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Bivariate Exponentiated Modified Weibull Extension Distribution

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Abstract: In this paper, a new bivariate exponentiated modified Weibull extension distribution (BEMWE) is introduced. This model is of Marshall-Olkin type. The marginals of the new bivariate distribution have exponentiated modified Weibull extension distribution which proposed by Sarhan et al.[1]. The joint probability density function and the joint cumulative distribution function are given in closed forms. Several properties of this distribution have been discussed. The maximum likelihood estimators of the parameters are derived. One real data set is analyzed using the new bivariate distribution, which shows that the new model can be used quite effectively in fitting and analyzing real lifetime data than the bivariate generalized Gompertz distribution (BGG) model.

Keywords: Joint probability density function, Conditional probability density function, Maximum likelihood estimators, Fisher information matrix.

1 Introduction

Recently, Sarhan et al. [1] has defined a new four-parameter distribution referred to as exponentiated modified Weibull extension (EMWE) distribution. Sarhan et al. [1] defined the (EMWE) distribution by exponentiating the new modified Weibull extension (MWE) distribution which discussed by Xie et al. [2] as was done for the exponentiated weibull (EW) distribution by Mudholkar et al. [3]. They observed that exponential distribution, generalized exponential distribution [4], Gompertz distribution [5], generalized Gompertz (GG) distribution [6], exponentiated Weibull (EW) distribution [7], Weibull extension model of Chen [8], modified Weibull extension (MWE) distribution [9] and etc distributions can be obtained as special cases of the (EMWE) distribution.

The objective of this paper is to provide a new bivariate distribution, whose marginals are (EMWE) distributions which referred to as bivariate exponentiated modified Weibull extension (BEMWE) distribution. It is obtained using a method similar to that used to obtain Marshall-Olkin bivariate exponential model Marshall and Olkin [10,11,12,13].

The paper is organized as follows. Section 2 presents the shock model yielding the (BEMWE) distribution. Also, the joint cumulative distribution function, the joint probability density function, the marginal probability density functions and the conditional probability density functions of (BEMWE) distribution are derived in Section 2. In Section 3 sum reliability studies are obtained. Section 4 presents the the marginal expectation of the (BEMWE) distribution. Section 5 obtains the parameters estimation using MLE. In section 6 a numerical results are obtained using real data. Finally, a conclusion for the results is given in Section 7.

2 Bivariate exponentiated modified Weibull extension distribution

In this section we introduce the BEMWE distribution using a similar method to that which was used by Marshall and Olkin [10]. We start with the joint cumulative function of the proposed bivariate distribution and so used it to derive the corresponding joint probability density function. Finally The marginal probability density functions and conditional probability density functions of this distribution are also derived. Let X be a random variable has univariate EMWE

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distribution with parameters $\gamma, \alpha, \beta, \lambda > 0$, then the corresponding cumulative distribution function (CDF) is given by

$$F(x) = \left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha x)^{\beta}} - 1\right)}\right]^{\gamma}, \quad x \geq 0, \tag{1}$$

and the probability density function (PDF) takes the following form

$$f(x) = \gamma\lambda\beta e^{(\alpha x)^{\beta}} (\alpha x)^{\beta-1} e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha x)^{\beta}} - 1\right)} \left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha x)^{\beta}} - 1\right)}\right]^{\gamma-1}, \quad x \geq 0. \tag{2}$$

2.1 Joint cumulative distribution function

Suppose that U_i $(i = 1,2,3)$ are three independent random variables such that U_i ∼EMWE $(\gamma_i, \alpha, \beta, \lambda)$. Define $X_1 = max\{U_1, U_3\}$ and $X_2 = max\{U_2, U_3\}$. Then, the bivariate vector (X_1, X_2) has a bivariate exponentiated modified Weibull extension distribution, with parameters $(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)$. Let us denote it by BEMWE$(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)$. The following interpretation can be provided for the BEMWE model.

Shock model: Assum thate there exists a three independent sources of shocks. Suppose these shocks are affecting a system with two components. It is assumed that the shock from source 1 reaches the system and destroys component 1 immediately, the shock from source 2 reaches the system and destroys component 2 immediately, while if the shock from source 3 hits the system it destroys both the components immediately. Let U_i denote the inter-arrival times, between the shocks in source i, $i = 1,2,3$, which follow the distribution EMWE. If X_1, X_2 denote the survival times of the components, then the bivariate vector (X_1, X_2) follows the BEMWE model.

We now study the joint cumulative distribution function of the bivariate random vector (X_1, X_2) in the following lemma.

Lemma 2.1. The joint CDF of (X_1, X_2) is given by

$$F_{BEMWE}(x_1, x_2) = \left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha x_1)^{\beta}} - 1\right)}\right]^{\gamma_1} \left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha x_2)^{\beta}} - 1\right)}\right]^{\gamma_2} \left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{(\alpha z)^{\beta}} - 1\right)}\right]^{\gamma_3}, \tag{3}$$

where $z = min(x_1, x_2)$.

proof: Since the joint CDF of the random variables X_1 and X_2 is defined as

$$\begin{aligned} F(x_1, x_2) &= P(X_1 \leq x_1, X_2 \leq x_2) \\ &= P(max\{U_1, U_3\} \leq x_1, max\{U_2, U_3\} \leq x_2) \\ &= P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq min(x_1, x_2)). \end{aligned}$$

As the random variables U_i $(i = 1,2,3)$ are mutually independent, we directly obtain

$$F_{BEMWE}(x_1, x_2) = P(U_1 \leq x_1) P(U_2 \leq x_2) P(U_3 \leq min(x_1, x_2))$$

$$= F_{EMWE}(x_1; \gamma_1, \alpha, \beta, \lambda) F_{EMWE}(x_2; \gamma_2, \alpha, \beta, \lambda) F_{EMWE}(z; \gamma_3, \alpha, \beta, \lambda). \tag{4}$$

Substituting from (1) into (4), we obtain (3), which completes the proof of the lemma 2.1.

