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Upper and Lower Solutions to a Coupled System of Nonlinear Fractional Differential Equations

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Abstract: In this article, we investigate upper and lower solutions to a system of coupled nonlinear fractional differential equations with multi-point coupled boundary conditions. Using monotone type iterative techniques combined with the upper and lower solutions method, some necessary and sufficient conditions are developed to guarantee existence of multiple solutions of the problem. The main result is demonstrated by analyzing two test problems.

Keywords: Coupled system, coupled boundary conditions, fractional derivatives, monotone iterative technique, upper and lower solutions.

1 Introduction

During the last few decades a huge amount of work is devoted to the study of fractional calculus. Being nonlocal in nature fractional calculus serves as a tool of 21st century for analyzing many natural and scientific phenomena. Among others, some of the recent fields in which fractional calculus proved to be efficient tool, are engineering sciences [1,2,3], psychological and life sciences [4,5], food science [7], electrodynamics [8,9], diffusion processes [10], control systems and dynamics [11,12], waves dynamics and marine sciences [13,14,15], solid mechanics [16,17] and Heat transform models [18,19,20,21]. In addition to the above mentioned applications, there are several applications of fractional calculus within different fields of mathematics itself. For example, the fractional operators are useful for the analytic investigation of various spacial functions [22,23].

The extensive application of fractional calculus or specifically fractional differential equations(FDEs), motivates the interest of many scientist around the globe to study different aspects of FDEs. Reviewing the current literature on FDEs leads us to the conclusion that among others, numerical approximation of solutions, exitances results of positive solutions and modeling applied problems in terms of FDEs are the most studied fields of fractional calculus. Consequently a huge amount of work is available and is still growing.

Numerical approximation of solutions of FDEs are relatively difficult as compare to integer order differential equations. Some recently developed spectral methods which can be successfully used to solve most of FDEs are given in [24,25,26,27,28,29]. Numerically approximation of solution of coupled system of FDEs can be found in [30,31,32,33,34]. Iterative methods of monotone type are also used to develop numerical methods for initial and boundary value problems of FDEs [53,54,55,56,57,58,59,60,61].

A numerical method will work only if the solution of the problem exists. In this regard, many authors devoted there time to study the existence theory of positive solutions, for more details we refer the reader to [39,40]. Among others, some of the well known approaches used in existence theory are classical fixed point theorems of Cone expansion and Banach contraction mapping principles[41,42,43,44,45]. Some results on the impulsive boundary conditions can be found in [46]. Existence of multiple solutions for nonlocal boundary value problems is also discussed in [47]. More recently iterative techniques (combined with upper and lower solution method) are extensively applied to study the existence of multiple positive of different types of boundary value problems. Beyond simplicity, the brevity of such approach is its applicability to large number of problems. For example, these techniques are successfully applied to develop conditions for existence of multiple solutions for ordinary and FDEs[48,49,50,51,52].

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Recently, Xu and Liu [62], applied an iterative technique of monotone type coupled with upper and lower solution method to develop sufficient conditions of multiple solutions to FDEs of the form

$$\begin{cases} \mathscr{D}^{\alpha}u(t)+f(t,v(t),\mathscr{I}^{\beta}v(t))=0, \quad t\in[0,1],\\ \mathscr{D}^{\beta}v(t)+g(t,u(t),\mathscr{I}^{\alpha}u(t))=0, \quad t\in[0,1],\\ \mathscr{I}^{3-\alpha}u(0)=\mathscr{D}^{\alpha-2}u(0)=u(1)=0,\\ \mathscr{I}^{3-\beta}v(0)=\mathscr{D}^{\beta-2}v(0)=v(1)=0, \end{cases}$$

where $2<\alpha,\ \beta\leq 3,\ f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are nonlinear functions satisfying Caratheodory conditions while $\mathscr{D}$ and $\mathscr{I}$ are Riemann-Liouville differential and integral operator respectively. The above results motivates our interest to study nonlocal multi point coupled boundary conditions of the more generalized problem given as

