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# Robustness Study for the Parameter of Power Function Distribution

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**Abstract:** Sequential testing procedure is developed for shape parameter of power function distribution. Sequential Probability Ratio Test (SPRT) is developed for testing the hypothesis regarding the (shape) parameter of power function distribution. The robustness of SPRT is studied. The expressions for the operating characteristic (OC) and average sample number (ASN) functions are derived and the results presented through tables and graphs.

**Keywords:** Power Function Distribution · Robustness · OC and ASN functions· SPRT.

## 1 Introduction

Power function distribution is mostly used in depiction of population data where sample size is predictable. This distribution is also counted as a life testing model to fit the failure data. Power function distribution is a specific case of Pareto distribution. Meniconi and Barry (1995)[5] discussed the application of Power function distribution. They proved that the Power function distribution is the best distribution to check the reliability of any electrical component. In reliability/survival testing this distribution measures hazard rate in specified time interval.

The pioneering work of sequential test is proposed by Wald (1947)[12], who developed Sequential Probability Ratio Test (SPRT) for testing a simple null hypothesis against a simple alternative hypothesis. As a measure of the performance of SPRT, Wald obtained the expression for the Operating Characteristic (OC) and Average Sample Number (ASN) functions. The robustness of the SPRT, dealing with the testing of the parameters under consider has undergone a change has been studied by various authors, while dealing with different probabilistic models useful in reliability/survival analysis. Epstein and Sobel (1955)[2] applied the SPRT for testing the simple null hypothesis against simple alternative hypothesis regarding the parameter of exponential distribution. Harter and Moore (1976)[3] conducted Monte Carlo study to investigate the robustness of SPRT for exponential distribution. Montagne and Singpurwalla (1985)[6] generalized the results of Harter and Moore (1976)[3]from Weibull to a class of distribution having an increasing failure rate. They obtained inequalities for OC and ASN functions in order to demonstrate the robustness. For sequential tests and their robustness for the parameters of some families of continuous distribuions, one may refers to Phatarfod (1971)[9], Chaturvedi, Kumar and Surinder (2000)[1], Surinder and Naresh (2009)[10] Parameshwar, Nagaraj and Gudaganavar (2010)[8] and Surinder and Mukesh (2013)[11].

In this paper, we have developed the Sequential Testing Procedure for the parameter of power function distribution. In section 2, we describe the basic introduction of power function distribution. In section 3, we develop the SPRT and obtained the expressions for the OC and ASN functions for parameter. In section 4, we have also studied the robustness of the SPRT when the parameter under consideration has undergone a change. Finally, we illustrate implementation of SPRT for power function distribution, in Section 5.

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## 2 Set Up of the Problem

Let the random variable(rv)$X$ follows the Power function distribution presented by the probability density function (pdf)

$$f(x) = \frac{C(x-a)^{(C-1)}}{(b-a)^C}; \quad a < x \leq b \tag{1}$$

where $C$ is the shape parameter $C > 0$ and $a, b$ are the boundary points. Taking $a = 0$ the pdf converted in to

$$f(x) = \frac{Cx^{(C-1)}}{b^C}; \quad 0 < x \leq b \tag{2}$$

Given a sequence of observation $X_1, X_2, X_3, \cdots, X_n$ from (2.2), suppose one wish to test the simple null hypothesis$H_0 = C = C_0$ against simple alternative hypothesis $H_1 = C = C_1$. The expression for the OC and ASN functions are obtained and their behaviour is studied through graphs and tables.