2.2 Joint probability density function

The following theorem gives the joint pdf of the X_1 and X_2 which is the joint pdf of BEMWE $(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)$.

Theorem 2.1. If the joint CDF of X_1 and X_2 is as in (3), then the joint pdf of X_1 and X_2 takes the form

$$f_{BEMWE}(x_1,x_2)=\begin{cases} f_1(x_1,x_2) & \text{if } x_1<x_2\\ f_2(x_1,x_2) & \text{if } x_2<x_1\\ f_0(x,x) & \text{if } x_1=x_2=x\end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(x_1,x_2) &= f_{EMWE}(x_2;\gamma_2,\alpha,\beta,\lambda)f_{EMWE}(x_1;\gamma_1+\gamma_3,\alpha,\beta,\lambda)\\ &=\gamma_2(\gamma_1+\gamma_3)\lambda^2\beta^2e^{(\alpha x_2)^\beta}(\alpha x_2)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2-1}\\ &\quad\times e^{(\alpha x_1)^\beta}(\alpha x_1)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1+\gamma_3-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} f_2(x_1,x_2) &= f_{EMWE}(x_1;\gamma_1,\alpha,\beta,\lambda)f_{EMWE}(x_2;\gamma_2+\gamma_3,\alpha,\beta,\lambda)\\ &=\gamma_1(\gamma_2+\gamma_3)\lambda^2\beta^2e^{(\alpha x_1)^\beta}(\alpha x_1)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1-1}\\ &\quad\times e^{(\alpha x_2)^\beta}(\alpha x_2)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2+\gamma_3-1} \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_3(x,x) &= \frac{\gamma_3}{\gamma_1+\gamma_2+\gamma_3}f_{EMWE}(x_2;\gamma_1+\gamma_2+\gamma_3,\alpha,\beta,\lambda)\\ &=\gamma_3\lambda\beta e^{(\alpha x)^\beta}(\alpha x)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}. \end{aligned} \tag{8}$$

Proof: Let us first assume that $x_1<x_2$. Then, the expression for $f_1(x_1,x_2)$ can be simply obtained by differentiating the joint CDF $F_{BEMWE}(x_1,x_2)$ given in (3) with respect to x_1 and x_2. Similarly, we find the expression of $f_2(x_1,x_2)$ when $x_2<x_1$. But $f_3(x,x)$ can not be derived in a similar method. For this reason, we use the following identity to derive $f_3(x,x)$.

$$\int_0^\infty\int_0^{x_2}f_1(x_1,x_2)dx_1dx_2+\int_0^\infty\int_0^{x_1}f_2(x_1,x_2)dx_2dx_1+\int_0^\infty f_3(x,x)dx=1. \tag{9}$$

Let

$$I_1=\int_0^\infty\int_0^{x_2}f_1(x_1,x_2)dx_1dx_2 \quad\text{and}\quad I_2=\int_0^\infty\int_0^{x_1}f_2(x_1,x_2)dx_2dx_1.$$

One can find that

$$I_1=\int_0^\infty\gamma_2\lambda\beta e^{(\alpha x_2)^\beta}(\alpha x_2)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}dx_2 \tag{10}$$

and

$$I_2=\int_0^\infty\gamma_1\lambda\beta e^{(\alpha x_1)^\beta}(\alpha x_1)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}dx_1. \tag{11}$$

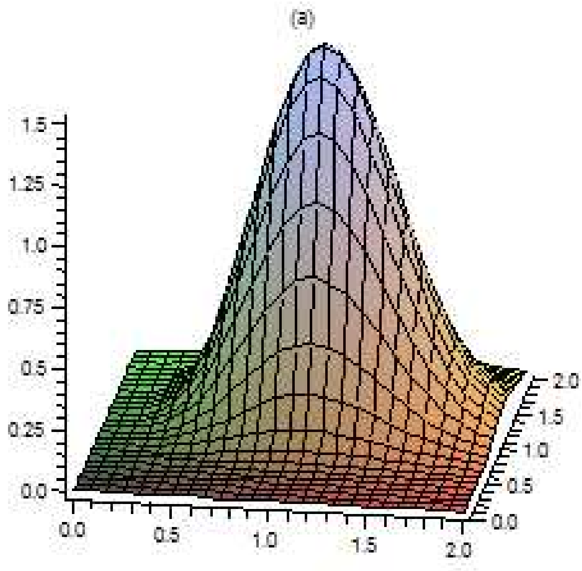
Substituting from (10) and (11) into (9) we obtain

$$\begin{aligned} \int_0^\infty f_3(x,x)dx &= 1-I_1-I_2\\ &=\int_0^\infty(\gamma_1+\gamma_2+\gamma_3)\lambda\beta e^{(\alpha x)^\beta}(\alpha x)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}dx\\ &\quad-\int_0^\infty\gamma_2\lambda\beta e^{(\alpha x)^\beta}(\alpha x)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}dx\\ &\quad-\int_0^\infty\gamma_1\lambda\beta e^{(\alpha x)^\beta}(\alpha x)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1}dx. \end{aligned}$$