$$\begin{cases} \mathscr{D}^{\alpha}u(t)+f(t,u(t),v(t))=0, \quad \mathscr{D}^{\beta}v(t)+g(t,u(t),v(t))=0, \quad 0<t<1,\\ u(0)=v(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\delta_i v(\eta_i),\ v(1)=\sum_{i=1}^{m-2}\lambda_i u(\xi_i), \end{cases} \tag{1}$$

where $1<\alpha,\beta\leq 2,\ \eta_i,\xi_i(i=1,2,...m-2)\in(0,1),\ \sum_{i=1}^{m-2}\delta_i\eta_i<1,\ \sum_{i=1}^{m-2}\gamma_i\xi_i<1,\ f,g:(0,1)\times\mathbb{R}^2\to\mathbb{R}^2$ are nonlinear continuous functions and $\mathscr{D}$ represents Caputo's fractional derivative. We use various tools of applied analysis to develop sufficient conditions for existence of multiple solutions of the problem. These results are demonstrated with the help of some test problems.

The rest of the article is organized as follows: In section 1, some preliminaries results of fractional calculus and functional analysis are presented. Section 2 is devoted to main results of the paper. In section 3, we provide some test examples which demonstrates our main results. The last section is devoted to a short conclusion.

2 Background Materials

The current portion of the article is related to some important definitions and well known results of fractional calculus and functional analysis. The concerned background materials can be found in [2,3,35,48,49,50,51,52].

Definition 1. *Let $\alpha>0$ and $z:[a,+\infty)\to\mathbb{R}$. Then the fractional order integral of $z(t)$ is defined by*

$$\mathscr{I}^{\alpha}_{a+}z(t)=\int_a^t\frac{(t-s)^{1-\alpha}z(s)}{\Gamma(\alpha)}ds,$$

where $\alpha\in\mathbb{R}_+$ and Γ represents Gamma function. The above definition is meaningful for all function such that integral exists.

Definition 2. *The α order derivative of a function $z(t)$ on the interval $[a,b]$ in Caputo's sense is defined as*

$$\mathscr{D}^{\alpha}_{a+}z(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}z^{(n)}(s)\,ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ represents the integer part of α.

Lemma 1. *[41] If $\mathscr{D}^{\alpha}z(t)=0$, be a differential equation, where $z\in C(0,1)\bigcap L(0,1)$, then in view of Definition (1), we have*

$$\mathscr{I}^{\alpha}\mathscr{D}^{\alpha}z(t)=z(t)+C_0+C_1t+C_2t^2+...+C_{n-1}t^{n-1},$$

for any $C_i\in\mathbb{R},\ \ i=0,1,2,...,n-1$.

Note: We denote $\mathscr{X}=C^2([0,1],\mathbb{R}),\ \mathscr{Y}=C([0,1],(0,\infty))$ through out this paper.

Definition 3. *$(x_0,y_0)\in\mathscr{X}\times\mathscr{X}$ is called lower solution of the CBVP (1) if*

$$\begin{cases} \mathscr{D}^{\alpha}u_0(t)+f(t,u_0(t),v_0(t))\geq 0, \quad \mathscr{D}^{\beta}v_0(t)+g(t,u_0(t),v_0(t))\geq 0, \quad 0<t<1,\\ u_0(0)\leq 0,v_0(0)\leq 0, \quad u_0(1)\leq\sum_{i=1}^{m-2}\delta_i v_0(\eta_i),\ v(1)\leq\sum_{i=1}^{m-2}\lambda_i u_0(\xi_i). \end{cases} \tag{2}$$

Similarly $(\mu_0,\nu_0)\in\mathscr{X}\times\mathscr{X}$ is called an upper solution if it satisfies the inequalities in reversed order. Moreover we assume that $(x_0,y_0),(\mu_0,\nu_0)\in\mathscr{X}\times\mathscr{X}$ are ordered lower and upper solutions respectively of CBVP (1).

We define the ordered sector

$$\Theta = \{(u,v) \in \mathscr{X} \times \mathscr{X} : (x_0,y_0) \le (u,v) \le (\mu_0, v_0), t \in [0,1]\}. \tag{3}$$

For further study, we make the following assumptions which will be use through out in this paper.

