Applied Mathematics & Information Sciences **5**(2) (2011), 220-238
– An International Journal

Qualitative Behaviour of a Model of an SIRS Epidemic: Stability and Permanence

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We consider a classical model of a SIRS epidemic in an open population. The positivity and permanence are studied and explicit formulæ are obtained by which the eventual lower bound of the density of infectives can be computed. The stability of the model is studied. We mainly use the Lyapunov functional to established the global stability of disease-free and endemic equilibrium points for both the deterministic and stochastic models. In addition we illustrate the dynamic behaviour of the deterministic and stochastic models via a numerical example.

Keywords: model of SIRS epidemic, Local and global stability, Lyapunov functional.

1 Introduction

Epidemiology is the study of the spread of infectious diseases with the objective to trace factors that contribute to their dynamics and stabilities. Formerly and recently models, which have the population subclasses (i) the susceptibles (S), (ii) the infectives (I) and (iii) the removals (R), have been studied by a number of authors (see for example Bailey [2], Tornatore [12], Beretta and Takeuchi [3], Zhang and Teng [13]). The basic and important research subjects for recent studies are the existence of the threshold values which distinguish whether the disease dies out, the stability of the disease-free and the endemic equilibria, permanence and extinction. Most of these works deal with local stability

PGLL thanks the University of KwaZulu-Natal and the National research Foundation of South Africa for their continued support. Any opinions expressed in this paper should not be construed as being of either institution.

of equilibria of the deterministic model in a closed population. There are very few works which study both deterministic and stochastic stabilities of the model.

In this paper we consider a model of an SIRS epidemic which is the extension of the classical SIR model (see [5] and [13]) for which it is assumed that all newborn are susceptibles and the population grows at a rate $b > 0$. The susceptibles, infectives and removals die at different rates, d_1, d_2 and d_3. It is biologically natural to assume that $d_1 < \min(d_2, d_3)$. In addition we suppose that an individual in the class (R) can be cured and becomes a new susceptible or infected another time, respectively, with rates γ_2 and γ_3. Mathematically the model that we consider is defined as follows. At time t the variables, $S(t)$, $I(t)$ and $R(t)$, represent, respectively, the density of susceptibles, infected and removed individuals. The epidemial process is thus completely determined by $\{(S(t), I(t), R(t)); t \geq 0\}$ while its dynamics is governed by the system of ordinary differential equations

$$\begin{cases} S' = b - \beta SI - d_1 S + \gamma_3 R, \\ I' = \beta SI - (d_2 + \gamma_1)\, I + \gamma_2 R, \\ R' = \gamma_1 I - (d_3 + \gamma_2 + \gamma_3)\, R, \end{cases} \tag{1}$$

with the initial conditions $S\,(0) = S_0$, $I\,(0) = I_0$ and $R\,(0) = R_0$, where the parameter β is the average number of contacts per infective per unit of time. As is known, systems like (1) are very important mathematical models which describe epidemiological dynamics. As mentioned in the first paragraph, the most basic and important questions to ask for these systems in the theory of mathematical epidemiology concern permanence and stability. Recently Ma *et al* [8] and Zhand and Teng [13] studied an SIRS system with time delay. Under certain conditions they proved the permanence of the disease and stability. Motivated by the above works the present analysis aims to establish some conditions on the positivity, boundness of solution and permanence of the epidemic. By using methods in [8] we obtain explicit formulæ of the eventual lower bound of infectious individuals and the total size of the epidemic. Applying the Lyapunov functional, we give some sufficient conditions for local and global stabilities of the deterministic model governed by (1) and of its stochastic version, which is obtained by random perturbation of the deterministic model.

The organization of this paper is as follows. In the next Section we give the equilibrium points of the model. In Section 3 we establish the positivity and we give the sufficient conditions of the ultimate boundedness of system (1). In Section 4 we give the sufficient and necessary conditions for the local and global stability of the disease-free and endemic equilibria. In Section 5 the stochastic stability of the disease-free and endemic equilibria is proved. In Section 6 numerical simulations are performed to complement the analytical results. Finally in Section 7 a brief discussion of the main results and some ideas for future research are given.