## 3 Testing the Hypothesis for C

The SPRT for testing simple null hypothesis against simple alternative hypothesis, corresponding ratio is

$$Z_i = log[\frac{f(x_i, C_1)}{f(x_i, C_0)}] = log(\frac{C_1}{C_0}) + (C_1 - C_0)logx_i + (C_0 - C_1)logb \tag{3}$$

Now we choose two numbers A and B such that $0 < B < 1 < A$. At the $n^{th}$ stage accept $H_0$ if $\Sigma Z_i \leq (logB)$, reject $H_0$ if $\Sigma Z_i \geq (logA)$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. $\alpha \varepsilon (0,1)$ and $\beta \varepsilon (0,1)$ are $TypeI^{st}$ and $TypeII^{nd}$ errors respectively, where $A = (\frac{1-\beta}{\alpha})$ and $B = (\frac{\beta}{1-\alpha})$.
The OC function of SPRT is given by

$$L(C) = \frac{(A^h - 1)}{(A^h - B^h)} \tag{4}$$

where h is the non zero solution of the given equation $E(e^{Z_i}) = 1$
or

$$\int_0^b [\frac{f(x_i, C_1)}{f(x_i, C_0)}]^h f(x_i, C) dx_i = 1 \tag{5}$$

After solving we get,

$$h = \frac{C[(\frac{C_1}{C_0})^h - 1]}{(C_1 - C_0)}$$

On substituting $d = (C_1 - C_0), a = (\frac{C_1}{C_0})$ we get

$$Ca^h = hd + C \tag{6}$$

Taking logarithm in both sides in (3.4),

$$hlog(a) + log(C) = log(hd + C)$$

$$hlog(a) = log(1 + \frac{hd}{C})$$

$$hloga = \frac{hd}{C} - \frac{h^2d^2}{2C^2} + \frac{h^3d^3}{3C^3}$$

Taking approximation upto $3^{rd}$ degree of h,we get quadratic equation

$$2d^3h^2 - 3Cd^2h + 6C^2d - 6C^3loga = 0 \tag{7}$$

The roots are

$$h = \frac{3Cd^2 \pm \sqrt{48C^3d^3 \log a + 39C^2d^4}}{4d^3}$$

ASN for power function distribution

$$E(N/C) = \frac{L(C)logB + [1-L(C)]logA}{E(Z_i)} \tag{8}$$

Now calculate $E(Z_i)$ from (3.1),where $E(Z_i) \neq 0$,

$$E(Z_i) = log(\frac{C_1}{C_0}) + (C_1 - C_0)\int_0^b logx_i f(x)dx_i + (C_0 - C_1)logb$$

$$= log(\frac{C_1}{C_0}) + (C_1 - C_0)\int_0^b logx_i \frac{Cx_i^{(C-1)}}{b^C}dx_i + (C_0 - C_1)logb$$

$$E(Z_i) = \log(\frac{C_1}{C_0}) - \frac{(C_1 - C_0)}{C} \tag{9}$$

## 4 Robustness of SPRT for the Scale Parameter of Power Function Distribution

Let us suppose that the parameter $C$ has undergone a change then the pdf in (2.2) become $f(x_i, C, b_1)$ . To study the robustness of SPRT developed in section 4 with respect to OC function, consider $h$ as the solution of the equation

$$\int_0^{b_1}[\frac{f(x_i, C_1, b)}{f(x_i, C_0, b)}]^h f(x_i, C, b_1)dx_i = 1 \tag{10}$$

After solving we get

$$(\frac{b_1}{b})^{h(C_1-C_0)}C(\frac{C_1}{C_0})^h = h(C_1 - C_0) + C$$

Let $\phi = (\frac{b_1}{b}), a = (\frac{C_1}{C_0}), d = (C_1 - C_0)$
The equation is written as

$$C(a^h)(\phi^{hd}) = hd + C$$

$$(a^h)(\phi^{hd}) = [\frac{hd}{C} + 1] \tag{11}$$

Taking logarithm in both sides in (4.2) and we get results

$$\frac{h^2d^3}{3C^3} - \frac{hd^2}{2C^2} + \frac{d}{C} - \log(a) - d\log(\phi) = 0 \tag{12}$$

