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Exact and Approximate Solutions of Fractional Diffusion Equations with Fractional Reaction Terms

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Abstract: In this paper, we consider fractional reaction-diffusion equations with linear and nonlinear fractional reaction terms in a semi-infinite domain. Using q-Homotopy Analysis Method, solutions to these equations are obtained in the form of general recurrence relations. Closed form solutions in the form of the Mittag-Leffler function are perfectly obtained in the case with linear fractional reaction term due to the ability to control the auxiliary parameter h. Series solution is obtained for the case of nonlinear fractional reaction term. Numerical analysis is presented for this case to display the fast convergent rate of the series solution obtained. q-HAM is a relatively simple and powerful method and has advantages over some other methods which we discuss and demonstrate for some initial value problems.

Keywords: Reaction-diffusion, Mittag-Lefler function, fractional derivative, q-Homotopy analysis method.

1 Introduction

Mathematical models to describe the process through which the concentration of one or two substances are dispersed in space changes caused by two processes (reaction and diffusion) are referred to as reaction-diffusion models. The reaction-diffusion models have wide range of applications in modelling population evolution, chemical reactions, epidemic spreading and also in combustion theory and pattern formation. Reaction-diffusion approach is also used to describe electric transport systems such as plasmas or semiconductors under some appropriate circumstances. See [1,2,3] for detail. Certainly, obtaining and studying solutions to these types of models is very important.

Physical systems are increasingly being modeled by the fractional calculus due its ability to incorporate memory effects. For example, in [4], Darcys law was modified by Caputo to incorporates the memory term to model transport through porous media. Other examples are, astrophysics [5], meteorology [6], reactive flows [7] and semiconductors [8], and ground water flow [9].

Typically, for such models like these, there is need to solve a partial differential equation of fractional order (FPDE). The commonly used analytical methods to obtain solutions of these equations are mostly restricted to linear systems and in the case of non-linear equations, numerical techniques are usually employed see [10]. There are some approximate methods used for both linear and non-linear FPDE such as variational iteration method, VIM, Adomian decomposition method, ADM, generalized differential transform method, GDTM see [11,12,13]. Of recent, in [14,15,16,17,18,19], a modified homotopy analysis method was introduced and has potential applications in a wide range of systems. This modified method is called q-homotopy analysis method (q-HAM)

In this paper, we present the application of q-HAM to initial value problems of the fractional reaction-diffusion equations. The aim of this work is to obtain solutions in the form of recurrence relations, and closed form solutions in terms of the Mittag-Leffler function where possible especially in the case with linear fractional reaction term due to the ability to control the auxiliary parameter h. For the case with nonlinear fractional reaction term, series solution is obtained. Numerical analysis is presented for this case to display the fast convergent rate of the series solution obtained. Caputo's fractional derivative is adopted in this work.

Definition 1.*The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \geq 0$, of a function $g \in L^1(a,b)$ is given as*

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$$I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}g(\tau)d\tau,\ t>0,\ \alpha>0, \tag{1}$$

where Γ is the Gamma function and $I^0g(t)=g(t)$.

Definition 2. *The fractional derivative in the Caputo's sense is defined as [20],*

$$\mathscr{D}^{\alpha}g(t)=I^{n-\alpha}D^ng(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}g^{(n)}(\tau)d\tau, \tag{2}$$

where $n-1<\alpha\leq n$, $n\in\mathbb{N}$, $t>0$.

Lemma 1. *Let $t\in(a,b]$. Then*

$$\left[I_a^{\alpha}(t-a)^{\beta}\right](t)=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha},\qquad \alpha\geqslant 0,\quad \beta>0. \tag{3}$$

Definition 3. *The Mittag-Leffler function for two parameters is defined as*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad \alpha,\beta,z\in\mathscr{C},\quad Re(\alpha)\geq 0. \tag{4}$$

