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Relative Strongly h-Convex Functions and Integral Inequalities

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Abstract: In this work we introduce the class of functions relative strongly h-convex functions and we show inequalities of Hermite-Hadamard- Fejér type.

Keywords: h-convex functions, Hermite-Hadamard, Fejér

1 Introduction

It is well known [12] that modern analysis directly or indirectly involve the applications of convexity.

Several generalizations have been introduced in recent years and extensions of the classical notion of convex function and in the theory of inequalities are produced important contributions in this regard. This research deals with some inequalities related to the renowned works, on classical convexity, of Charles Hermite [5], Jaques Hadamard [4] and Lipót Fejér [3]. The inequalities of Hermite-Hadamard and Fejér have been object of intense investigation and have produced many applications. In this paper we establish the notion of relative strongly h-convex function, properties and some results related with these inequalities mentioned above. The Hermite-Hadamard inequality gives us a estimate of the (integral) mean value of a convex function; more precisely:

Theorem 1([4]). *Let f be a convex function on $[a,b]$, with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

In [3], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1) as follows:

Theorem 2. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and let $a,b \in I$ with $a<b$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx &\le \int_a^b f(x)p(x)dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \end{aligned} \tag{2}$$

where $p:[a,b] \to \mathbb{R}$ is non negative, integrable and symmetric with respect to $(a+b)/2$.

2 Preliminaries

In [11] Noor introduced and studied a new class of convex set and convex function with respect to an arbitrary function; which are called relative convex set and relative convex function respectively, as follows.

Let K be a nonempty closed set in a real Hilbert spaces H.

Definition 1([11]). *Let Kg be any set in H. The set Kg is said to be relative convex (g-convex) with respect to an arbitrary function $g: H \to H$ such that*
$(1-t)u+tg(v) \in Kg, \quad \forall u,v \in H : u, g(v) \in Kg, \quad t \in [0,1]$.

Note that every convex set is relative convex, but the converse is not true.

Definition 2([11]). *A function $f: Kg \to H$ is said to be relative convex, if there exists an arbitrary function $g: H \to H$ such that*

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$$f((1-t)u+tg(v)) \leq (1-t)f(u)+tf(g(v))$$

for all $u,v \in H : u,g(v) \in Kg$ *and* $t \in [0,1]$.

Clearly every convex function is relative convex, but the converse is not true. The reader interested in the relative convex functions can consult the references [9, 11]. In [10] Noor established some Hadamard's type inequality for relative convex functions as follows:

Theorem 3([10]). *Let* $f : Kg = [a,g(b)] \to \mathbb{R}$ *be a relative convex function. Then, we have*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{(g(b)-a)}\int_a^{g(b)} f(x)\,dx \leq \frac{f(a)+f(g(b))}{2}.$$

Noor in [8] introduced the class of relative h-convex functions and also discussed some special cases, in addition established some Hermite-Hadamard type inequalities related to relative h-convex functions.

Definition 3([8]). *A function* $f : Kg \to H$ *is said to be relative h-convex function with respect to two functions* $h : [0,1] \to (0,+\infty)$ *and* $g : H \to H$ *such that* Kg *is a relative convex set, if*

$$f((1-t)u+tg(v)) \leq h(1-t)f(u)+h(t)f(g(v))$$

$\forall u,v \in H : u,g(v) \in Kg, \quad t \in (0,1)$.

Theorem 4([8]). *Let* $f : Kg \to \mathbb{R}$ *be a relative h-convex function, such that* $h(\frac{1}{2}) \neq 0$*, then, we obtain*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{(g(b)-a)}\int_a^{g(b)} f(x)\,dx \leq [f(a)+f(g(b))]\int_0^1 h(t)dt.$$

Strongly convex functions have been introduced by Polyak in [13]. Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. These functions play an important role in optimization theory and mathematical economics ([7, 14]). In [1] H. Angulo, J. Giménez, A. Moros and K. Nikodem established some Hadamard's Type inequality for strongly h-convex functions, this result generalizes the Hermite-Hadamard-type inequalities obtained by N. Merentes and K. Nikodem in [6] for strongly convex functions, as follows:

Definition 4. *Let* $(X, \|\cdot\|)$ *be a real normed space,* D *stands for a convex subset of* X*,* $h : (0,1) \to (0,\infty)$ *is a given function and* c *is a positive constant. We say that a function* $f : D \to \mathbb{R}$ *is* strongly h-convex with module c *if*

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y)-ct(1-t)\|x-y\|^2 \tag{3}$$

for all $x,y \in D$ *and* $t \in (0,1)$.