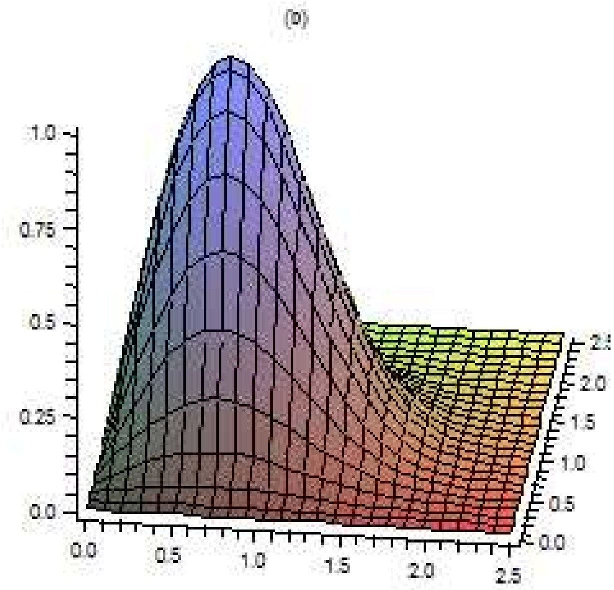
Thus,

$$f_3(x,x) = \gamma_3 \lambda \beta e^{(\alpha x)^\beta} (\alpha x)^{\beta-1} e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)} \left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3-1},$$

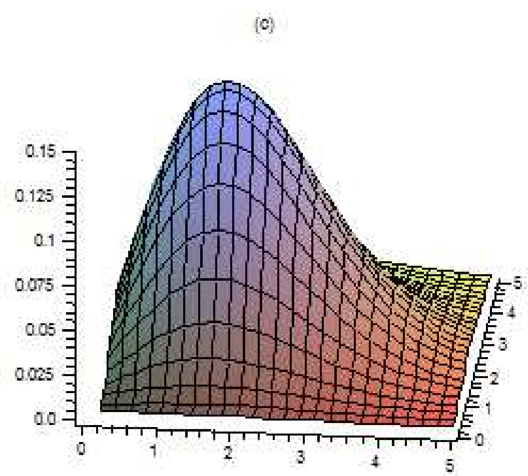
which completes the proof.



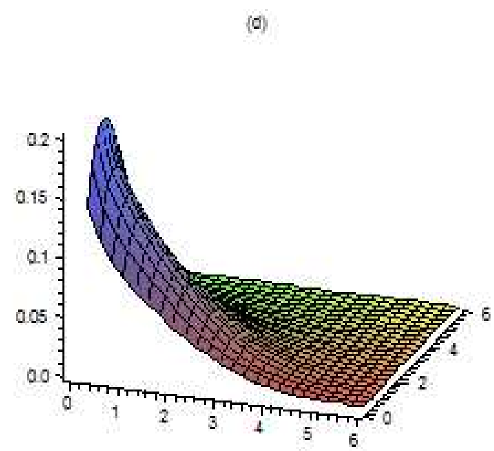
(a) $\alpha = \beta = \lambda = 1$ and $\gamma_1 = \gamma_2 = \gamma_3 = 5$.



(b) $\alpha = \beta = \lambda = 1$ and $\gamma_1 = \gamma_2 = \gamma_3 = 2$.



(c) $\alpha = 2, \beta = \lambda = 0.5$ and $\gamma_1 = \gamma_2 = \gamma_3 = 3$.



(d) $\alpha = 2, \beta = \lambda = 0.5$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$.

Fig. 1: A plot of the joint PDF of BEMWE $(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)$ given in (5), for different values of $(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)$.

From Fig. 1. We note that, the joint PDF of BEMWE can take different shapes depending on the values of its parameters.

2.3 Marginal probability density functions

The following theorem gives the marginal probability density functions of X_1 and X_2.

Theorem 2.2. The marginal probability density functions of X_i $(i=1,2)$ is given by

$$f_{X_i}(x_i) = f_{EMWE}(x_i;\gamma_i+\gamma_3,\alpha,\beta,\lambda),\quad x_i>0,\ i=1,2$$
$$= (\gamma_i+\gamma_3)\lambda\beta e^{(\alpha x_i)^\beta}(\alpha x_i)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{(\gamma_i+\gamma_3)-1}. \tag{12}$$

Proof: The marginal cumulative distribution function for X_i is

$$F(x_i) = P(X_i \le x_i) = P(max\{U_i,U_3\}\le x_i) = P(U_i\le x_i, U_3\le x_i).$$

As the random variables U_i $(i=1,2)$ and U_3 are mutually independent, we directly obtain

$$F(x_i) = P(U_i\le x_i)P(U_3\le x_i)$$
$$= \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_i}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_3}$$
$$= \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_i+\gamma_3} = F_{EMWE}(x_i;\gamma_i+\gamma_3,\alpha,\beta,\lambda). \tag{13}$$

From which we readily derive the pdf of X_i , $f(x_i)=\frac{\partial}{\partial x_i}F(x_i)$, as in (12).

2.4 Conditional probability density functions

The following theorem gives the conditional probability density functions of (X_1,X_2).