2 The Equilibrium Points

The equilibria of (1) are the solutions of the system

$$\begin{cases} b - \beta SI - d_1 S + \gamma_3 R = 0 \\ \beta SI - (d_2 + \gamma_1) I + \gamma_2 R = 0 \\ \gamma_1 I - (d_3 + \gamma_2 + \gamma_3) R = 0. \end{cases} \tag{2}$$

It is evident that (2) has two solutions, $E_0 = (b/d_1, 0, 0)$ which called the disease-free equilibrium point and another solution, $E^* = (S^*, I^*, R^*)$, where

$$\begin{cases} S^* = \dfrac{1}{\beta}.\left(d_2 + \gamma_1 - \dfrac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3}\right), \\ I^* = \left(\beta S^* - \dfrac{\gamma_1\gamma_3}{d_3 + \gamma_2 + \gamma_3}\right)^{-1} (b - d_1 S^*), \\ R^* = \dfrac{\gamma_1}{d_3 + \gamma_2 + \gamma_3} I^* \end{cases}$$

$E^\star$ is positive if and only if $\frac{\beta b}{d_1} > d_2 + \gamma_1 - \dfrac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3}$. Under the last condition E^* is called the endemic-disease point.

3 Positivity, Boundedness and Permanence

3.1 Positivity and boundedness of the solution

The application of the classical theory of Ordinary Differential Equations implies that for every set of initial data, (S_0, I_0, R_0), there exists a unique solution, $(S(t), I(t), R(t))$, defined in the maximal open interval $(-T, T)$ with $T > 0$.

Proposition 3.1. *Let* (S, I, R) *be the solution of* (1).

(i) *If* $S_0 > 0,\ I_0 > 0$ *and* $R_0 > 0$, *then* $S(t) > 0,\ I(t) > 0$ *and* $R(t) > 0$ *for every* $t \in [0, T)$.

(ii) *The solution* (S, I, R) *is defined in* $[0, \infty)$ *and* $\limsup_{t\to\infty} N(t) \le \dfrac{b}{d_1}$, *where* $N(t) = S(t) + I(t) + R(t)$.

Proof. We firstly prove (i). To do this we suppose that there exists $t_0 \in (0, T)$ such that $S(t_0) = 0,\ S^{'}(t_0) \le 0$ and $S(t) > 0$ for $t \in [0, t_0)$. Then $I(t) > 0$ for $t \in [0, t_0]$. If this be not the case, there exists $t_1 \in [0, t_0]$ such that $I(t_1) = 0,\ I^{'}(t_1) \le 0$ and $I(t) > 0$ for$t \in [0, t_1)$. Integration of the third equation of (1) leads to

$$R(t) = R_0 \exp\left[-(d_3 + \gamma_2 + \gamma_3) t\right]$$
$$+\gamma_1 \int_0^t \exp\left[-(d_3 + \gamma_2 + \gamma_3)(t - \tau)\right] I(\tau)\, \mathrm{d}\tau > 0 \quad \text{for } t \in [0, t_1].$$

Then $I^{'}(t_1) = \gamma_2 R(t_1) > 0$. This is a contradiction. Hence $R(t) > 0$ for every $t \in [0, t_0]$. Therefore $S^{'}(t_0) = b + \gamma_3 R(t_0) > 0$, but this leads to a contradiction to the supposition that $S^{'}(t_0) \leq 0$, which completes the proof of (i).

For (ii) we note that $N^{'} = b - d_1 S - d_2 I - d_3 R \leq b - d_1 N$. By integrating the last inequality we obtain

$$\begin{aligned} N(t) &\leq \frac{b}{d_1}\left(1 - \mathrm{e}^{-d_1 t}\right) \text{ for every } t \in [0, T) \\ &\leq \frac{2b}{d_1}. \end{aligned}$$

The solution (S, I, R) is bounded in the interval $[0, T)$. Therefore $N(t) \leq \frac{b}{d_1}\left(1 - \mathrm{e}^{-d_1 t}\right)$ for every $t \in [0, \infty)$. Finally $\limsup_{t\to\infty} N(t) \leq \frac{b}{d_1}$. □

Remark Following the same method used to demonstrate Proposition 3.1, we see that system (1) with the initial conditions $S_0 \geq 0$, $I_0 \geq 0$ and $R_0 \geq 0$ has a nonnegative solution defined in all $\mathbb{R}$ and the set $\Omega = \left\{(S, I, R) / S > 0,\ I > 0,\ \ R > 0 \text{ and } S + I + R \leq \frac{b}{d_1}\right\}$ is invariant by (1).