Theorem 5. *Let* $h : (0,1) \to (0,\infty)$ *be a given function. If a function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is Lebesgue integrable and strongly h-convex with module* $c > 0$*, then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq (f(a)+f(b))\int_0^1 h(t)dt - \frac{c}{6}(b-a)^2, \end{aligned} \tag{4}$$

for all $a,b \in I$*,* $a < b$.

3 Main results

In this section, we present the class of relative strongly h-convex functions and discuss some important properties, in addition discuss some Hermite-Hadamard-Fejér type inequalities related to relative strongly h-convex functions.

Definition 5. *A function* $f : Kg \to H$ *is said to be relative strongly h-convex function with module* $c > 0$ *with respect to two functions* $h : [0,1] \to (0,+\infty)$ *and* $g : H \to H$ *such that* Kg *is a relative convex set, if*

$$\begin{aligned} &f((1-t)u+tg(v)) \\ &\leq h(1-t)f(u)+h(t)f(g(v))-ct(1-t)\|u-g(v)\|^2, \end{aligned} \tag{5}$$

$\forall u,v \in H : u,g(v) \in Kg, \quad t \in (0,1)$.

Remark.
1. If we take $h(t)=t$ in (5), then we have the definition of relative strongly convex function with module c.
2. If we take $h(t)=t^s$ in (5), then the definition of relative strongly h-convex function with module c reduces to the definition of relative strongly s-convex function with module c.
3. If we take $h(t)=t^{-1}$ in (5), then the definition of relative strongly h-convex function with module c reduces to the definition of relative strongly Godunova-Levin function with module c.
4. If we take $h(t)=1$ in (5), then we have the definition of relative strongly P-convex function with module c.
5. If we take $g(x)=x$ in (5), then we have the definition of strongly h-convex function.
6. If we take $g(x)=x$, $h(t)=t$ in (5), then we have the definition of strongly convex function with module c.

We will present some properties for the class of relative strongly h-convex function.

Theorem 6. *Let $h_i : [0,1] \to (0,+\infty)$, $i = 1,2$ be any two functions, $\alpha \geq 0$. If $f_i : Kg \to H$, is relative strongly h_i-convex function with module $c_i > 0$, then*

(a) $f_1 + f_2$ is strongly h-convex function with module $c > 0$ where $h = max\{h_1, h_2\}$ y $c = c_1 + c_2$.
(b) αf_1 is relative strongly h_1-convex function with module c where $c = \alpha c_1$.

Proof. (a). Since each $f_i : Kg \to H$ is relative strongly h_i-convex function with module c_i, then $\forall u, v \in H : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&(f_1+f_2)((1-t)u+tg(v))\\
&= f_1((1-t)u+tg(v)) + f_2((1-t)u+tg(v))\\
&\leq h_1(1-t)f(u) + h_1(t)f(g(v)) - c_1t(1-t)\|u-g(v)\|^2\\
&\quad + h_2(1-t)f(u) + h_2(t)f(g(v)) - c_2t(1-t)\|u-g(v)\|^2\\
&\leq h(1-t)(f_1+f_2)(u) + h(t)(f_1+f_2)(g(v))\\
&\quad -(c_1+c_2)(t(1-t))\|u-g(v)\|^2\\
&\leq h(1-t)(f_1+f_2)(u) + h(t)(f_1+f_2)(g(v))\\
&\quad -c(t(1-t))\|u-g(v)\|^2
\end{aligned}$$

where $h = max\{h_1, h_2\}$ and $c = c_1 + c_2$.