(A_1) $0 < \sum_{i=1}^{m-2} \delta_i \sum_{i=1}^{m-2} \lambda_i < 1$.
(A_2) $f,g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$.
(A_3) $f(t,u,v)$ is non-decreasing in v and there exist $U(t) \in \mathscr{Y}$ such that $f(t,u_1,v) - f(t,u_2,v) \le -U(u_1 - u_2)$, $U \ge 0$.
(A_4) $g(t,u,v)$ is non-decreasing in u and there exist $V(t) \in \mathscr{Y}$ such that $g(t,u,v_1) - g(t,u,v_2) \le -V(v_1 - v_2)$, $V \ge 0$.

Lemma 2.*[53].Let $h(t) \in \mathscr{X}$ and $\gamma(t) \in \mathscr{Y}$. If $h(t)$ satisfies the inequality*

$$-\mathscr{D}^{\alpha} h(t) \le -\gamma(t)h(t), t \in (0,1) \quad \text{such that} \quad h(0) \le 0, \; h(1) \le 0, \tag{4}$$

then

$$h(t) \le 0, \; \text{for all } t \in [0,1].$$

Lemma 3.*(Comparison theorem) Let $U(t), V(t) \in \mathscr{Y}$ be given, assume that $u(t), v(t)$ satisfy*

$$\begin{cases} -\mathscr{D}^{\alpha} u(t) \le -U(t)u(t), \; -\mathscr{D}^{\beta} v(t) \le -V(t)v(t) t \in (0,1), \\ u(0) \le 0, v(0) \le 0, \; u(1) \le \sum_{i=1}^{m-2} \delta_i v_0(\eta_i), \; v(1) \le \sum_{i=1}^{m-2} \lambda_i u_0(\xi_i), \\ \text{then} \quad u(t) \le 0, v(t) \le 0, \; \text{for all } t \in [0,1]. \end{cases} \tag{5}$$

*Proof.*For proof see the Lemma (3) of [53].

Note that if $U(t), V(t) \in \mathscr{Y}$ satisfies

$$\begin{aligned} &-\mathscr{D}^{\alpha} u(t) = -U(t)u(t), \; -\mathscr{D}^{\beta} v(t) = -V(t)v(t), t \in (0,1), \\ &u(0) = 0, v(0) = 0, u(1) = \sum_{i=1}^{m-2} \delta_i v(\xi_i), v(1) = \sum_{i=1}^{m-2} \lambda_i u(\eta_i). \\ &\text{Then } \textit{wehave } u(t) = v(t) = 0, \; \text{for all } t \in [0,1]. \end{aligned} \tag{6}$$

3 Main Results

Now we are in the position to develop our main results. The following lemma is important.

Lemma 4.*Let $y(t), w(t) \in C[0,1]$, then the corresponding integral representation of (1) is given by*

$$\begin{aligned} &u(t) = \int_0^1 \mathscr{K}_{11}(t,s)y(s)ds + \int_0^1 \mathscr{K}_{12}(t,s)w(s)ds, \\ &v(t) = \int_0^1 \mathscr{K}_{21}(t,s)w(s)ds + \int_0^1 \mathscr{K}_{22}(t,s)y(s)ds, \\ &\text{where} \\ &\mathscr{K}_{11}(t,s) = \mathscr{G}_{\alpha}(t,s) + \frac{t \sum_{i=1}^{m-2} \delta_i \eta_i \sum_{i=1}^{m-2} \lambda_i}{\Delta} \mathscr{G}_{\alpha}(\xi_i, s), \; \mathscr{K}_{12} = \frac{t \sum_{i=1}^{m-2} \delta_i}{\Delta} \mathscr{G}_{\beta}(\eta_i, s), \\ &\mathscr{K}_{21}(t,s) = \mathscr{G}_{\beta}(t,s) + \frac{t \sum_{i=1}^{m-2} \lambda_i \xi_i \sum_{i=1}^{m-2} \delta_i}{\Delta_i} \mathscr{G}_{\beta}(\eta_i, s), \; \mathscr{K}_{22} = \frac{t \sum_{i=1}^{m-2} \lambda_i}{\Delta} \mathscr{G}_{\alpha}(\xi_i, s), \\ &\Delta = 1 - \sum_{i=1}^{m-2} \lambda_i \xi_i \sum_{i=1}^{m-2} \delta_i \eta_i. \end{aligned} \tag{7}$$

