $T = 0$ are those of degenerate fermions near the Fermi surface. Besides this we should consider the mixing of colored quarks. A purely formal clarification explains the excitations as a result of the Bogoliubov canonical transformation taking the interacting Hamiltonian with pairs of oppositely directly momenta for the ground state operators into a simple sum over all finite momenta of the quasi-particle number operators with a modified energy spectrum containing various pieces of the interactions. These excitations arise in color space in much the same way that the spin waves appear in the similar operators for atomic physics. Clearly these excitations are not so simply represented in the $SU(3)_c$ space in the actual colorless ground state of the hadrons, which has a singlet representation. However, the octet states are mostly those involving only two colors in the Gell-Mann matrices – except for λ_8. Thus by taking linear combinations of λ_3 and λ_8 we may write down *nine* matrices for three pairs of $SU(2)$ color pair states which have been constructed from the octet states. Each pair may be treated analogously to the spin waves. All three together could be imagined to be waves propagating in three perpendicular directions, which are clearly not independent of each other. However, over very short distances a large range of momenta could be accommodated, which within these values of momenta the oppositely directed pair of constituents are present.

Finally, we conclude by noting that the effects of the inter-particle interaction for both the Bose-Einstein condensate and the BCS superconductor causes the entanglement. This fact is characterized by the finite momentum contributions to the condensation process which offer the Bogoliubov canonical transformation results in the excitation spectra. It is the coefficients of this transformation that appear in the quantum entropy density of entanglement. Thus we have noted that for both the composite Bose-Einstein condensate and the superconductor with its Cooper pairing structure the entanglement arises from the interaction between the oppositely directed momenta of the constituent particles. For the parameters we used here the maximum entropy for the excitations in the two condensates is compatible with the value for the mixing of two quarks at zero temperature. Therefore, the entropy arising from the excitations in the condensates BCS and BEC, $S \leq \ln 4$, depend upon the momentum space.

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