(b). Let $\alpha \in \mathbb{R}$. As $f_1 : Kg \to H$ is relative strongly h_1-convex function with module c_1 $\forall u, v \in H : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&(\alpha f_1)((1-t)u+tg(v))\\
&= \alpha f_1((1-t)u+tg(v))\\
&\leq \alpha\left(h_1(1-t)f(u) + h_1(t)f(g(v)) - c_1t(1-t)\|u-g(v)\|^2\right)\\
&\leq h_1(1-t)\alpha f(u) + h_1(t)\alpha f(g(v)) - \alpha c_1 t(1-t)\|u-g(v)\|^2.
\end{aligned}$$

Therefore αf_1 is relative strongly h_1-convex function with module c where $c = \alpha c_1$.

Proposition 1. *If $f : Kg \to H$, is relative strongly convex function with module $c > 0$ and $h : [0,1] \to (0,+\infty)$, $h(t) \geq t$, then f is relative strongly h-convex function with module c.*

Proof. Given that f is relative strongly convex function then $\forall u, v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&f((1-t)u+tg(v))\\
&\leq (1-t)f(u) + tf(g(v)) - ct(1-t)\|u-g(v)\|^2\\
&\leq h(1-t)f(u) + h(t)f(g(v)) - ct(1-t)\|u-g(v)\|^2.
\end{aligned}$$

Therefore f is relative strongly h-convex function with module c.

Proposition 2. *If $f : Kg \to H$, is relative strongly h-convex function with module c and $h : [0,1] \to (0,+\infty)$, $h(t) \leq t$, then f is relative strongly convex function with module $c > 0$.*

Proof. Since f is relative strongly h-convex function with module c then $\forall u, v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&f((1-t)u+tg(v))\\
&\leq h(1-t)f(u) + h(t)f(g(v)) - ct(1-t)\|u-g(v)\|^2\\
&\leq (1-t)f(u) + tf(g(v)) - ct(1-t)\|u-g(v)\|^2.
\end{aligned}$$

Therefore f is relative strongly convex function with module $c > 0$.

Proposition 3. *Let $h_i : [0,1] \to (0,+\infty)$, $i = 1,2$ be any function such that $h_1(t) \leq h_2(t)$ for $t \in [0,1]$. If $f : Kg \to H$ is relative strongly h_1-convex function with module c_1 then f is relative strongly h_2-convex function with module c with $0 < c_2 \leq c$.*

Proof. Given that f is relative strongly h_1-convex function with module c_1 then $\forall u, v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&f((1-t)u+tg(v))\\
&\leq h_1(1-t)f(u) + h_1(t)f(g(v)) - c_1t(1-t)\|u-g(v)\|^2\\
&\leq h_2(1-t)f(u) + h_2(t)f(g(v)) - ct(1-t)\|u-g(v)\|^2.
\end{aligned}$$

Therefore f is relative strongly h_2-convex function with module c with $0 < c_2 \leq c$.

Proposition 4. *If $f_n : Kg \to H$, is a sequence of functions which pointwise converge to $f : Kg \to H$ and $h_n : [0,1] \to (0,+\infty)$, is a sequence of functions which pointwise converge to $h : [0,1] \to (0,+\infty)$ so there is a $k > 0$ such that f_n is relative strongly h_n-convex function with module c_n for $n \geq k$, then f is relative strongly h-convex function with module c, where $c = \lim\limits_{n\to+\infty} c_n$.*

Proof. As each f_n is relative strongly h_n-convex function with module c_n then $\forall u, v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned}
&f((1-t)u+tg(v))\\
&= \lim_{n\to+\infty} f_n((1-t)u+tg(v))\\
&\leq \lim_{n\to+\infty}\left(h_n(1-t)f_n(u) + h_n(t)f_n(g(v)) - c_nt(1-t)\|u-g(v)\|^2\right)\\
&\leq h(1-t)f(u) + h(t)f(g(v)) - ct(1-t)\|u-g(v)\|^2.
\end{aligned}$$

Therefore f is relative strongly h-convex function with module c, where $c = \lim\limits_{n\to+\infty} c_n$.