The functions $\mathscr{G}_\alpha(t,s), \mathscr{G}_\beta(t,s)$ *are defined as*

$$\mathscr{G}_\alpha(t,s) = \begin{cases} \dfrac{t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le s \le t \le 1, \\ \dfrac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le t \le s \le 1. \end{cases} \tag{8}$$

and

$$\mathscr{G}_\beta(t,s) = \begin{cases} \dfrac{t(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} & 0 \le s \le t \le 1, \\ \dfrac{t(1-s)^{\beta-1}}{\Gamma(\beta)} & 0 \le t \le s \le 1. \end{cases} \tag{9}$$

*Proof.*Applications of $\mathscr{I}^\alpha, \mathscr{I}^\beta$ on both sides of (1), and using the corresponding homogenous boundary conditions $u(0) = 0, v(0) = 0, u(1) = 0, v(1) = 0$, we obtain

$$u_c(t) = t\mathscr{I}^\alpha y(1) - \mathscr{I}^\alpha y(t) = \int_0^1 \mathscr{G}_\alpha(t,s)y(s)ds, \qquad v_c(t) = t\mathscr{I}^\beta w(1) - \mathscr{I}^\beta w(t) = \int_0^1 \mathscr{G}_\beta(t,s)w(s)ds, \tag{10}$$

the equivalent system of integral equations corresponding to (1) is given by

$$u(t) = u(1)t + \int_0^1 \mathscr{G}_\alpha(t,s)y(s)ds, \qquad v(t) = v(1)t + \int_0^1 \mathscr{G}_\beta(t,s)w(s)ds, \ t \in [0,1]. \tag{11}$$

Using the above estimates along with coupled m-point boundary conditions, we get

$$v(1) = \sum_{i=1}^{m-2} \lambda_i u(\xi_i) = \sum_{i=1}^{m-2} \lambda_i \xi_i u(1) + \sum_{i=1}^{m-2} \lambda_i \int_0^1 \mathscr{G}_\alpha(\xi_i,s)y(s)ds, \tag{12}$$

and

$$u(1) = \sum_{i=1}^{m-2} \delta_i v(\eta_i) = \sum_{i=1}^{m-2} \delta_i \eta_i v(1) + \sum_{i=1}^{m-2} \delta_i \int_0^1 \mathscr{G}_\beta(\eta_i,s)w(s)ds. \tag{13}$$

Further simplification, and using $\Delta = 1 - \sum_{i=1}^{m-2} \lambda_i \xi_i \sum_{i=1}^{m-2} \delta_i \eta_i$, we get

$$u(1) = \frac{1}{\Delta}\left[t(\sum_{i=1}^{m-2} \delta_i \eta_i) \sum_{i=1}^{m-2} \lambda_i \int_0^1 \mathscr{G}_\alpha(\xi_i,s)y(s)ds + t\sum_{i=1}^{m-2} \delta_i \int_0^1 \mathscr{G}_\beta(\eta_i,s)w(s)ds\right], \tag{14}$$

and

$$v(1) = \frac{1}{\Delta}\left[t(\sum_{i=1}^{m-2} \lambda_i \xi_i) \sum_{i=1}^{m-2} \delta_i \int_0^1 \mathscr{G}_\beta(\eta_i,s)w(s)ds + t\sum_{i=1}^{m-2} \lambda_i \int_0^1 \mathscr{G}_\alpha(\xi_i,s)y(s)ds\right]. \tag{15}$$

Using (14) and (15) in (13) and (12), and using the values of $u(1), v(1)$ in (11), we get the desire results.