ASN function for SPRT can be obtained as follows

$$E(N/C) = \frac{L(C)logB + [1-L(C)]logA}{E_{b_1}(Z_i)}$$

$$E_{b_1}(Z_i) = \int_o^{b_1} Z f(x_i, C, b_1)dx_i \tag{13}$$

$$E_{b_1}(Z_i) = log(\frac{C_1}{C_0}) + (C_0 - C_1)logb + (C_1 - C_0)\int_0^{b_1} logx_i f(x_i, C, b_1)dx_i$$

$$= log(\frac{C_1}{C_0}) + (C_0 - C_1)logb + (C_1 - C_0)[logb_1 - \frac{1}{C}]$$

$$E_{b_1}(Z_i) = \log(a) - d\log(\frac{1}{\phi}) + \frac{d}{C} \tag{14}$$

where $\phi = (\frac{b_1}{b}), a = (\frac{C_1}{C_0}), d = (C_1 - C_0)$

## 5 Implementation of SPRT for power function distribution

The nature of SPRT in case of power function distribution is described as, let $X_1, X_2, X_3, \cdots, X_n$ be identically independent distributed r.v's from power function function distribution (2.2), where $C > 0$,we wish to test the simple null hypothesis$H_0 = C = C_0$ against simple alternative hypothesis $H_1 = C = C_1 (> C_0)$ having pre-assigned $0 < \alpha, \beta < 1$. which are (A and B) defined in section 3 and $Z_i$ defined as

$$Z_i = log[\frac{f(x_i, C_1)}{f(x_i, C_0)}] = log(\frac{C_1}{C_0}) + (C_1 - C_0) log x_i + (C_0 - C_1) log b \tag{15}$$

Let $n(\geq), i = 1, 2, ...,$. The SPRT summarized and simplified to the following. Let$Y(n) = \sum X_i$ and N=first integer $(n \geq 1)$ for which inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with constants

$$c_1 = \frac{ln(B)}{(C_1 - C_0)}, c_1 = \frac{ln(A)}{(C_1 - C_0)}), d = \frac{ln[(\frac{C_1}{C_1}) b^{(C_0 - C_1)}]}{(C_1 - C_0)}$$

At the stoping stage of sampling, if $Y(N) \leq c_1 + dN$ we accept $H_0$ and if $Y(N) \geq c_2 + dN$ we reject $H_0$ for different values of N, where A and B are the fixed constants. **Fig.3** shows acceptance and rejection region for null hypothesis, when $H_0 = C_0 = 20, H_1 = C_1 = 25, b = 1, \alpha = \beta = 0.05$. The calculated values constants are $c_1 = -0.9108, c_2 = 0.59713$ and $d = 0.01938$. from these constants if $Y(N) \leq 0.019N - 0.9108$ we accept $H_0$ and if $Y(N) \geq 0.019N + 0.59714$ we accept $H_1$.At the intermediate stage, we continue sampling.