Proposition 5. *Let $f_i : Kg \to H$,$g : H \to H$, $h_i : [0,1] \to (0,+\infty)$, be with $i = 1,2$. If f_i is relative strongly h_i-convex function with module c_i with $i = 1,2$ then $f(x) = max\{f_1(x), f_2(x)\}$ is relative strongly h-convex function with module c, where $h(t) = max\{h_1(t), h_2(t)\}$.*

*Proof.*Since each f_i is relative strongly h_i-convex function with module c_i then $\forall u,v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we have

$$\begin{aligned} & f_i((1-t)u+tg(v)) \\ \le\ & h_i(1-t)f_i(u)+h_i(t)f_i(g(v))-c_it(1-t)\|u-g(v)\|^2 \\ \le\ & h(1-t)f(u)+h(t)f(g(v))-ct(1-t)\|u-g(v)\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} & f((1-t)u+tg(v)) \\ =\ & max\{f_1((1-t)u+tg(v)), f_2((1-t)u+tg(v))\} \\ \le\ & h(1-t)f(u)+h(t)f(g(v))-ct(1-t)\|u-g(v)\|^2. \end{aligned}$$

Therefore $f(x) = max\{f_1(x), f_2(x)\}$ is relative strongly h-convex function with module c, where $h(t) = max\{h_1(t), h_2(t)\}$.

Proposition 6.*Let $h : [0,1] \to (0,+\infty)$ be a given function. If $g : H \to H$, is invertible and $f : Kg \to H$ is relative strongly h-convex function with module c, then f is strongly h-convex function with module c.*

*Proof.*Since f is relative strongly h-convex function with module c then $\forall u,v \in Kg : u, g(v) \in Kg$ and $t \in (0,1)$ we get

$$\begin{aligned} & f((1-t)u+tv) \\ =\ & f((1-t)u+t(g(g^{-1}(v)))) \\ \le\ & h(1-t)f(u)+h(t)f(g(g^{-1}(v)))-c_1t(1-t)\|u-g(g^{-1}(v))\|^2 \\ \le\ & h(1-t)f(u)+h(t)f(v)-ct(1-t)\|u-(v)\|^2. \end{aligned}$$

Therefore f is strongly h-convex function with module c.

Note that the previous theorem shows us that if $g : H \to H$, is invertible then the set of the relative strongly h-convex functions with module c is contained in the set of the strongly h-convex functions with module c.

Proposition 7.*If $f : [a, g(b)] \to \mathbb{R}$ is a relative strongly h-convex function with module c and $h : [0,1] \to (0,+\infty)$ is an upper bounded function then f is an upper bounded function.*

*Proof.*For any $x = (1-t)a+tg(b) \in [a, g(b)]$ we obtain

$$\begin{aligned} f(x) &\le h(1-t)f(a)+h(t)f(g(b))-c_1t(1-t)(a-g(b))^2 \\ &\le Mf(a)+Mf(g(b))-ct(1-t)(a-g(b))^2 \\ &\le M(f(a)+f(g(b))). \end{aligned}$$

Therefore f is an upper bounded function.

Theorem 7.*Let $h : (0,1) \to (0,\infty)$ be a given function. If a function $f : I \to \mathbb{R}$ is Lebesgue integrable and relative strongly h-convex with module $c > 0$, then*

$$\begin{aligned} & \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \\ \le\ & (f(a)+f(g(b)))\int_0^1 h(t)dt - \frac{c}{6}(g(b)-a)^2, \end{aligned} \tag{6}$$

for all $a, g(b) \in I$, $a < g(b)$.