2 Method of Solution: q-Homotopy Analysis Method

Given consideration to differential equation of the form

$$N\left[\mathscr{D}_t^{\alpha}c(x,t)\right]-g(x,t)=0, \tag{5}$$

where N denotes non-linear operator, $\mathscr{D}_t^{\alpha}$ is the the Caputo fractional derivative, g is a given function and c is an unknown function. To generalize the original homotopy method, the zeroth-order deformation equation is constructed as

$$(1-nq)L\left(\sigma(x,t;q)-c_0(x,t)\right)=qhH(x,t)\left(N[\mathscr{D}_t^{\alpha}\sigma(x,t;q)]-g(x,t)\right), \tag{6}$$

where $n\geqslant 1$, $q\in\left[0,\frac{1}{n}\right]$ denotes the so-called embedded parameter, $h\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $H(x,t)$ is a non-zero auxiliary function.

When $q=0$ and $q=\frac{1}{n}$, we have equation (6) to be

$$\sigma(x,t;0)=c_0(x,t)\quad \textit{and}\quad \sigma\left(x,t;\frac{1}{n}\right)=c(x,t), \tag{7}$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\sigma(x,t;q)$ varies from the initial guess $c_0(x,t)$ to the solution $c(x,t)$.

If $c_0(x,t)$, L, h, $H(x,t)$ are chosen appropriately, solution $\sigma(x,t;q)$ of equation(6) exists for $q\in\left[0,\frac{1}{n}\right]$.

The Taylor series expansion of $\sigma(x,t;q)$ gives

$$\sigma(x,t;q)=c_0(x,t)+\sum_{m=1}^{\infty}c_m(x,t)q^m, \tag{8}$$

where

$$c_m(x,t)=\frac{1}{m!}\frac{\partial^m\sigma(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{9}$$

Assume that the auxiliary linear operator L, the initial guess c_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series 8 converges at $q=\frac{1}{n}$, then we have

$$c(x,t)=c_0(x,t)+\sum_{m=1}^{\infty}c_m(x,t)\left(\frac{1}{n}\right)^m. \tag{10}$$

Let the vector c_n be define as follows:

$$\mathbf{c}_n = \{c_0(x,t), c_1(x,t), \cdots, c_n(x,t)\}. \tag{11}$$

Differentiating equation (6) m-times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have what is known as the m^{th}-order deformation equation, Liao [21]

$$L\left[c_m(x,t) - \chi_m^* c_{m-1}(x,t)\right] = hH(x,t)\mathscr{R}_m\left(\mathbf{c}_{m-1}\right) \tag{12}$$

with initial conditions

$$c_m^{(k)}(x,0) = 0, \quad k = 0,1,2,...,m-1, \tag{13}$$

where

$$\mathscr{R}_m\left(\mathbf{c}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left(N[\mathscr{D}_t^\alpha \sigma(x,t;q)] - g(x,t)\right)}{\partial q^{m-1}}\bigg|_{q=0} \tag{14}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1, \\ n & otherwise, \end{cases} \tag{15}$$

3 Recurrence Relation for Fractional Reaction-Diffusion

We take into consideration the fractional reaction-diffusion equation with constant diffusivity, and reaction term f, in a semi-infinite domain $x > 0$,

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,u(x,t)), \quad 0 < \beta, x > 0, t > 0 \tag{16}$$

subjects to the initial condition

$$u(x,0) = p(x), \tag{17}$$

where $\dfrac{\partial^\beta}{\partial t^\beta} := \mathscr{D}_t^\beta$ is the Caputo fractional derivatives of order β and D is the diffusivity.

In order to solve this system using q-HAM, we choose the linear operator

$$L[\sigma(x,t;q)] = \mathscr{D}_t^\alpha \sigma(x,t;q) \tag{18}$$

having property that $L[c_1] = 0$, where c_1 is constant; and initial approximation $u_0(x,t) = h(x)$. So, the non-linear operator can then defined as,

$$N[\sigma(x,t;q)] = \mathscr{D}_t^\beta \sigma(x,t;q) - D\sigma_{xx}(x,t;q) + f(x,\sigma(x,t)). \tag{19}$$