Theorem 2.3. The conditional probability density function of X_i given $X_j=x_j$, $(i,j=1,2,i\neq j)$ is given by

$$f_{X_i|X_j}(x_i\mid x_j)=\begin{cases} f^{(1)}_{X_i|X_j}(x_i\mid x_j) & \text{if } 0<x_i<x_j \\ f^{(2)}_{X_i|X_j}(x_i\mid x_j) & \text{if } 0<x_j<x_i \\ f^{(3)}_{X_i|X_j}(x_i\mid x_j) & \text{if } x_i=x_j>0 \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i\mid x_j)=\frac{\gamma_j(\gamma_i+\gamma_3)\lambda\beta e^{(\alpha x_i)^\beta}(\alpha x_i)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_i+\gamma_3-1}}{(\gamma_j+\gamma_3)\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_j)^\beta}-1)}\right]^{\gamma_i+\gamma_3-1}},$$

$$f^{(2)}_{X_i|X_j}(x_i\mid x_j)=\gamma_i\lambda\beta e^{(\alpha x_i)^\beta}(\alpha x_i)^{\beta-1}e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_i-1}$$

and

$$f^{(3)}_{X_i|X_j}(x_i\mid x_j) = \frac{\gamma_3}{\gamma_i+\gamma_3}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\right]^{\gamma_i}.$$

Proof: The proof follows immediately by substituting the joint probability density function of (X_1,X_2) given in (6), (7) and (8) and the marginal probability density function given in (12), using the relation

$$f_{X_i|X_j}(x_i\mid x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_i}(x_i)},\quad (i=1,2).$$

3 Reliability studies

In this section, we present the joint survival function of (X_1,X_2), the CDF of the random variable $Y = max\{X_1,X_2\}$ and the CDF of the random variable $W = \min\{X_1,X_2\}$.

3.1 Joint survival function

In this subsection, we derive the joint survival function of (X_1,X_2) in a compact form.

Theorem 3.1. The joint survival function of (X_1,X_2) is given by

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} S_1(x_1,x_2) & \text{if } x_1 < x_2 \\ S_2(x_1,x_2) & \text{if } x_2 < x_1 \\ S_0(x,x) & \text{if } x_1 = x_2 = x \end{cases} \tag{14}$$

where

$$S_1(x_1,x_2) = 1 - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2+\gamma_3} - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1+\gamma_3}\left(1-\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2}\right),$$

$$S_2(x_1,x_2) = 1 - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1+\gamma_3} - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2+\gamma_3}\left(1-\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1}\right)$$

and

$$S_0(x,x) = 1 - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_3}\left(\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1} + \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_2} - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x)^\beta}-1)}\right]^{\gamma_1+\gamma_2}\right).$$

Proof: The joint survival function of (X_1,X_2) can be obtained from the following relation

$$S_{X_1,X_2}(x_1,x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1,x_2). \tag{15}$$

Substituting from (3) and (13) in (15), we get

$$S_{X_1,X_2}(x_1,x_2) = 1 - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1+\gamma_3} - \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2+\gamma_3} + \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_1)^\beta}-1)}\right]^{\gamma_1}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_2)^\beta}-1)}\right]^{\gamma_2}\left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha z)^\beta}-1)}\right]^{\gamma_3}, \tag{16}$$

where $z = min(x_1,x_2)$. From (16) we can be obtained simply the expressions of $S_1(x_1,x_2)$, $S_2(x_1,x_2)$ and $S_0(x_1,x_2)$ for $x_1 < x_2$, $x_2 < x_1$ and $x_1 = x_2 = x$ respectively, which completes the proof.

Comment 3.1. Basu [14] defined the bivariate failure rate [15] function $h(x_1,x_2)$ for the random vector (X_1,X_2) as the following relation

$$h_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{S_{X_1,X_2}(x_1,x_2)}. \tag{17}$$

We can obtained the bivariate failure rate function $h(x_1,x_2)$ for the random vector (X_1,X_2) by substituting from (5) and (14) in (17).

Lemma 3.1. The CDF of the random variable $Y = max\{X_1,X_2\}$ is given as

$$F_Y(y) = \left[1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha y)^\beta}-1)}\right]^{\gamma_1+\gamma_2+\gamma_3}. \tag{18}$$

Proof: Since

$$F_Y(y) = P(Y \leq y) = P(max\{X_1, X_2\} \leq y) = P(X_1 \leq y, X_2 \leq y)$$
$$= P(max\{U_1, U_3\} \leq y, max\{U_2, U_3\} \leq y) = P(U_1 \leq y, U_2 \leq y, U_3 \leq y),$$

where the random variables U_i $(i = 1, 2, 3)$ are mutually independent, we directly obtain

$$F_Y(y) = P(U_1 \leq y)P(U_2 \leq y)P(U_3 \leq y)$$
$$= F_{EMWE}(y; \gamma_1, \alpha, \beta, \lambda) F_{EMWE}(y; \gamma_2, \alpha, \beta, \lambda) F_{EMWE}(y; \gamma_3, \alpha, \beta, \lambda). \quad (19)$$

Substituting from (1) in (19), we get (18) which completes the proof of the lemma 3.1.

Comment 3.2. From lemma 3.1. we can say that, if X_1 and X_2 are independent EMWE random variables then $max\{X_1, X_2\}$ is also EMWE random variable.