Now consider the following problem.

$$\begin{cases} \mathscr{D}^\alpha u(t) - U(t)u(t) + y(t) = 0, t \in (0,1), 1 < \alpha \le 2, \\ \mathscr{D}^\beta v(t) - V(t)v(t) + w(t) = 0, t \in (0,1), 1 < \beta \le 2, \\ u(0) = v(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \delta_i v(\xi_i), v(1) = \sum_{i=1}^{m-2} \lambda_i u(\eta_i), \end{cases} \tag{16}$$

where $U(t), V(t) \in \mathscr{Y}$.
We define an operator $\mathscr{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ by

$$\begin{aligned} \mathscr{T}(u,v) = \Big(&-\int_0^1 \mathscr{K}_{11}(t,s)U(s)u(s)ds - \int_0^1 \mathscr{K}_{12}(t,s)V(s)v(s)ds, \\ &-\int_0^1 \mathscr{K}_{21}(t,s)V(s)v(s)ds - \int_0^1 \mathscr{K}_{22}(t,s)U(s)u(s)ds \Big) + (\mu, \nu), \end{aligned} \tag{17}$$

where

$$\mu = \int_0^1 \mathscr{K}_{11}(t,s)y(s)ds + \int_0^1 \mathscr{K}_{12}(t,s)w(s)ds, \ \nu = \int_0^1 \mathscr{K}_{21}(t,s)w(s)ds + \int_0^1 \mathscr{K}_{22}(t,s)y(s)ds.$$

The following Lemma guarantees the unique solution of (16).

Lemma 5. *Let $U(t), V(t) \in \mathscr{Y}$, then the unique fixed point for the operator $\mathscr{T}$ is the corresponding unique solutions of* (16).

Proof. In view of Fredholm theorem the operator (17) is completely continuous also in view of Lemma (3) the operator equation $\mathscr{T}(u,v) = (u,v)$ has only the zero solution. Hence the operator equations defined in (17), for $(\mu, v) \in \mathscr{X} \times \mathscr{X}$ has a unique fixed point which is the unique solution of the problem.

Lemma 6. *Assume that $(A_1) - (A_4)$ holds. Then there exist a monotone sequences $\{(u_n, v_n)\}$ of ordered lower and $\{(\mu_n, v_n)\}$ of order upper solutions of* (1) *in sector Θ such that*

$$\{(u_n, v_n)\} \to \{(u_0, v_0)\},\ \{(\mu_n, v_n)\} \to \{(\mu_0, v_0)\} \text{ as } n \to \infty,$$

where (u_0, v_0) is initial approximation of ordered lower solution and (μ_0, v_0) is initial approximation of ordered upper solution of (1).

Proof. Let us define a sequence by

$$\begin{aligned} &\mathscr{D}^\alpha u_{n+1}(t) + f(t, u_n(t), v_n(t)) = U[u_{n+1}(t) - u_n(t)], \quad t \in (0,1), \\ &\mathscr{D}^\beta v_{n+1}(t) + g(t, u_n(t), v_n(t)) = V[v_{n+1}(t) - v_n(t)], \quad t \in (0,1), \\ &u_{n+1}(0) = 0,\ v_{n+1}(0) = 0, u_{n+1}(1) = \sum_{i=1}^{m-2} \delta_i v_{n+1}(\xi_i), v_{n+1}(1) = \sum_{i=1}^{m-2} \lambda_i u_{n+1}(\eta_i). \end{aligned} \tag{18}$$

Then in view of Lemma (5) the above system has a unique solutions (u_{n+1}, v_{n+1}). Using $n = 0$ in (18) we get

$$\begin{aligned} &\mathscr{D}^\alpha u_1(t) + f(t, u_0(t), v_0(t)) = U[u_1(t) - u_0(t)], \quad t \in (0,1), \\ &\mathscr{D}^\beta v_1(t) + g(t, u_0(t), v_0(t)) = V[v_1(t) - v_0(t)], \quad t \in (0,1), \\ &u_1(0) = 0,\ v_1(0) = 0, u_1(1) = \sum_{i=1}^{m-2} \delta_i v_1(\xi_i), v_1(1) = \sum_{i=1}^{m-2} \lambda_i u_1(\eta_i). \end{aligned} \tag{19}$$