Next, we construct the zeroth order deformation equation,

$$(1-nq)L\left[\sigma(x,t;q) - u_0(x,t)\right] = qhH(x,t)N\left[\mathscr{D}_t^\beta \sigma(x,t;q)\right]. \tag{20}$$

Choosing $H(x,t) = 1$ we obtain the mth-order deformation equation to be,

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = h\mathscr{R}_m\left(\mathbf{u}_{m-1}\right), \tag{21}$$

with initial condition $u_m(x,0) = 0$, for $m \geqslant 1$. χ_m^* is as defined in (15), and $\mathscr{R}_m$ is given by,

$$\mathscr{R}_m\left(\mathbf{u}_{m-1}\right) = \mathscr{D}_t^\beta u_{m-1} - Du_{(m-1)xx} + f(x,u_{m-1}(x,t)). \tag{22}$$

We thus obtain the recurrent relation for $u_m(x,t)$, for $m \geqslant 1$,

$$u_m(x,t) = \chi_m^* u_{m-1} + hI_t^\beta\left[\mathscr{R}_m\left(\mathbf{u}_{m-1}\right)\right]. \tag{23}$$

4 Examples Involving Linear Fractional Reaction Terms

In this section, we consider particular choices of the source term, namely as linear fractional reaction terms of the form $f(x,u(x,t)) = -\frac{\partial^\gamma}{\partial x^\gamma}(r(x)u(x,t))$ for $\gamma \geq 1$, and $r(x)$ is a function of x (to be specified).

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^\gamma}{\partial x^\gamma}\left(r(x)u(x,t)\right), 0<\beta,\gamma\leq 1, x>0, t>0. \tag{24}$$

4.1 Example 1

Problem (24) is considered here with $D=1$ and $r(x) = -x$ given as

$$(P_1)\begin{cases} \dfrac{\partial^\beta u(x,t)}{\partial t^\beta} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial^\gamma}{\partial x^\gamma}(xu(x,t)), \quad 0<\beta,\gamma\leq 1, x>0, t>0, \\ u(x,0) = x^\alpha. \end{cases} \tag{25}$$

Using initial approximation $u_0(x,t) = x^\alpha$, we obtain components of the solution using q-HAM recurrent relation in (23) successively as follows

$$\begin{aligned} u_1(x,t) &= \chi_1^* u_0 + hI_t^\beta\left[\mathscr{D}_t^\beta u_0 - u_{0xx} - D_x^\gamma(xu_0)\right] \\ &= -\alpha(\alpha-1)hx^{\alpha-2}\frac{t^\beta}{\Gamma(1+\beta)} - \frac{h\Gamma(\alpha+2)x^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\frac{t^\beta}{\Gamma(1+\beta)} \end{aligned} \tag{26}$$

$$\begin{aligned} u_2(x,t) &= \chi_2^* u_1 + hI_t^\beta\left[\mathscr{D}_t^\beta u_1 - u_{1xx} - D_x^\gamma(xu_1)\right] \\ &= \alpha(1-\alpha)(n+h)hx^{\alpha-2}\frac{t^\beta}{\Gamma(1+\beta)} - \frac{(n+h)h\Gamma(\alpha+2)x^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\frac{t^\beta}{\Gamma(1+\beta)} \\ &\quad +\alpha(\alpha-1)(\alpha-2)(\alpha-3)h^2x^{\alpha-4}\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\frac{(\alpha-\gamma+1)(\alpha-\gamma)\Gamma(\alpha+2)h^2x^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma+2)}\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\alpha(\alpha-1)\Gamma(\alpha)h^2x^{\alpha-\gamma-1}\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\frac{h^2\Gamma(\alpha+2)\Gamma(\alpha-\gamma+3)x^{\alpha-2\gamma+2}}{\Gamma(\alpha-\gamma+2)\Gamma(\alpha-2\gamma+3)}\frac{t^{2\beta}}{\Gamma(1+2\beta)}. \end{aligned} \tag{27}$$

Following the same procedure, $u_m(x,t)$ for $m = 3,4,5,\cdots$ can be obtained.