3.2 Permanence of the Epidemic

Lemma 3.1. *Let (S, I, R) be the solution of system (1). If there exists a sequence (t_n) such that $t_n \to \infty$, $S(t_n) \to l$, $I(t_n) \to 0$ and $R(t_n) \to 0$, then $l = \frac{b}{d_1}$.*

Proof. We have $0 \leq l \leq \frac{b}{d_1}$. Suppose that $0 \leq l < \frac{b}{d_1}$. Since $S(t_n) \to l,\ I(t_n) \to 0$ and $R(t_n) \to 0$, it follows that $(l, 0, 0) \in W(S_0, I_0, R_0)$ which is the set of $W -$ limit. Consider (S, I, R), the solution of (1) with the initial condition $(l, 0, 0)$. Therefore $(S(t), I(t), R(t)) \in W(S_0, I_0, R_0)$ for every t because the set $\in W(S_0, I_0, R_0)$ is invariant by (1). It is easy to verify that $S(t) = \frac{b}{d_1} + \left(l - \frac{b}{d_1}\right)\mathrm{e}^{-d_1 t}$ and $I(t) = R(t) = 0$ for every $t \in \mathbb{R}$. Since $0 \leq l < \frac{b}{d_1}$, contrary to the positivity of S in all $\mathbb{R}$ (Remark 2), we have $S(t) < 0$ for $t \to -\infty$. □

Proposition 3.2. *Let (S, I, R) be the solution of (1) such that $\frac{\beta b}{d_1} > d_2 + \gamma_1 - \frac{\gamma_1 \gamma_2}{d_3 + \gamma_2 + \gamma_3}$. Then $I_\infty > 0$. If I is bounded below by the real positive number m, then $R_\infty \geq \frac{\gamma_1 m}{d_3 + \gamma_2 + \gamma_3}$.*

Proof. Suppose that $I_\infty = 0$. We have in this case two possibilities: $\lim_{t\to\infty} I(t) = 0$ or $0 = \liminf_{t\to\infty} I(t) < \limsup_{t\to\infty} I(t)$. When $\lim_{t\to\infty} I(t) = 0$, on account of the proof of Proposition 3.1 we have $R(t) \underset{t\to\infty}{\to} 0$ and $S(t) \underset{t\to\infty}{\to} \frac{b}{d_1}$. Then for ε sufficiently small and

t sufficiently large $S(t) > \frac{b}{d_1}(1-\varepsilon)$ and

$$\begin{aligned} I' &= \beta SI - (d_2+\gamma_1) I + \gamma_2 R \\ &> \left[\beta \frac{b}{d_1}(1-\varepsilon) - (d_2+\gamma_1)\right] I + \gamma_2 R \\ &= \left[\beta \frac{b}{d_1}(1-\varepsilon) - (d_2+\gamma_1)\right] I + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3} I - \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R'. \end{aligned}$$

Therefore for t sufficiently large

$$\left(I + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R\right)' > \left[\beta \frac{b}{d_1}(1-\varepsilon) - (d_2+\gamma_1) + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}\right] I. \qquad (4)$$

Since $\frac{\beta b}{d_1} > d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$, we can choose ε sufficiently small such that

$$\beta \frac{b}{d_1}(1-\varepsilon) - (d_2+\gamma_1) + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3} > 0.$$

Then for t sufficiently large $\left(I + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R\right)' > 0$. However, $I + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R > 0$ and $\left(I + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R\right) \underset{t\mapsto\infty}{\to} 0$. This is a contradiction in this case. If $0 = \liminf_{t\to\infty} I(t) < \limsup_{t\to\infty} I(t)$, then there exists a sequence $(t_n)_n$ such that $I(t_n) \underset{n\to\infty}{\to} 0$ and $I'(t_n) = 0$. From the second equation of (1) $\gamma_2 R(t_n) = (d_2+\gamma_1) I(t_n) - \beta S(t_n) I(t_n)$. Since $(S(t_n))_n$ is bounded, it follows that $R(t_n) \underset{n\to\infty}{\to} 0$ and $\liminf_{t\to\infty} R(t) = 0$ $(R(t) > 0)$. Therefore $\liminf_{t\to\infty}\left[I(t) + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R(t)\right] = 0$. Hence there exists a sequence, denoted also $(t_n)_n$, such that

$$I(t_n) + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R(t_n) \underset{n\to\infty}{\to} 0 \text{ and } I'(t_n) + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R'(t_n) = 0.$$