Lemma 3.2. The CDF of the random variable $W = \min\{X_1, X_2\}$ is given as

$$F_W(w) = \left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha w)^\beta} - 1)}\right]^{\gamma_1} + \left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha w)^\beta} - 1)}\right]^{\gamma_2} - \left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha w)^\beta} - 1)}\right]^{\gamma_1 + \gamma_2 + \gamma_3}. \quad (20)$$

Proof: Since

$$F_W(w) = P(W \leq w) = P(\min\{X_1, X_2\} \leq w) = 1 - P(\min\{X_1, X_2\} > w)$$
$$= 1 - P(X_1 > w, X_2 > w) = 1 - S(w, w). \quad (21)$$

Substituting from (14) in (21), we get

$$F_W(w) = F_{X_1}(w) + F_{X_2}(w) - F_{X_1, X_2}(w, w). \quad (22)$$

Substituting from (3) and (13) in (22), we get (20) which completes the proof of the lemma 3.2.

4 The marginal expectation

In this section, we derive the marginal expectation of X_i $(i = 1, 2)$. The following theorem gives the rth moments of X_i $(i = 1, 2)$ as infinite series expansion.

Theorem 3.1. The rth moment of X_i $(i = 1, 2)$ is given by:

$$E(X_i^r) = \frac{(\gamma_i + \gamma_3)\lambda}{\alpha^{1-\beta}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{(\gamma_i + \gamma_3) - 1}{j} \frac{(-1)^{j+k} \lambda^k (j+1)^k}{\alpha^{k+\beta+r}(k+1)^{\beta+r} k!} e^{\frac{\lambda(j+1)}{\alpha}} \Gamma(\frac{r}{\beta} + 1). \quad (23)$$

Proof: We will start with the known definition of the rth moment of the random variables X_i with pdf $f(x_i)$ given by

$$E(X_i^r) = \int_0^{\infty} x_i^r f_{X_i}(x_i) dx_i.$$

Substituting for $f_{X_i}(x_i)$ from (12), we get

$$E(X_i^r) = \frac{(\gamma_i + \gamma_3)\lambda\beta}{\alpha^{1-\beta}} \int_0^{\infty} x_i^{r+\beta-1} e^{(\alpha x_i)^\beta} e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)} \left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)}\right]^{(\gamma_i + \gamma_3) - 1} dx_i. \quad (24)$$

Since $0 < e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)} < 1$ for $x > 0$, then by using the binomial series expansion of $\left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)}\right]^{(\gamma_i + \gamma_3) - 1}$ given by

$$\left[1 - e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)}\right]^{(\gamma_i + \gamma_3) - 1} = \sum_{j=0}^{\infty} \binom{(\gamma_i + \gamma_3) - 1}{j} (-1)^j e^{-\frac{j\lambda}{\alpha}(e^{(\alpha x_i)^\beta} - 1)}. \quad (25)$$

Substituting from (25) into (24), we get

$$E(X_i^r) = \frac{(\gamma_i+\gamma_3)\lambda\beta}{\alpha^{1-\beta}} \sum_{j=0}^{\infty} \binom{(\gamma_i+\gamma_3)-1}{j} (-1)^j e^{\frac{\lambda(j+1)}{\alpha}} \int_0^\infty x_i^{r+\beta-1} e^{(\alpha x_i)^\beta} e^{-\frac{\lambda(j+1)}{\alpha} e^{(\alpha x_i)^\beta}} dx_i.$$

Using the series expansion of $e^{-\frac{\lambda(j+1)}{\alpha} e^{(\alpha x_i)^\beta}}$, one gets

$$E(X_i^r) = \frac{(\gamma_i+\gamma_3)\lambda\beta}{\alpha^{1-\beta}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{(\gamma_i+\gamma_3)-1}{j} \frac{(-1)^{j+k}\lambda^k (j+1)^k}{\alpha^k k!} e^{\frac{\lambda(j+1)}{\alpha}} \int_0^\infty x_i^{r+\beta-1} e^{(k+1)(\alpha x_i)^\beta} dx_i.$$

Let $y=(k+1)(\alpha x_i)^\beta$ in the above integral, then we can get

$$E(X_i^r) = \frac{(\gamma_i+\gamma_3)\lambda}{\alpha^{1-\beta}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{(\gamma_i+\gamma_3)-1}{j} \frac{(-1)^{j+k}\lambda^k (j+1)^k}{\alpha^{k+\beta+r}(k+1)^{\beta+r} k!} e^{\frac{\lambda(j+1)}{\alpha}} \int_0^\infty y^{\frac{r}{\beta}} e^y dy. \quad (26)$$

Since, $\Gamma(z) = x^z \int_0^\infty e^{xt} t^{z-1} dt \quad, z>0, x>0$, then

$$\int_0^\infty y^{\frac{r}{\beta}} e^y dy = \Gamma(\frac{r}{\beta}+1). \quad (27)$$

Substituting from (27) into (26), we get (23). This completes the proof.

5 Maximum liklihood estimators

In this section, we use the method of maximum likelihood to estimate the unknown parameters of the BEMWE distribution. Consider constant values to the parameters α andβ so, we want to estimate the other parameters $\gamma_1,\gamma_2,\gamma_3$and λ. Suppose that we have a sample of size n ,of the form $\{(x_{11},x_{21}),(x_{12},x_{22}),...,(x_{1n},x_{2n})\}$ from BEMWE distribution. We use the following notation

$I_1=\{x_{1i}<x_{2i}\}, \quad I_2=\{x_{1i}>x_{2i}\}, \quad I_3=\{x_{1i}=x_{2i}=x_i\}, \quad I=I_1\cup I_2\cup I_3, |I_1|=n_1, \quad |I_2|=n_2, \quad |I_3|=n_3$, and $n_1+n_2+n_3=n$.