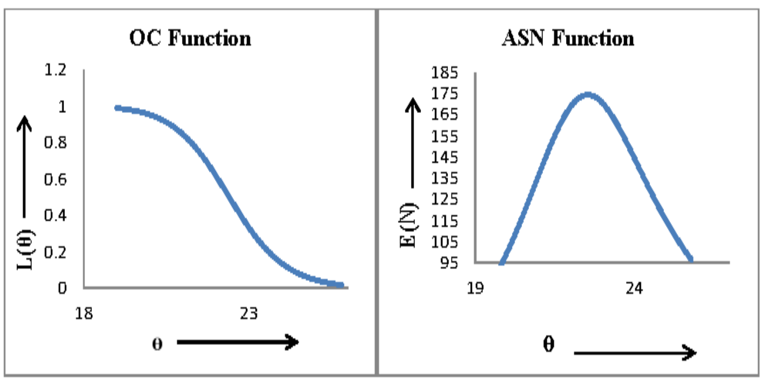


**Fig. 1:** OC function and ASN function for testing when $C_1 = 20,\ C_2 = 25,\ b = 1, \alpha = \beta = 0.05$.

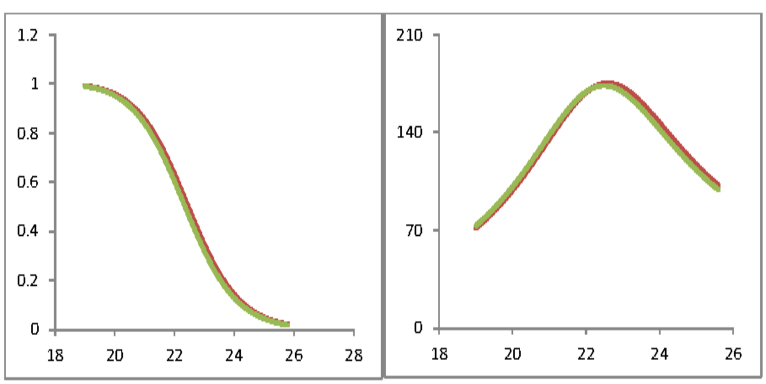

**Fig. 2:** Robustness of OC and ASN functions for testing $C_0 = 20,\ C_1 = 25,\ b = 1, \alpha = \beta = 0.05,\ p = \phi$.

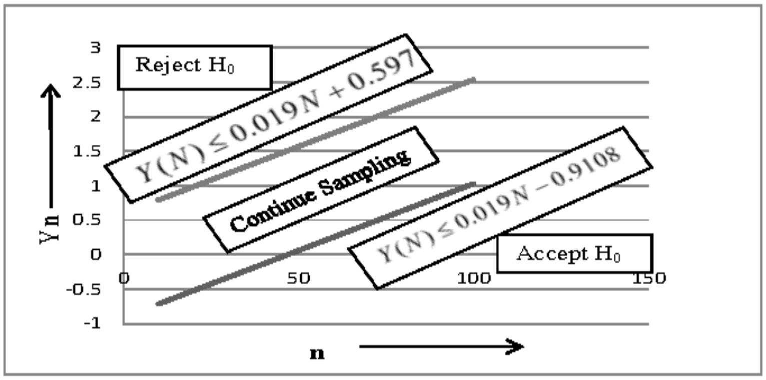


**Fig. 3:** Acceptance and rejection region for testing $C_0 = 20,\ C_1 = 25,\ b = 1, \alpha = \beta = 0.05,\ p = \phi$.

Table: 1 OC and ASN functions for testing $C_0 = 20, C_1 = 25, b = 1$ and $\alpha = \beta = 0.5, p = \phi$.