To order to show that $(u_0, v_0) \le (u_1, v_1) \le (\mu_0, v_0)$ we set $\phi(t) = u_0(t) - u_1(t),\ \psi(t) = v_0(t) - v_1(t)$ in (19), which implies

$$\begin{aligned} &\mathscr{D}^\alpha \phi(t) = \mathscr{D}^\alpha [u_0(t) - u_1(t)] \le -U\phi(t),\ \mathscr{D}^\beta \psi(t) = \mathscr{D}^\beta [v_0(t) - v_1(t)] \le -V\psi(t),\ t \in (0,1), \\ &\phi(0) \le 0,\ \psi(0) \le 0,\ \phi(1) \le \sum_{i=1}^{m-2} \delta_i \psi(\xi_i),\ \psi(1) \le \sum_{i=1}^{m-2} \lambda_i \phi(\eta_i). \end{aligned} \tag{20}$$

Hence by lemma (3) $\mathscr{D}^\alpha \phi(t) \le -U\phi(t),\ \mathscr{D}^\beta \psi(t) \le -V\psi(t) \Rightarrow \phi(t) \le 0,\ \psi(t) \le 0.$

Hence $u_0(t) \le u_1(t),\ v_0(t) \le v_1(t)$. Thus $(u_0, v_0) \le (u_1, v_1),\ t \in [0,1]$. Similarly setting $\phi(t) = u_1(t) - \mu_1(t),\ \psi(t) = v_1(t) - v(t)$ in (19), we get $(u_1, v_1) \le (\mu_1, v_1),\ t \in [0,1]$. Next to prove that $(\mu_1, v_1) \le (\mu_0, v_0)$. Setting $\phi(t) = \mu_1(t) - \mu_0(t),\ \psi(t) = v_1(t) - v_0(t)$, we can easily obtain that $\mu_1(t) \le \mu_0(t),\ v_1(t) \le v_0(t)$ we have $(\mu_1, v_1) \le (\mu_0, v_0)$. Thus after collecting these relations, we have

$$(u_0, v_0) \le (u_1, v_1) \le (\mu_1, v_1) \le (\mu_0, v_0).$$

Now we assume that $k > 1$, and set $\phi(t) = u_k - u_{k+1},\ \psi(t) = v_k - v_{k+1}$ in system (18), implies that

$$\begin{aligned} &\mathscr{D}^\alpha \phi(t) = \mathscr{D}^\alpha [u_k(t) - u_{k+1}(t)] \le -U\phi(t),\ \mathscr{D}^\beta \psi(t) = \mathscr{D}^\beta [v_k(t) - v_{k+1}(t)] \le -V\psi(t),\ t \in (0,1), \\ &\phi(0) \le 0,\ \psi(0) \le 0,\ \phi(1) \le \sum_{i=1}^{m-2} \delta_i \psi(\xi_i),\ \psi(1) \le \sum_{i=1}^{m-2} \lambda_i \phi(\eta_i). \end{aligned} \tag{21}$$

Thus, by Lemma (3) we have $\phi(t) \le 0,\ \psi(t) \le 0$. Hence $u_k \le u_{k+1},\ v_k \le v_{k+1} \Rightarrow (u_k, v_k) \le (u_{k+1}, v_{k+1})$. Similarly we can prove that $((u_{k+1}, v_{k+1}) \le (\mu_{k+1}, v_{k+1})$.

Now using the corresponding system of Fredholm integral equations we have

$$\begin{aligned} u_{n+1} &= \int_0^1 \mathscr{K}_{11}(t,s) f(s, u_n, v_n) ds + \int_0^1 \mathscr{K}_{12}(t,s) g(s, u_n, v_n) ds, \\ v_{n+1} &= \int_0^1 \mathscr{K}_{21}(t,s) g(s, u_n, v_n) ds + \int_0^1 \mathscr{K}_{22}(t,s) f(s, u_n, v_n) ds. \end{aligned} \tag{22}$$

The above estimates implies that (u,v) is a solution of the system (22). Next we prove that $(u_0,v_0),(\mu_0,v_0)$ are ordered extremal solutions of (1). Assume that (x,y) be another solution different from $(u_0,v_0),(\mu_0,v_0)$ such that for some $n\in Z^+$, we have $(u_k,v_k)\le(x_k,y_k)\le(\mu_k,v_k),\ t\in[0,1]$. We set $\phi(t)=u_{k+1}-x_k,\ \psi(t)=v_{k+1}-y_k$, then by Lemma(2) and Theorem(3), we have $\phi(t)\le 0,\psi(t)\le 0$, hence $u_{k+1}\le x_k,\ v_{k+1}\le y_k$ for all $t\in[0,1]$. Similarly we can show that $x_k\le \mu_{k+1}, y_k\le v_{k+1}, \forall k\in Z^+$. Thus $(u_k,v_k)\le(x_k,y_k)\le(\mu_k,v_k)$, applying limit $k\to\infty$ we have $(u_0,v_0)\le(x_0,y_0)\le(\mu_0,v_0)$, hence maximal and minimal solutions follows in the sector Θ. This completes the proof.