So, the expression of the series solution given by q-HAM can be written in the form

$$\begin{aligned} u(x,t;n;h) &= x^\alpha + \sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \\ &= x^\alpha - \alpha(\alpha-1)h(2n+h)x^{\alpha-2}\frac{t^\beta}{\Gamma(1+\beta)} - \frac{(2n+h)h\Gamma(\alpha+2)x^{\alpha-\gamma+1}}{n^2\Gamma(\alpha-\gamma+2)}\frac{t^\beta}{n\Gamma(1+\beta)} \\ &\quad +\alpha(\alpha-1)(\alpha-2)(\alpha-3)h^2x^{\alpha-4}\frac{t^{2\beta}}{\Gamma(1+2\beta)} + \alpha(\alpha-1)\Gamma(\alpha)h^2x^{\alpha-\gamma-1}\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\frac{(\alpha-\gamma+1)(\alpha-\gamma)\Gamma(\alpha+2)h^2x^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma+2)}\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\frac{h^2\Gamma(\alpha+2)\Gamma(\alpha-\gamma+3)x^{\alpha-2\gamma+2}}{\Gamma(\alpha-\gamma+2)\Gamma(\alpha-2\gamma+3)}\frac{t^{2\beta}}{\Gamma(1+2\beta)} + \cdots. \end{aligned} \tag{28}$$

Hence, equation (28) is the solution to the problem (25). This solution is given in terms of convergence parameter h and n.

Case I: When $\alpha=0$, $\gamma=1$ and $n=1$, we choose $h=-1$ and obtain a closed form solution in form of Mittag-Leffler function of one parameters

$$u(x,t)=1+\frac{t^{\beta}}{\Gamma(1+\beta)}+\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\cdots=E_{\beta}(t^{\beta}). \tag{29}$$

This agrees with the solution obtained by VIM [11], and by GDTM [13]. We confirm that for the case when $\beta=1$ we obtain $u(x,t)=E_1(t)=e^t$.

Case II: When $\alpha=1$, $\gamma=1$ and $n=1$, we choose $h=-1$ and obtain a closed form solution in form of Mittag-Leffler function of one parameters

$$u(x,t)=x+2x\frac{t^{\beta}}{\Gamma(1+\beta)}+4x\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\cdots=xE_{\beta}(2t^{\beta}). \tag{30}$$

This agrees with the solution obtained by VIM [11], and by GDTM [13]. We confirm that when $\beta=1$ we obtain $u(x,t)=xE_1(2t)=xe^{2t}$.

Case III: When $\alpha=2$, $\gamma=1$ and $n=1$, we choose $h=-1$ and obtain a closed form solution in form of Mittag-Leffler function of one parameters

$$\begin{aligned}u(x,t)&=x^2\left(1+\frac{3t^{\beta}}{\Gamma(1+\beta)}+\frac{9t^{2\beta}}{\Gamma(1+2\beta)}+\cdots\right)+\left(1+\frac{3t^{\beta}}{\Gamma(1+\beta)}+\frac{9t^{2\beta}}{\Gamma(1+2\beta)}+\cdots\right)\\&\quad-\left(1+\frac{t^{\beta}}{\Gamma(1+\beta)}+\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\cdots\right)\\&=x^2E_{\beta}(3t^{\beta})+E_{\beta}(3t^{\beta})-E_{\beta}(t^{\beta}).\end{aligned} \tag{31}$$

This agrees with the one obtained in [11] by VIM, and in [13] by GDTM. When $\beta=1$ as

$$u(x,t)=x^2E_1(3t)+E_1(3t)-E_1(t)=(x^2+1)e^{3t}-e^t. \tag{32}$$

4.2 Example 2

Problem (24) is considered here with $D=1$, $r(x)=e^{-x}$ and $\gamma=1$ given as

$$(P_2)\begin{cases}\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}=\frac{\partial^2u(x,t)}{\partial x^2}-\frac{\partial}{\partial x}\left(e^{-x}u(x,t)\right), \quad 0<\beta\le 1, x>0, t>0,\\ u(x,0)=e^x.\end{cases} \tag{33}$$