We have $I(t_n) \underset{n\to\infty}{\to} 0$ and $R(t_n) \underset{n\to\infty}{\to} 0$ since $(S(t_n))_n$ is bounded. There exists, then, a subsequence, denoted also $(t_n)_n$, such that $(S(t_n))_n$ is convergent and by using Lemma 1 we deduce that $S(t_n) \underset{n\to\infty}{\to} \frac{b}{d_1}$. Applying the inequality (4) we get, for n sufficiently large,

$$\begin{aligned} 0 &= I'(t_n) + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R'(t_n) \\ &> \left[\beta \frac{b}{d_1}(1-\varepsilon) - (d_2+\gamma_1) + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}\right] I(t_n) > 0. \end{aligned}$$

This is also a contradiction.

We now turn to prove the second result. If I is bounded below by $m > 0$, then $R' = \gamma_1 I - (d_3+\gamma_2+\gamma_3) R \geq \gamma_1 m - (d_3+\gamma_2+\gamma_3) R$. Integrating the above inequality we get

$$R(t) \geq \frac{\gamma_1 m}{d_3+\gamma_2+\gamma_3} + \left(R_0 - \frac{\gamma_1 m}{d_3+\gamma_2+\gamma_3}\right) \exp\left[-(d_3+\gamma_2+\gamma_3) t\right].$$

The result follows when t tends towards ∞. □

Theorem 3.1. *Let* (S, I, R) *be the solution of* (1) *such that* $\frac{\beta b}{d_1} > d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$. *Then* $\liminf_{t\to\infty} I(t) > \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$, *where* λ *and* Λ *satisfy*

$$\frac{\beta b}{d_1 + \lambda\beta}\left(1 - \mathrm{e}^{-(d_1+\beta\Lambda)}\right) > d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{2} d_3 + \gamma_2 + \gamma_3. \tag{5}$$

Proof. Since $\frac{\beta b}{d_1} > d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$, there exist λ sufficiently small and Λ sufficiently large such that (5) is satisfied. Firstly we claim that there exists $t_0 > 0$ such that $I(t_0) > \lambda$. If this be not the case, $I(t) < \lambda$ for every $t > 0$. Then

$$S' = b - \beta SI - d_1 S + \gamma_3 R > b - (d_1 + \lambda\beta) S.$$

Integrating the above inequality we obtain for every $t > 0$ that

$$S(t) \geq \frac{b}{d_1 + \lambda\beta} + \left(S_0 - \frac{b}{d_1 + \lambda\beta}\right) \mathrm{e}^{-(d_1+\lambda\beta)t}.$$

Therefore for every $t > \Lambda$ $S(t) \geq \frac{b}{d_1+\lambda\beta}\left(1 - \mathrm{e}^{-(d_1+\lambda\beta)\Lambda}\right) \equiv S^{\Delta}$. Combining the second and third equations in (1) we see that for every $t > \Lambda$

$$\begin{aligned}\left(I + \frac{\gamma_2}{d_3 + \gamma_2 + \gamma_3} R\right)' &= \beta SI - (d_2 + \gamma_1) I + \frac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3} I \\ &> \left(\beta S^{\Delta} - (d_2 + \gamma_1) + \frac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3}\right) I^{\Delta},\end{aligned}$$

where $I^{\Delta} = \inf_{t\geq 0} I(t)$. By Proposition 3.1 we have $I^{\Delta} > 0$. Since

$$\beta S^{\Delta} - (d_2 + \gamma_1) + \frac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3} > 0$$