Based on the observations, the likelihood function of the sample of size n given by:

$$l(\gamma_1,\gamma_2,\gamma_3,\alpha,\beta,\lambda) = \prod_{i=1}^{n_1} f_1(x_{1i},x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i},x_{2i}) \prod_{i=1}^{n_3} f_3(x_i,x).$$

The log-likelihood function can be written as

$$\begin{aligned}
L(\gamma_1,\gamma_2,\gamma_3,\alpha,\beta,\lambda) =\; & n_1 \ln\left(\gamma_2(\gamma_1+\gamma_3)\lambda^2\beta^2\right) + \sum_{i=1}^{n_1}(\alpha x_{1i})^\beta - \frac{\lambda}{\alpha}\sum_{i=1}^{n_1}(e^{(\alpha x_{1i})^\beta}-1) - \frac{\lambda}{\alpha}\sum_{i=1}^{n_1}(e^{(\alpha x_{2i})^\beta}-1) \\
& +(\gamma_2-1)\sum_{i=1}^{n_1}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}) + (\beta-1)\sum_{i=1}^{n_1}\ln(\alpha^2 x_1 x_2) + (\gamma_1+\gamma_3-1)\sum_{i=1}^{n_1}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}) \\
& +\sum_{i=1}^{n_1}(\alpha x_{2i})^\beta + n_2\ln\left(\gamma_1(\gamma_2+\gamma_3)\lambda^2\beta^2\right) + \sum_{i=1}^{n_2}(\alpha x_{1i})^\beta - \frac{\lambda}{\alpha}\sum_{i=1}^{n_2}(e^{(\alpha x_{1i})^\beta}-1) + \sum_{i=1}^{n_2}(\alpha x_{2i})^\beta \\
& -\frac{\lambda}{\alpha}\sum_{i=1}^{n_2}(e^{(\alpha x_{2i})^\beta}-1) + (\gamma_1-1)\sum_{i=1}^{n_2}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}) + (\gamma_2+\gamma_3-1)\sum_{i=1}^{n_2}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}) \\
& +(\beta-1)\sum_{i=1}^{n_2}\ln(\alpha^2 x_1 x_2) + n_3\ln(\gamma_3\lambda\beta) - \frac{\lambda}{\alpha}\sum_{i=1}^{n_3}(e^{(\alpha x_i)^\beta}-1) + \sum_{i=1}^{n_3}(\alpha x_i)^\beta + (\beta-1)\sum_{i=1}^{n_3}\ln(\alpha x_i) \\
& +(\gamma_1+\gamma_2+\gamma_3-1)\sum_{i=1}^{n_3}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}).
\end{aligned} \quad (28)$$

Computing the first partial derivatives of (28) with respect to γ_1,γ_2 and γ_3 and setting the results equal zeros, we get the likelihood equations as in the following form

$$\frac{\partial L}{\partial \gamma_1} = \frac{n_1}{\gamma_1+\gamma_3} + \sum_{i=1}^{n_1}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}) + \frac{n_2}{\gamma_1} + \sum_{i=1}^{n_2}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}) + \sum_{i=1}^{n_3}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}), \quad (29)$$

$$\frac{\partial L}{\partial \gamma_2}=\frac{n_1}{\gamma_2}+\sum_{i=1}^{n_1}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)})+\frac{n_2}{\gamma_2+\gamma_3}+\sum_{i=1}^{n_2}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)})+\sum_{i=1}^{n_3}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}), \tag{30}$$

$$\frac{\partial L}{\partial \gamma_3}=\frac{n_1}{\gamma_1+\gamma_3}+\sum_{i=1}^{n_1}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)})+\frac{n_2}{\gamma_2+\gamma_3}+\sum_{i=1}^{n_2}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)})+\frac{n_3}{\gamma_3}+\sum_{i=1}^{n_3}\ln(1-e^{-\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}) \tag{31}$$

and

$$\begin{aligned}\frac{\partial L}{\partial \lambda} &= \frac{2n_1}{\lambda}-\frac{1}{\alpha}\sum_{i=1}^{n_1}(e^{(\alpha x_{1i})^\beta}-1)-\frac{1}{\alpha}\sum_{i=1}^{n_1}(e^{(\alpha x_{2i})^\beta}-1)+\frac{\gamma_2-1}{\alpha}\sum_{i=1}^{n_1}\frac{(e^{(\alpha x_{2i})^\beta}-1)}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1}+\frac{\gamma_1+\gamma_3-1}{\alpha}\sum_{i=1}^{n_1}\frac{(e^{(\alpha x_{1i})^\beta}-1)}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1}\\ &+\frac{2n_2}{\lambda}-\frac{1}{\alpha}\sum_{i=1}^{n_2}(e^{(\alpha x_{1i})^\beta}-1)-\frac{1}{\alpha}\sum_{i=1}^{n_2}(e^{(\alpha x_{2i})^\beta}-1)+\frac{\gamma_1-1}{\alpha}\sum_{i=1}^{n_2}\frac{(e^{(\alpha x_{1i})^\beta}-1)}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1}+\frac{\gamma_2+\gamma_3-1}{\alpha}\sum_{i=1}^{n_2}\frac{(e^{(\alpha x_{2i})^\beta}-1)}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1}+\frac{n_3}{\lambda}\\ &-\frac{1}{\alpha}\sum_{i=1}^{n_3}(e^{(\alpha x_i)^\beta}-1)+\frac{\gamma_1+\gamma_2+\gamma_3-1}{\alpha}\sum_{i=1}^{n_3}\frac{(e^{(\alpha x_i)^\beta}-1)}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}-1}.\end{aligned} \tag{32}$$

To get the MLEs of the parameters $\gamma_1,\gamma_2,\gamma_3$ and λ , we have to solve the above system of four non-linear equations. The solution of equations (29), (30) ,(31)and (32) are not easy to solve, so numerical technique is needed to get the MLEs.