Also, using the initial approximation $u_0(x,t)=e^x$, we obtain components of the solution using q-HAM recurrent relation in (23) successively as follows

$$\begin{aligned}u_1(x,t)&=\chi_1^*u_0+hI_t^{\beta}\left[\mathscr{D}_t^{\beta}u_0-u_{0xx}+e^{-x}u_0-e^{-x}u_{0x}\right]\\&=-he^x\frac{t^{\beta}}{\Gamma(1+\beta)}\end{aligned} \tag{34}$$

$$\begin{aligned}u_2(x,t)&=\chi_2^*u_1+hI_t^{\beta}\left[\mathscr{D}_t^{\beta}u_1-u_{1xx}+e^{-x}u_1-e^{-x}u_{1x}\right]\\&=-(n+h)he^x\frac{t^{\beta}}{\Gamma(1+\beta)}+h^2e^x\frac{t^{2\beta}}{\Gamma(1+2\beta)}\end{aligned} \tag{35}$$

$$\begin{aligned}u_3(x,t)&=\chi_2^*u_2+hI_t^{\beta}\left[\mathscr{D}_t^{\beta}u_2-u_{2xx}+e^{-x}u_2-e^{-x}u_{2x}\right]\\&=-(n+h)^2he^x\frac{t^{\beta}}{\Gamma(1+\beta)}+2(n+h)h^2e^x\frac{t^{2\beta}}{\Gamma(1+2\beta)}\\&\quad-h^3e^x\frac{t^{3\beta}}{\Gamma(1+3\beta)}.\end{aligned} \tag{36}$$

Then the series solution to equation (33) obtained from q-HAM method is

$$
\begin{aligned}
u(x,t;n;h) &= e^x + \sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \\
&= e^x - he^x \frac{t^\beta}{n\Gamma(1+\beta)} - (n+h)he^x \frac{t^\beta}{n^2\Gamma(1+\beta)} + h^2e^x \frac{t^{2\beta}}{n^2\Gamma(1+2\beta)} \\
&\quad -(n+h)^2he^x \frac{t^\beta}{n^3\Gamma(1+\beta)} + 2(n+h)h^2e^x \frac{t^{2\beta}}{n^3\Gamma(1+2\beta)} \\
&\quad -h^3e^x \frac{t^{3\beta}}{n^3\Gamma(1+3\beta)} + \cdots .
\end{aligned} \tag{37}
$$

Also, equation (37) is an appropriate solution to the problem (33) and is given in terms of convergence parameter h and n.

Case I: When $\alpha = 0$ and $n = 1$, we choose $h = -1$ and obtain a closed form solution in form of Mittag-Leffler function of one parameters

$$u(x,t) = e^x \left(1 + \frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{2\beta}}{\Gamma(1+2\beta)} + \cdots\right) = e^x E_\beta(t^\beta). \tag{38}$$

For this particular case, our result is in perfect agreement with the one obtained in [13] by GDTM.

The exact solution is obtained by our method for classical case when $\beta = 1$ as

$$u(x,t) = e^x E_1(t) = e^{x+t}. \tag{39}$$

5 Example Involving Nonlinear Fractional Reaction Term

5.1 Analytical Result

In this section, we consider equation (24) with nonlinear fractional reaction given as

$$
(P_1') \begin{cases} \dfrac{\partial^\beta u(x,t)}{\partial t^\beta} = \dfrac{\partial^2 u(x,t)}{\partial x^2} - x^{-\gamma}\dfrac{\partial^\gamma}{\partial x^\gamma}\left((u(x,t))^2\right), \quad 0 < \beta, \gamma \leq 1, x > 0, t > 0, \\ u(x,0) = x^a, \quad a > 0. \end{cases} \tag{40}
$$

Some adjustment in the q-HAM algorithm given in section(3) is required here due to the nonlinear term present. So,

$$\mathscr{R}_m(\mathbf{u}_{m-1}) = \mathscr{D}_t^\beta u_{m-1} - u_{(m-1)xx} + x^{-\gamma}\frac{\partial^\gamma}{\partial x^\gamma}\left[\sum_{k=0}^{m-1} u_k u_{m-1-k}\right]. \tag{41}$$