, we deduce that $I(t) + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R \quad (t) \underset{t\to\infty}{\to} \infty$, which contradicts the fact that $\left(I + \frac{\gamma_2}{d_3+\gamma_2+\gamma_3} R\right)$ is bounded (Proposition 3.1). Hence there exists $t_0 > 0$ such that $I(t_0) > \lambda$ and we cannot have $I(t) < \lambda$ for large t. Therefore we have two possibilities: $I(t) > \lambda$ for large t or I oscillates about λ. We claim in the second case that $I(t) > \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$. If I oscillates about λ such that $I(t_1) = I(t_2) = \lambda$ and $I(t) < \lambda$ for $t \in (t_1, t_2)$, we have $I' = \beta SI - (d_2 + \gamma_1) I + \gamma_2 R > -(d_2 + \gamma_1) I$. It follows by the integration of the previous inequality on $[t_1, t_2]$ that $I(t) \geq I(t_1) \mathrm{e}^{-(d_2+\gamma_1)t} \geq R\lambda \mathrm{e}^{-(d_2+\gamma_1)(t_2-t_1)}$. If $t_2 - t_1 < \Lambda$, then $I(t) > \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$ for every $t \in [t_1, t_2]$. If this be not the case, we have $I(t) > \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$ in $[t_1, t_1 + h]$. We claim that the above inequality remains true in the interval $[t_1 + h, t_2]$. If this be not the case, there exist T_1 and T_2 such that $[T_1, T_2] \subset [t_1 + h, t_2]$ and

$$\begin{cases} I(T_1) = I(T_2) = \lambda \mathrm{e}^{-(d_1+\beta)\Lambda} \\ I(t) < \lambda \mathrm{e}^{-(d_1+\beta)\Lambda} \text{ for } t \in (T_1, T_2) \\ I'(T_1) < 0 < I'(T_2). \end{cases}$$

Using Proposition 2 we have

$$\begin{aligned} I'(T_1) &= \beta S(T_1) I(T_1) - (d_2+\gamma_1) I(T_1) + \gamma_2 R(T_1) \\ &> \left[\beta S^{\Delta} - (d_2+\gamma_1)\right] \lambda \mathrm{e}^{-(d_1+\beta)\Lambda} + \gamma_2 R(T_1) \\ &> \left[\beta S^{\Delta} - (d_2+\gamma_1)\right] \lambda \mathrm{e}^{-(d_1+\beta)\Lambda} + \frac{\lambda_1\lambda_2 I^{\Delta}}{d_2+\gamma_2+\gamma_3}. \end{aligned}$$

Since $0 < I^{\Delta} = \inf_{t\geq 0} I(t) \leq I(t) < \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$ for $t \in [T_1, T_2]$, it follows that

$$I'(T_1) \geq \left(\beta S^{\Delta} - (d_2+\gamma_1) + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}\right) I^{\Delta} > 0.$$

This contradicts the fact that $I'(T_1) < 0$. Therefore $I(t) > \lambda \mathrm{e}^{-(d_1+\beta)\Lambda}$ for t sufficiently large. This completes the proof of the Theorem. □

4 Deterministic Stability

4.1 The local stability of the disease-free point

Theorem 4.1. *The disease-free point,* $\left(\frac{b}{d_1}, 0, 0\right)$*, is locally asymptotically stable for* (1) *if and only if* $\frac{\beta b}{d_1} < d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$.

Proof. Let $u = (u_1, u_2, u_3) = (S - \frac{b}{d_1}, I, R)$. By (1) the t-derivative of u is

$$\begin{cases} u_1' = -\beta\left(u_1 + \frac{b}{d_1}\right) u_2 - d_1 u_1 + \gamma_3 u_3 \\ u_2' = \beta\left(u_1 + \frac{b}{d_1}\right) u_2 - (d_2+\gamma_1) u_2 + \gamma_2 u_3 \\ u_3' = \gamma_1 u_2 - (d_3+\gamma_2+\gamma_3) u_3. \end{cases} \tag{6}$$

Linearising the system (6) at the point $(0,0,0)$ we obtain $u' = Mu$, where

$$M = \begin{pmatrix} -d_1 & -\frac{\beta b}{d_1} & \gamma_3 \\ 0 & \frac{\beta b}{d_1} - (d_2+\gamma_1) & \gamma_2 \\ 0 & \gamma_1 & -(d_3+\gamma_2+\gamma_3) \end{pmatrix}.$$