5.1 *Asymptotic confidence bounds*

In this subsection we consider the approximate confidence intervals of the parameters $\gamma_1,\gamma_2,\gamma_3$ and λ by using variance covariance matrix I_0^{-1}see Lawless [16], where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1}=-\begin{pmatrix}\frac{\partial^2 L}{\partial \gamma_1^2} & \frac{\partial^2 L}{\partial \gamma_1\partial\gamma_2} & \frac{\partial^2 L}{\partial \gamma_1\partial\gamma_3} & \frac{\partial^2 L}{\partial \gamma_1\partial\lambda}\\ \frac{\partial^2 L}{\partial \gamma_2\partial\gamma_1} & \frac{\partial^2 L}{\partial \gamma_2^2} & \frac{\partial^2 L}{\partial \gamma_2\partial\gamma_3} & \frac{\partial^2 L}{\partial \gamma_2\partial\lambda}\\ \frac{\partial^2 L}{\partial \gamma_3\partial\gamma_1} & \frac{\partial^2 L}{\partial \gamma_3\partial\gamma_2} & \frac{\partial^2 L}{\partial \gamma_3^2} & \frac{\partial^2 L}{\partial \gamma_3\partial\lambda}\\ \frac{\partial^2 L}{\partial \lambda\partial\gamma_1} & \frac{\partial^2 L}{\partial \lambda\partial\gamma_2} & \frac{\partial^2 L}{\partial \lambda\partial\gamma_3} & \frac{\partial^2 L}{\partial \lambda^2}\end{pmatrix}^{-1}=\begin{pmatrix}Var(\hat{\gamma}_1) & Cov(\hat{\gamma}_1,\gamma_2) & Cov(\hat{\gamma}_1,\hat{\gamma}_3) & Cov(\hat{\gamma}_1,\hat{\lambda})\\ Cov(\hat{\gamma}_2,\hat{\gamma}_1) & Var(\hat{\gamma}_2) & Cov(\hat{\gamma}_2,\hat{\gamma}_3) & Cov(\hat{\gamma}_2,\hat{\lambda})\\ Cov(\hat{\gamma}_3,\hat{\gamma}_1) & Cov(\hat{\gamma}_3,\hat{\gamma}_2) & Var(\hat{\gamma}_3) & Cov(\hat{\gamma}_3,\hat{\lambda})\\ Cov(\hat{\lambda},\hat{\gamma}_1) & Cov(\hat{\lambda},\hat{\gamma}_2) & Cov(\hat{\lambda},\hat{\gamma}_3) & Var(\hat{\lambda})\end{pmatrix}. \tag{33}$$

The derivatives in I_0^{-1} are given as follows

$$\begin{aligned}\frac{\partial^2 L}{\partial \gamma_1^2} &= -\frac{n_1}{(\gamma_1+\gamma_3)^2}-\frac{n_2}{\gamma_1{}^2},\ \frac{\partial^2 L}{\partial\gamma_1\partial\gamma_3}=-\frac{n_1}{(\gamma_1+\gamma_3)^2},\ \frac{\partial^2 L}{\partial\gamma_2^2}=-\frac{n_1}{\gamma_2^2}-\frac{n_2}{(\gamma_2+\gamma_3)^2},\\ \frac{\partial^2 L}{\partial\gamma_2\partial\gamma_3} &= -\frac{n_2}{(\gamma_2+\gamma_3)^2},\ \frac{\partial^2 L}{\partial\gamma_3^2}=-\frac{n_1}{(\gamma_1+\gamma_3)^2}-\frac{n_2}{(\gamma_2+\gamma_3)^2}-\frac{n_3}{\gamma_3^2},\ \frac{\partial^2 L}{\partial\gamma_1\partial\gamma_2}=0,\\ \frac{\partial^2 L}{\partial\lambda\partial\gamma_1} &= \frac{1}{\alpha}\left[\sum_{i=1}^{n_1}\frac{e^{(\alpha x_{1i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1}+\sum_{i=1}^{n_2}\frac{e^{(\alpha x_{1i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1}+\sum_{i=1}^{n_3}\frac{e^{(\alpha x_i)^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}-1}\right],\\ \frac{\partial^2 L}{\partial\lambda\partial\gamma_2} &= \frac{1}{\alpha}\left[\sum_{i=1}^{n_1}\frac{e^{(\alpha x_{2i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1}+\sum_{i=1}^{n_2}\frac{e^{(\alpha x_{2i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1}+\sum_{i=1}^{n_3}\frac{e^{(\alpha x_i)^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}-1}\right],\\ \frac{\partial^2 L}{\partial\lambda\partial\gamma_3} &= \frac{1}{\alpha}\left[\sum_{i=1}^{n_1}\frac{e^{(\alpha x_{1i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1}+\sum_{i=1}^{n_2}\frac{e^{(\alpha x_{2i})^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1}+\sum_{i=1}^{n_3}\frac{e^{(\alpha x_i)^\beta}-1}{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}-1}\right],\end{aligned}$$