*Proof.*Take $(1-t)a+tg(b) \in [a, g(b)]$. Then, the relative strong h-convexity of f implies

$$\begin{aligned} & f((1-t)a+tg(b)) \\ \le\ & h(1-t)f(a)+h(t)f(g(b))-ct(1-t)(a-g(b))^2. \end{aligned}$$

Integrating over the interval $(0,1)$, we get

$$\begin{aligned} & \int_0^1 f((1-t)a+tg(b))dt \\ \le\ & \int_0^1 \left(h(1-t)f(a)+h(t)f(g(b))-ct(1-t)(a-g(b))^2\right)dt \\ \le\ & f(a)\int_0^1 h(1-t)dt + f(g(b))\int_0^1 h(t)dt \\ & -c\|a-g(b)\|^2\int_0^1 t(1-t)dt. \end{aligned}$$

By a simple calculation and using the change of the variable, we obtain

$$\begin{aligned} & \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \\ \le\ & (f(a)+f(g(b)))\int_0^1 h(t)dt - \frac{c}{6}(g(b)-a)^2. \end{aligned} \tag{7}$$

*Remark.*If $g : \mathbb{R} \to \mathbb{R}$ is right-invertible, $f : K_g = [a, g(b)] \to \mathbb{R}$ is a relative strongly h-convex function and f is Lebesgue integrable then f is strongly h-convex and we get the following result is a counterpart of the Hermite-Hadamard inequalities for relative strongly h-convex functions.

$$\begin{aligned} & \frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+g(b)}{2}\right)+\frac{c}{12}(g(b)-a)^2\right] \\ \le\ & \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \\ \le\ & (f(a)+f(g(b)))\int_0^1 h(t)dt - \frac{c}{6}(g(b)-a)^2, \end{aligned} \tag{8}$$

and when g is the identity function then the result (8) coincides with the Theorem 4.1 in [1].

4 A refinement of the Hermite-Hadamard type inequalities

In this section we present a refinement of the right-hand side of the Hermite-Hadamard type inequalities (8) for relative strongly h-convex functions. A similar result for strongly convex functions can be found in [2, Theorem 5].

Theorem 8.*Let $h : (0,1) \to (0,\infty)$ be a given function. If a function $f : I \to \mathbb{R}$ is Lebesgue integrable and relative* *strongly h-convex with module* $c>0$, *then*

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\\ &\le \frac{\left(1+2h\left(\frac{1}{2}\right)\right)}{2}(f(a)+f(g(b)))\int_0^1 h(t)dt\\ &-\left[\frac{1}{4}\int_0^1 h(t)dt+\frac{1}{24}\right]c(g(b)-a)^2, \qquad (9)\end{aligned}$$

for all $a,g(b)\in I$, $a<g(b)$.

*Proof.*Applying the Theorem 7 in the intervals $\left[a,\frac{a+g(b)}{2}\right]$ and $\left[\frac{a+g(b)}{2},g(b)\right]$ we obtain

$$\begin{aligned}&\frac{2}{g(b)-a}\int_a^{\frac{a+g(b)}{2}} f(x)dx\\ &\le \left(f(a)+f\left(\frac{a+g(b)}{2}\right)\right)\int_0^1 h(t)dt-\frac{c}{6}\frac{(g(b)-a)^2}{4}, \qquad (10)\end{aligned}$$

and

$$\begin{aligned}&\frac{2}{g(b)-a}\int_{\frac{a+g(b)}{2}}^{g(b)} f(x)dx\\ &\le \left(f\left(\frac{a+g(b)}{2}\right)+f(g(b))\right)\int_0^1 h(t)dt-\frac{c}{6}\frac{(g(b)-a)^2}{4}. \qquad (11)\end{aligned}$$

Summing up these inequalities we get

$$\begin{aligned}&\frac{2}{g(b)-a}\int_a^{g(b)} f(x)dx\\ &\le \left(f(a)+2f\left(\frac{a+g(b)}{2}\right)+f(g(b))\right)\int_0^1 h(t)dt-\frac{2c}{6}\frac{(g(b)-a)^2}{4}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\\ &\le \frac{\left(f(a)+2f\left(\frac{a+g(b)}{2}\right)+f(g(b))\right)}{2}\int_0^1 h(t)dt-\frac{c}{6}\frac{(g(b)-a)^2}{4}.\end{aligned}$$