Using initial approximation $u_0(x,t) = x^a$, we obtain components of the solution using q-HAM recurrent relation in (23) successively with the new $\mathscr{R}_m$ as follows

$$\begin{aligned} u_1(x,t) &= \chi_1^* u_0 + hI_t^\beta \left[\mathscr{D}_t^\beta u_0 - u_{0xx} + x^{-\gamma} \frac{\partial^\gamma}{\partial x^\gamma} \left(u_0^2\right) \right] \\ &= -a(a-1)hx^{a-2} \frac{t^\beta}{\Gamma(1+\beta)} + \frac{h\Gamma(2a+1)x^{2a}}{\Gamma(2a+\gamma+1)} \frac{t^\beta}{\Gamma(1+\beta)} \end{aligned} \tag{42}$$

$$\begin{aligned} u_2(x,t) &= \chi_2^* u_1 + hI_t^\beta \left[\mathscr{D}_t^\beta u_1 - u_{1xx} + 2x^{-\gamma} \frac{\partial^\gamma}{\partial x^\gamma} (u_0 u_1) \right] \\ &= -a(a-1)(n+h)hx^{a-2} \frac{t^\beta}{\Gamma(1+\beta)} + \frac{(n+h)h\Gamma(2a+1)x^{2a}}{\Gamma(2a+\gamma+1)} \frac{t^\beta}{\Gamma(1+\beta)} \\ &\quad + a(a-1)(a-2)(a-3)h^2 x^{a-4} \frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad + \frac{2a(2a-1)\Gamma(2a+1)h^2 x^{2a-2}}{\Gamma(2a+\gamma+1)\Gamma(3a+\gamma+1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad - \frac{2a(a-1)\Gamma(2a-1)h^2 x^{2a-2}}{\Gamma(2a+\gamma-1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad + \frac{2h^2\Gamma(2a+1)\Gamma(3a+1)x^{3a}}{\Gamma(2a+\gamma+1)\Gamma(3a+\gamma+1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)}. \end{aligned} \tag{43}$$

Following the procedure, we can obtain $u_m(x,t)$ for $m = 3, 4, 5, \cdots$ using Mathematica software.

Hence the series solution to equation (40) obtained by q-HAM is written as

$$\begin{aligned} u(x,t;n;h) &= x^a + \sum_{i=1}^{\infty} u_i(x,t;n;h) \left(\frac{1}{n}\right)^i \\ &= x^a - a(a-1)hx^{a-2} \frac{t^\beta}{n\Gamma(1+\beta)} + \frac{h\Gamma(2a+1)x^{2a}}{n\Gamma(2a+\gamma+1)} \frac{t^\beta}{\Gamma(1+\beta)} \\ &\quad - a(a-1)(n+h)hx^{a-2} \frac{t^\beta}{n^2\Gamma(1+\beta)} + \frac{(n+h)h\Gamma(2a+1)x^{2a}}{n^2\Gamma(2a+\gamma+1)} \frac{t^\beta}{\Gamma(1+\beta)} \\ &\quad + a(a-1)(a-2)(a-3)h^2 x^{a-4} \frac{t^{2\beta}}{n^2\Gamma(1+2\beta)} \\ &\quad + \frac{2a(2a-1)\Gamma(2a+1)h^2 x^{2a-2}}{n^2\Gamma(2a+\gamma+1)\Gamma(3a+\gamma+1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad - \frac{2a(a-1)\Gamma(2a-1)h^2 x^{2a-2}}{n^2\Gamma(2a+\gamma-1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad + \frac{2h^2\Gamma(2a+1)\Gamma(3a+1)x^{3a}}{n^2\Gamma(2a+\gamma+1)\Gamma(3a+\gamma+1)} \frac{t^{2\beta}}{\Gamma(1+2\beta)} + \cdots. \end{aligned} \tag{44}$$

Equation (44) is the solution to the problem (40) in terms of convergence parameter h and n.