Table. 1. American Football League (NFL) data

X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	8.53	14.57	2.90	2.90	1.38	1.38
9.05	9.05	31.13	49.88	7.02	7.02	10.53	10.53
0.85	0.85	14.58	20.57	6.42	6.42	12.13	12.13
3.43	3.43	5.78	25.98	8.98	8.98	14.58	14.58
7.78	7.78	13.80	49.75	10.15	10.15	11.82	11.82
10.57	14.28	7.25	7.25	8.87	8.87	5.52	11.27
7.05	7.05	4.25	4.25	10.40	10.25	19.65	10.70
2.58	2.58	1.65	1.65	2.98	2.98	17.83	17.83
7.23	9.68	6.42	15.08	3.88	6.43	10.85	38.07
6.85	34.58	4.22	9.48	0.75	0.75		
32.45	42.35	15.53	15.53	11.63	17.37		

and

$$\frac{\partial^2 L}{\partial \lambda^2}=-\frac{2n_1}{\lambda^2}-\frac{\gamma_1+\gamma_3-1}{\alpha^2}\sum_{i=1}^{n_1}\frac{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}\left(e^{(\alpha x_{1i})^\beta}-1\right)^2}{\left(e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1\right)^2}-\frac{\gamma_2-1}{\alpha^2}\sum_{i=1}^{n_1}\frac{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}\left(e^{(\alpha x_{2i})^\beta}-1\right)^2}{\left(e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1\right)^2}-\frac{2n_2}{\lambda^2}-\frac{\gamma_2+\gamma_3-1}{\alpha^2}\times$$

$$\sum_{i=1}^{n_2}\frac{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}\left(e^{(\alpha x_{2i})^\beta}-1\right)^2}{\left(e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{2i})^\beta}-1)}-1\right)^2}-\frac{\gamma_1-1}{\alpha^2}\sum_{i=1}^{n_2}\frac{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}\left(e^{(\alpha x_{1i})^\beta}-1\right)^2}{\left(e^{\frac{\lambda}{\alpha}(e^{(\alpha x_{1i})^\beta}-1)}-1\right)^2}$$

$$-\frac{n_3}{\lambda^2}-\left(\frac{\gamma_1+\gamma_2+\gamma_3-1}{\alpha^2}\right)\sum_{i=1}^{n_3}\frac{e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}\left(e^{(\alpha x_i)^\beta}-1\right)^2}{\left(e^{\frac{\lambda}{\alpha}(e^{(\alpha x_i)^\beta}-1)}-1\right)^2}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters $\gamma_1, \gamma_2, \gamma_3$ and λ by using variance covariance matrix as in the following forms

$$\hat{\gamma}_i \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\gamma}_i)} \quad ,i=1,2,3 \text{ and } \hat{\lambda} \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\lambda})}.$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$th percentile of the standard normal distribution.

6 Data analysis

In this section we present the analysis of a bivariate real data set to illustrate that the BEMWE distribution can be used as a good lifetime model, comparing with the BGG distribution [17]. We will use the log-likeihood values(L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Bayesian information criterion (BIC) test statistic. The data set represents the American Football (National Football League) League data and it is obtained from the matches played on three consecutive weekends in 1986. The data were first published in 'Washington Post' and they are also available in Csorgo and Welsh [18]. It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the 'game time' to the first points scored by kicking the ball between goal posts, and X_2 represents the 'game time' to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game.

The data (scoring times in minutes and seconds) are represented in Table. 1. Here also all the data points are divided by 100 just for computational purposes. The variables have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal, (ii) $X_1 > X_2$, means the first score is an unconverted touchdown or safety, (iii) $X_1 = X_2$ means the first score is a converted touchdown.

Table. 2. The MLEs and the values of L, AIC, CAIC and BIC**.**

Model	MLEs	L	AIC	CAIC	BIC
BGG $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\lambda}, 0.1)$	$\hat{\gamma}_1 = 0.024, \hat{\gamma}_2 = 0.150,$ $\hat{\gamma}_3 = 0.310, \hat{\lambda} = 0.0044$	-260.5	529.0	530.03	268.07
BEMWE $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\lambda}, 0.1, 0.42)$	$\hat{\gamma}_1 = 0.212, \hat{\gamma}_2 = 1.315,$ $\hat{\gamma}_3 = 2.645, \hat{\lambda} = 0.096$	-239.86	487.36	488.39	247.25

Consider a constant value to the parameters α and β which take the values 0.1 and 0.42 respectively. In Table. 2. We present the MLEs of the unknown parameters of BGG and BEMWE models. We have also provide the values of L, AIC, CAIC and BIC for the two models. From the values of L, AIC, CAIC and BIC for the two models, it is clear that the BEMWE model fits the data better than the BGG model. By substituting the MLEs of unknown parameters in (33), we get estimation of the variance covariance matrix as

$$I_0^{-1} = \begin{pmatrix} 0.022067 & 0.001654 & -0.004712 & 0.0000852 \\ 0.001655 & 0.115258 & 0.01429 & 0.000472 \\ -0.004712 & 0.014298 & 0.21701 & 0.000923 \\ 0.0000852 & 0.000472 & 0.000923 & 0.0000259 \end{pmatrix}.$$

The 95% confidence intervals of $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$ and $\hat{\lambda}$ are (0,0.69414), (0.2311,2.3992), (1.18637, 4.10552) and (0,0.20584) respectively.

7 Conclusions

In this paper we have introduced the bivariate exponentiated modified Weibull extension distribution whose marginals are exponentiated modified Weibull extension distributions. We discussed some statistical properties of the new bivariate distribution. Maximum likelihood estimates of the new distribution are discussed and we provided the observed Fisher information matrix. One real data set is analyzed and finally we conclude that, the new model fits the given real data very well than the BGG model.

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