4 Examples

This section is devoted to study the following examples for upper and lower solutions to verify the aforementioned techniques.

*Example 1.*Consider the following system of coupled boundary values problem with coupled boundary conditions

$$\begin{cases} \mathscr{D}^{\frac{3}{2}}u(t)+\cos(t)-1-8u(t)+\dfrac{t^2}{4}v^2(t)=0,\ t\in(0,1),\\ \mathscr{D}^{\frac{5}{3}}v(t)+e^{\frac{t}{2}}+\dfrac{u^2(t)}{2}-v^2(t)=0, t\in(0,1),\\ u(0)=v(0)=0, \qquad u(1)=\displaystyle\sum_{i=1}^{m-2}\delta_i v(\eta_i),\ v(1)=\sum_{i=1}^{m-2}\lambda_i u(\xi_i),\\ \text{where }\displaystyle\sum_{i=1}^{m-2}\lambda_i<\frac{1}{100},\ \sum_{i=1}^{m-2}\delta_i<\frac{1}{50}. \end{cases} \tag{23}$$

Since

$$f(t,u,v)=\cos(t)-1-8u(t)+\frac{t^2}{4}v^2(t),\ g(t,u,v)=e^{\frac{t}{2}}+\frac{u^2(t)}{2}-v^2(t),$$

taking $(-1,-1)=(u_0,v_0)$ and $(2,2)=(\mu_0,v_0)$ be initial approximation of ordered lower and upper solutions respectively,then

$$\mathscr{D}^{\frac{3}{2}}u_0(t)+\cos(t)-1-8u_0(t)+\frac{t^2}{4}v_0^2(t)=\cos t-1+8+\frac{t^2}{4}\ge 0,\ t\in(0,1),$$

$$\mathscr{D}^{\frac{5}{3}}v_0(t)+e^{\frac{t}{2}}+8u_0(t)+\frac{u^2(t)}{2}-v_0^2(t)=\frac{2e^{\frac{t}{2}}-1}{2}\ge 0,\ t\in(0,1),$$

$$u_0(0)\le 0,\ v_0(0)\le 0,\ u_0(1)\le\sum_{i=1}^{m-2}\delta_i v_0(\eta_i),\ v_0(1)\le\sum_{i=1}^{m-2}\lambda_i u_0(\xi_i),$$

$$\text{where }\sum_{i=1}^{m-2}\lambda_i<\frac{1}{100},\ \sum_{i=1}^{m-2}\delta_i<\frac{1}{50}.$$

Similarly by taking $(2,2)=(\mu_0,v_0)$,

$$\mathscr{D}^{\frac{3}{2}}\mu_0(t)+\cos(t)-1-8\mu_0(t)+\frac{t^2}{4}v_0^2(t)=\cos t-17+t^2\le 0,\ t\in(0,1),$$

$$\mathscr{D}^{\frac{5}{3}}v_0(t)+e^{\frac{t}{2}}+8\mu_0(t)+\frac{\mu^2(t)}{2}-v_0^2(t)=e^{\frac{t}{2}}-2\le 0,\ t\in(0,1),$$

$$\mu_0(0)\ge 0,\ v_0(0)\ge 0,\ \mu_0(1)\ge\sum_{i=1}^{m-2}\delta_i v_0(\eta_i),\ v_0(1)\ge\sum_{i=1}^{m-2}\lambda_i\mu_0(\xi_i),$$

$$\text{where }\sum_{i=1}^{m-2}\lambda_i<\frac{1}{100},\ \sum_{i=1}^{m-2}\delta_i<\frac{1}{50}.$$

Hence it follows that $(-1,-1),\ (2,2)$ are ordered lower and upper solutions respectively of (CBVP) (23). Also assumptions (A_2) and (A_3) holds for $U(t)=8,V(t)=\frac{1}{4}$.