5.2 Numerical Results

Though a closed form is not obtained here for the case where nonlinear fractional reaction term is involved, this subsection is devoted to showing that the series solution obtained in subsection (5.1) is a good approximate solution to equation (40). Figure (1), Figure (2) and Figure (3) are plotted with $a = 2$, $\beta = \gamma = 1$, $h = -0.0001$, and $n = 1$

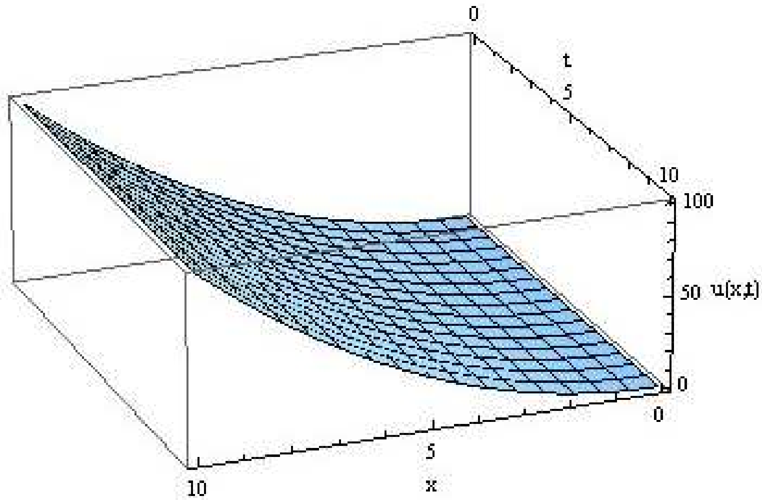

Fig. 1: q-HAM solution plot for U_0

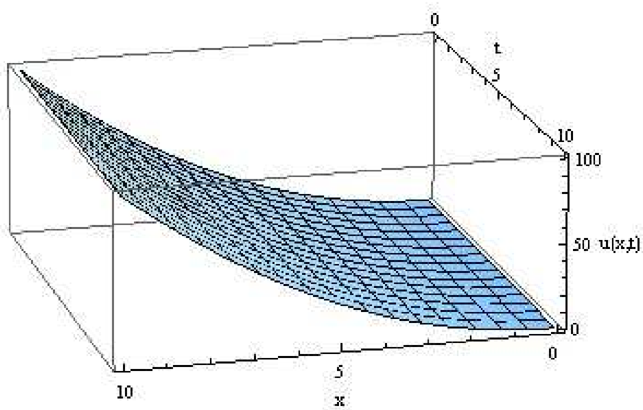

Fig. 2: q-HAM solution plot for U_1

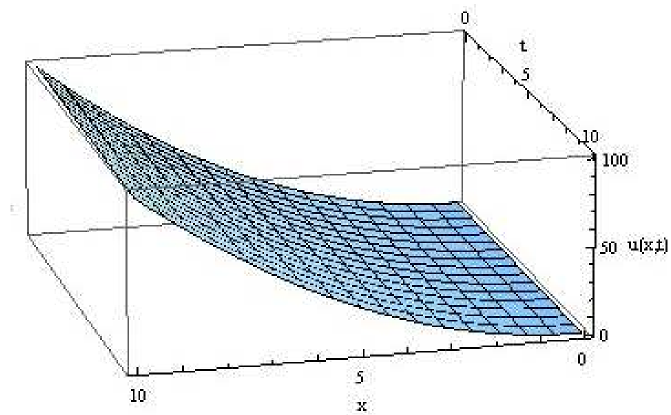

Fig. 3: q-HAM solution plot for U_2

*Remark.*The effects of different β and γ on the solution obtained are displayed in Figure (4) and Figure (5).

*Remark.*Our choice of h is not by chance but the help of h-curve given in Figure (6) using horizontal line test.