| $\phi = 1.0000$ | | | $\phi = 0.9999$ | | $\phi = 1.00011$ | |
|---|---|---|---|---|---|---|
| $\theta$ | $L(\theta)$ | $E(N)$ | $L(\theta)$ | $E(N)$ | $L(\theta)$ | $E(N)$ |
| 19.0 | 0.991363 | 72.31351 | 0.992172 | 71.53846 | 0.990406 | 73.17874 |
| 19.2 | 0.987318 | 76.99261 | 0.988400 | 76.14191 | 0.986043 | 77.94159 |
| 19.4 | 0.982016 | 82.06598 | 0.983439 | 81.13527 | 0.980344 | 83.10313 |
| 19.6 | 0.975152 | 87.55464 | 0.977001 | 86.54096 | 0.972990 | 88.68267 |
| 19.8 | 0.966368 | 93.47254 | 0.968737 | 92.37510 | 0.963604 | 94.69146 |
| 20.0 | 0.955244 | 99.82246 | 0.958246 | 98.64377 | 0.951753 | 101.1284 |
| 20.2 | 0.941309 | 106.5910 | 0.945068 | 105.3381 | 0.936951 | 107.9745 |
| 20.4 | 0.924045 | 113.7422 | 0.928695 | 112.4284 | 0.918671 | 115.1869 |
| 20.6 | 0.902911 | 121.2111 | 0.908590 | 119.8574 | 0.896371 | 122.6915 |
| 20.8 | 0.877375 | 128.8965 | 0.884214 | 127.5332 | 0.869529 | 130.3769 |
| 21.0 | 0.846965 | 136.6559 | 0.855074 | 135.3235 | 0.837704 | 138.0891 |
| 21.2 | 0.811339 | 144.3026 | 0.820789 | 143.0518 | 0.800600 | 145.6304 |
| 21.4 | 0.770356 | 151.6088 | 0.781158 | 150.4991 | 0.758145 | 152.7644 |
| 21.6 | 0.724155 | 158.3153 | 0.736244 | 157.4116 | 0.710572 | 159.2276 |
| 21.8 | 0.673220 | 164.1503 | 0.686438 | 163.5172 | 0.658462 | 164.7502 |
| 22.0 | 0.618398 | 168.8549 | 0.632498 | 168.5496 | 0.602763 | 169.0833 |
| 22.2 | 0.560876 | 172.2127 | 0.575530 | 172.2782 | 0.544740 | 172.0282 |
| 22.4 | 0.502091 | 174.0768 | 0.516919 | 174.5352 | 0.485879 | 173.4642 |
| 22.6 | 0.443597 | 174.3928 | 0.458205 | 175.2418 | 0.427737 | 173.3580 |
| 22.8 | 0.386905 | 173.1983 | 0.400925 | 174.4119 | 0.371786 | 171.7775 |
| 23.0 | 0.333337 | 170.6221 | 0.346460 | 172.1547 | 0.319273 | 168.8698 |
| 23.2 | 0.283912 | 166.8587 | 0.295914 | 168.6504 | 0.271124 | 164.8426 |
| 23.4 | 0.239300 | 162.1437 | 0.250047 | 164.1287 | 0.227908 | 159.9362 |
| 23.6 | 0.199819 | 156.7255 | 0.209262 | 158.8380 | 0.189855 | 154.3971 |
| 23.8 | 0.165483 | 150.8422 | 0.173644 | 153.0222 | 0.156907 | 148.4561 |
| 24.0 | 0.136072 | 144.7059 | 0.143025 | 146.9019 | 0.128790 | 142.3152 |
| 24.2 | 0.111204 | 138.4934 | 0.117056 | 140.6642 | 0.105093 | 136.1401 |
| 24.4 | 0.090408 | 132.3438 | 0.095283 | 134.4583 | 0.085330 | 130.0591 |
| 24.6 | 0.073178 | 126.3606 | 0.077204 | 128.3968 | 0.068991 | 124.1660 |
| 24.8 | 0.059011 | 120.6156 | 0.062312 | 122.5594 | 0.055583 | 118.5246 |
| 25.0 | 0.047437 | 115.1547 | 0.050128 | 116.9981 | 0.044647 | 113.1745 |
| 25.2 | 0.038032 | 110.0033 | 0.040215 | 111.7430 | 0.035771 | 108.1363 |
| 25.4 | 0.030422 | 105.1714 | 0.032186 | 106.8078 | 0.028597 | 103.4164 |
| 25.6 | 0.024288 | 100.6578 | 0.025708 | 102.1935 | 0.022819 | 99.01144 |
| 25.8 | 0.019357 | 96.45376 | 0.020498 | 97.89312 | 0.018179 | 94.91104 |

## 6 Conclusion

The values of OC and ASN functions for the cases $b < b_1, b = b_1, b > b_1$ are plotted in **Fig.2** respectively. From the **Fig.2** we observe that for $b < b_1$and $b > b_1$, the OC curve is not shifted to any other side of the curve. From the **Fig.2** it is clear that SPRT is robust, as the deviation in OC and ASN functions are insignificant. Thus we conclude that for the present model, the SPRT for testing the hypothesis regarding $C$, is highly robust for changes in the value of $b$ .

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