Now, using the relative strong h-convexity of f, we obtain

$$\begin{aligned}&f\left(\frac{a+g(b)}{2}\right)\\ &\le h\left(\frac{1}{2}\right)f(a)+h\left(\frac{1}{2}\right)f(g(b))-\frac{c}{4}(a-g(b))^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\\ &\le \frac{\left(1+2h\left(\frac{1}{2}\right)\right)}{2}(f(a)+f(g(b)))\int_0^1 h(t)dt\\ &-\frac{c}{4}(a-g(b))^2\int_0^1 h(t)dt-\frac{c}{6}\frac{(g(b)-a)^2}{4}\\ &=\frac{\left(1+2h\left(\frac{1}{2}\right)\right)}{2}(f(a)+f(g(b)))\int_0^1 h(t)dt\\ &-\left[\frac{1}{4}\int_0^1 h(t)dt+\frac{1}{24}\right]c(g(b)-a)^2.\end{aligned}$$

Corollary 1.*Under the same hypotheses of theorem 8, if* $h(\frac{1}{2})\le\frac{1}{2}$ *and* $\int_0^1 h(t)dt\ge\frac{1}{2}$ *we get*

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\\ &\le \frac{\left(1+2h\left(\frac{1}{2}\right)\right)}{2}(f(a)+f(g(b)))\int_0^1 h(t)dt-\\ &\left[\frac{1}{4}\int_0^1 h(t)dt+\frac{1}{24}\right]c(g(b)-a)^2\\ &\le (f(a)+f(g(b)))\int_0^1 h(t)dt-\frac{c}{6}(g(b)-a)^2.\end{aligned}$$

Corollary 2.*If we take* $g(b)=b$, *then we get the right-hand side of the inequality given in [1].*

Remark. 1.If we take $c=0$ and $h(\frac{1}{2})=\frac{1}{2}$ in the Theorem 9, then we have the right-hand side of the inequality given in [8, Theorem 16].

2.If we take $h(t)=t^s$ with $s\in[0,1]$ in Corollary 1, then we obtain

$$\int_0^1 t^s dt=\frac{1}{s+1}\ge\frac{1}{2}\Leftrightarrow 0\le s\le 1,$$

and

$$\begin{aligned}h\left(\frac{1}{2}\right)\le\frac{1}{2} \quad&\Leftrightarrow\quad \frac{1}{2^s}\le\frac{1}{2}\\ &\Leftrightarrow\quad 2\le 2^s\\ &\Leftrightarrow\quad s\ge 1\end{aligned}$$

thus, the theorem is valid only for $s=1$.

3.If we take $h(t)=t$ for $t\in(0,1)$ then the inequalities in the Corollary 1 reduce to

$$\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\le\frac{(f(a)+f(g(b)))}{2}-\frac{c}{6}(g(b)-a)^2,$$

these is the hermite-Hadamard type inequalities for relative strongly convex functions.

5 Fejér type inequalities

Now we will present a bound for the right hand side of (2). First, we prove the following result which is similar to Lemma 1 in [15].

Lemma 1.*If* $f:[0,\infty)\to\mathbb{R}$ *is a relative strongly h-convex function, with module* $c>0$, *then, for all* $x\in[a,g(b)]\subset[0,\infty)$ *there is* $\alpha_x\in[0,1]$ *such that*

$$\begin{aligned}&f(a+g(b)-x)\\ &\le h(1-\alpha_x)f(a)+h(\alpha_x)f(g(b))-c(x-a)(g(b)-x). \qquad (12)\end{aligned}$$

where $\alpha_x=\dfrac{x-a}{g(b)-a}$ *and* $1-\alpha_x=\dfrac{g(b)-x}{g(b)-a}$.