Moreover, we present another example as follow:

*Example 2.*Let us take the another system of coupled differential equations of coupled boundary values problem

$$\begin{cases} \mathscr{D}^{\frac{5}{3}}u(t)+4t^3[t-u(t)]^3-4t^3v^2(t)=0,\ t\in(0,1),\\ \mathscr{D}^{\frac{7}{4}}v(t)+6t^3[t-v(t)]^3-6t^3u^2(t)=0, t\in(0,1),\\ u(0)=v(0)=0, \qquad u(1)=\sum_{i=1}^{50}\delta_i v(\eta_i),\ v(1)=\sum_{i=1}^{50}\lambda_i u(\xi_i),\\ \text{where}\ \ m=52, \sum_{i=1}^{50}\lambda_i<\frac{1}{10}, \sum_{i=1}^{50}\delta_i<\frac{1}{20}. \end{cases} \tag{24}$$

Since

$$f(t,u,v)=4t^3[t-u(t)]^3-4t^3v^2(t),\ g(t,u,v)=6t^3[t-v(t)]^3-6t^3u^2(t),$$

taking $(-1,-1)=(u_0,v_0)$ and $(1,1)=(\mu_0,\nu_0)$ be initial approximation of ordered lower and upper solutions respectively,then

$$\mathscr{D}^{\frac{5}{3}}u_0(t)+4t^3[t-u_0(t)]^3-4t^3v_0^2(t)=4t^3[(t+1)^3-1]\geq 0,\ t\in(0,1),$$
$$\mathscr{D}^{\frac{7}{4}}v_0(t)+6t^3[t+v_0(t)]^3-6t^3u_0^2(t)=6t^3[(t+1)^3-1]\geq 0,\ t\in(0,1),$$
$$u_0(0)\leq 0,\ v_0(0)\leq 0,\ u_0(1)\leq\sum_{i=1}^{50}\delta_i v_0(\eta_i),\ v_0(1)\leq\sum_{i=1}^{50}\lambda_i u_0(\xi_i),$$
$$\text{where}\ \sum_{i=1}^{m-2}\lambda_i<\frac{1}{10}, \sum_{i=1}^{m-2}\delta_i<\frac{1}{20}.$$

Similarly by taking $(1,1)=(\mu_0,\nu_0)$,

$$\mathscr{D}^{\frac{5}{3}}\mu_0(t)+4t^3[t-\mu_0(t)]^3-4t^3\nu_0^2(t)=4t^3[(t-1)^3-1]\leq 0,\ t\in(0,1),$$
$$\mathscr{D}^{\frac{7}{4}}\nu_0(t)+6t^3[t-\nu_0(t)]^3-6t^3\mu_0^2(t)=6t^3[(t-1)^3-1]\leq 0,\ t\in(0,1),$$
$$\mu_0(0)\geq 0,\ \nu_0(0)\geq 0,\ \mu_0(1)\geq\sum_{i=1}^{50}\delta_i \nu_0(\eta_i),\ \nu_0(1)\geq\sum_{i=1}^{50}\lambda_i \mu_0(\xi_i),$$
$$\text{where}\ \sum_{i=1}^{m-2}\lambda_i<\frac{1}{10}, \sum_{i=1}^{m-2}\delta_i<\frac{1}{20}.$$

Hence it follows that $(-1,-1)$, $(1,1)$ are ordered lower and upper solutions respectively of (CBVP) (24). Also assumptions (A_2) and (A_3) holds for $U(t)=4t^3,\ V(t)=6t^3$.

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5 Conclusions

Monotone iterative technique plays a vital role to investigate the approximate solutions of DEs. With the help of the said technique, we successfully formed iterative sequences for the corresponding upper and lower solutions for the problem under consideration. From the constructions of the approximate sequences, it is clear that monotone iterative technique is a powerful tools to study multiplicity of solutions for non-linear system of differential equations of fractional order. The said technique can be similarly applied to find the approximate solutions for nonlinear partial fractional differential equations.

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