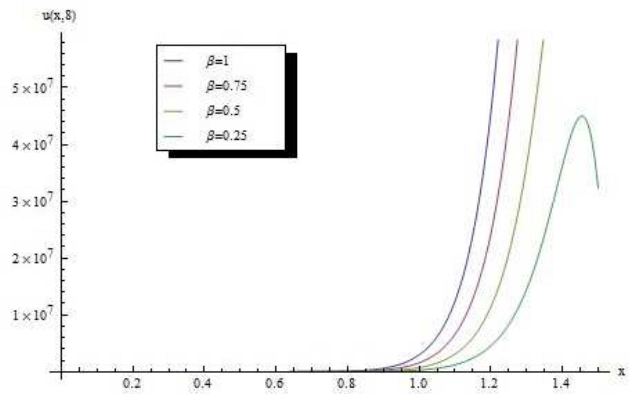

Fig. 4: U_2 plot for different β with $a = 19$, $\gamma = 1$, $h = -1.0$ and $n = 1$

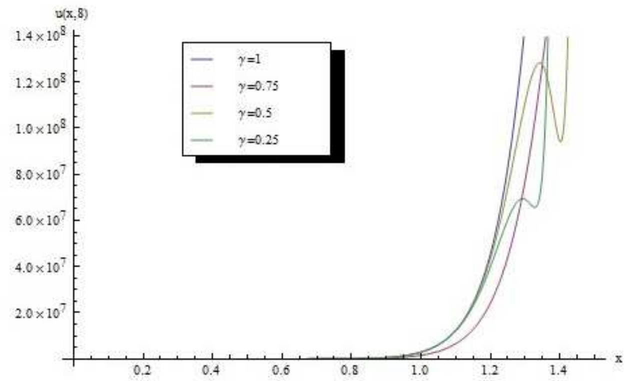

Fig. 5: U_2 plot for different γ with $a = 19$, $\beta = 1$, $h = -1.0$ and $n = 1$

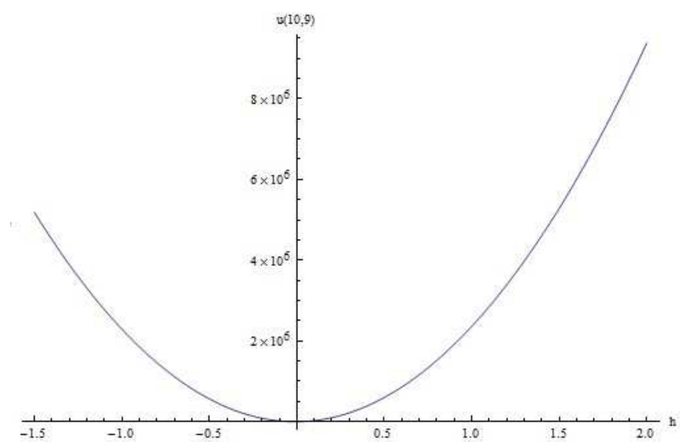

Fig. 6: h-Curve with $a = 2$, $\beta = \gamma = 1$ and $n = 1$

6 Conclusions

Solving mathematical models such as reaction-diffusion with significant applications into Physics and other field of studies is usually integral part of modern research. In a very elegant way, this paper has obtained solutions to reaction-diffusion with both linear and non-linear fractional reaction terms. Closed form solution in the form of Mittag-Leffler function is obtained in generally for the linear case including exact solutions for special cases. For the nonlinear case, a convergent series solution is obtained and the effect of the fractional orders are shown.

The need for a Lagrange multiplier in the case of VIM, and long calculations often required with the Adomian polynomials, and the assumption required in the generalized differential transform methods, are avoided in q-HAM. These considerations gives q-HAM significant advantage in many problems, and may therefore become popular among applied mathematicians and scientists in many different fields where it is applicable. Here, we have demonstrated the power of q-HAM method in obtaining solutions of the non-linear fractional reaction-diffusion equation with fractional reaction term in several cases. With appropriate choices of auxiliary parameter h and the fraction factor $\frac{1}{n}$ this method enables us to obtain solutions in the form of recurrence relations for initial value problems often quickly with little effort as compared with other methods. The fact that q-HAM can be used to solve non-linear systems with equal ease broadens its applicability and general usefulness significantly.

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