Proof. Since any $x \in [a, g(b)]$ can be written as

$$x = \alpha_x a + (1-\alpha_x) g(b),$$

for some $\alpha_x \in [0,1]$, where $\alpha_x = \dfrac{x-a}{g(b)-a}$, $1-\alpha_x = \dfrac{g(b)-x}{g(b)-a}$ and

$$a+g(b)-x = a+g(b)-\alpha_x a-(1-\alpha_x)g(b) = (1-\alpha_x)a+\alpha_x g(b),$$

we get

$$\begin{aligned} & f(a+g(b)-x) \\ = & f(a+g(b)-\alpha_x a-(1-\alpha_x)g(b)) \\ = & f((1-\alpha_x)a+\alpha_x g(b)) \\ \leq & h(1-\alpha_x)f(a)+h(\alpha_x)f(g(b))-c(x-a)(g(b)-x). \end{aligned}$$

The proof is completed.

Theorem 9. *Let $f:[0,\infty)\to\mathbb{R}$ be a relative strongly h-convex function with module $c>0$, which is integrable in $[a,g(b)]$, where $a, g(b)\in[0,\infty)$, $a<g(b)$, and let $p:[a,g(b)]\to\mathbb{R}$ be a non negative and integrable function which is symmetric with respect to $\frac{a+g(b)}{2}$, then*

$$\begin{aligned} & \int_a^{g(b)} f(x)p(x)dx \\ \leq & \int_a^{g(b)} [h(1-\alpha_x)f(a)+h(\alpha_x)f(g(b))-c(x-a)(g(b)-x)]\,p(x)dx. \end{aligned}$$

where $\alpha_x = \dfrac{x-a}{g(b)-a}$ and $1-\alpha_x = \dfrac{g(b)-x}{g(b)-a}$.

Proof. By the symmetry of p with respect to $\frac{a+g(b)}{2}$ and Lemma 1

$$\begin{aligned} & \int_a^{g(b)} f(x)p(x)dx \\ = & \frac{1}{2}\int_a^{g(b)} f(a+g(b)-x)p(a+g(b)-x)dx+\frac{1}{2}\int_a^{g(b)} f(x)p(x)dx \\ = & \frac{1}{2}\int_a^{g(b)} f(a+g(b)-x)p(x)dx+\frac{1}{2}\int_a^{g(b)} f(x)p(x)dx \\ \leq & \frac{1}{2}\int_a^{g(b)} [h(1-\alpha_x)f(a)+h(\alpha_x)f(g(b))-c(x-a)(g(b)-x)]\,p(x)dx \\ & +\frac{1}{2}\int_a^{g(b)} f(x)p(x)dx \\ \leq & \frac{f(a)}{2}\int_a^{g(b)} h(1-\alpha_x)p(x)dx+\frac{f(g(b))}{2}\int_a^{g(b)} h(\alpha_x)p(x)dx \\ & -\frac{1}{2}\int_a^{g(b)} c(x-a)(g(b)-x)p(x)dx+ \\ & \frac{1}{2}\int_a^{g(b)} f(x)p(x)dx, \end{aligned}$$

thus

$$\begin{aligned} & \frac{1}{2}\int_a^{g(b)} f(x)p(x)dx \\ = & \frac{1}{2}\int_a^{g(b)} f(a+g(b)-x)p(a+g(b)-x)dx \\ & +\frac{1}{2}\int_a^{g(b)} f(x)p(x)dx \\ \leq & \frac{f(a)}{2}\int_a^{g(b)} h(1-\alpha_x)p(x)dx+\frac{f(g(b))}{2}\int_a^{g(b)} h(\alpha_x)p(x)dx \\ & -\frac{1}{2}\int_a^{g(b)} c(x-a)(g(b)-x)p(x)dx. \end{aligned}$$

Remark. Notice that if $h(t)=1$ in Theorem 9 we, indeed, get

$$\begin{aligned} & \int_a^{g(b)} f(x)p(x)dx \\ \leq & \left(\frac{f(a)+f(g(b))}{2}\right)\int_a^{g(b)} p(x)dx \\ & -\frac{1}{2}\int_a^{g(b)} c(x-a)(g(b)-x)p(x)dx \end{aligned}$$

for relative strongly P-convex functions with module c.

We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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