The matrix M has three eigenvalues,

$$\begin{cases} \lambda_1 = -d_1 \\ \lambda_2 = \frac{1}{2}\left[\frac{\beta b}{d_1} - (d_2+\gamma_1) - (d_3+\gamma_2+\gamma_3) + \sqrt{\Delta}\right] \\ \lambda_3 = \frac{1}{2}\left[\frac{\beta b}{d_1} - (d_2+\gamma_1) - (d_3+\gamma_2+\gamma_3) - \sqrt{\Delta}\right] \end{cases}$$

with

$$\Delta = \left[\frac{\beta b}{d_1} - (d_2+\gamma_1) - (d_3+\gamma_2+\gamma_3)\right]^2 + 4\gamma_1\gamma_2.$$

The disease-free point $\left(\frac{b}{d_1}, 0, 0\right)$ is locally asymptotically stable if and only if the real parts of the eigenvalues are negative. This is equivalent to $\frac{\beta b}{d_1} < d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$. □

4.2 The global stability of the disease-free point

Theorem 4.2. *If* $\frac{\beta b}{d_1} < d_2 + \gamma_1 - \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$, *then the disease-free point,* $(\frac{b}{d_1}, 0, 0)$, *is globally asymptotically stable.*

In order to prove the above theorem we need the following results.

Lemma 4.1. *([11]) Let* D *be a bounded interval in* $\mathbb{R}$ *and* $g : (t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ *be a bounded and uniformly continuous function. Furthermore let* $x : (t_0, \infty) \to \mathbb{R}$ *be a solution of* $x' = g(t, x)$, *which is defined on the whole interval* (t_0, ∞) *. Then*

(i) $\liminf_{t\to\infty} g(t, x_\infty) \le 0 \le \limsup_{t\to\infty} g(t, x_\infty)$ *and*

(ii) $\liminf_{t\to\infty} g(t, x^\infty) \le 0 \le \limsup_{t\to\infty} g(t, x^\infty)$,
where $x^\infty = \limsup_{t\mapsto\infty} x(t)$ *and* $x_\infty = \liminf_{t\mapsto\infty} x(t)$.

Proof. From the second equation of (1) we have $I'(t) = g(t, I(t))$, where $g(t, I) = \beta S(t) I(t) - (d_2 + \gamma_1) I(t) + \gamma_2 R(t)$. Using (ii) of Lemma 2 we deduce that

$$0 \le \limsup_{t\to\infty} g(t, I^\infty)$$

or

$$0 \le \limsup_{t\to\infty} [\beta S(t) I^\infty - (d_2 + \gamma_1) I^\infty + \gamma_2 R(t)]$$

and hence

$$0 \le \beta S^\infty I^\infty - (d_2 + \gamma_1) I^\infty + \gamma_2 R^\infty.$$

Applying Lemma 2 to (1) we get

$$R^\infty \le \frac{\gamma_1}{d_3 + \gamma_2 + \gamma_3} I^\infty. \tag{7}$$

Therefore $0 \le \beta S^\infty I^\infty - (d_2 + \gamma_1) I^\infty + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3} I^\infty$. Use of the previous inequality and the fact that $S^\infty \le N^\infty \le \frac{b}{d_1}$ lead to

$$0 \le \left[\frac{\beta b}{d_1} - (d_2 + \gamma_1) + \frac{\gamma_1\gamma_2}{d_3 + \gamma_2 + \gamma_3}\right] I^\infty.$$

However, $\frac{\beta b}{d_1} - (d_2 + \gamma_1) + \frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3} < 0$. Hence $I^\infty = 0 = \lim_{t\to\infty} I(t)$ and by (7) we obtain $R^\infty = 0 = \lim_{t\to\infty} R(t)$. It remains to show that $\lim_{t\to\infty} S(t) = \frac{b}{d_1}$. To do this it is enough to see that $S_\infty \ge \frac{b}{d_1}$. Applying (i) of Lemma 2 to the equation of (1) we obtain $(b - \beta S_\infty I - d_1 S_\infty + \gamma_3 R)_\infty \le 0$. Then $b - \beta S_\infty I^\infty - d_1 S_\infty + \gamma_3 R_\infty \le 0$. Since $I^\infty = R_\infty = 0$, it follows that $b - d_1 S_\infty \le 0$. □

4.3 The stability of the endemic point

Theorem 4.3. *If* $\frac{\beta b}{d_1} > d_2+\gamma_1-\frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$*, then the endemic point is locally asymptotically stable. Moreover there exists an explicit attractive region* A *for the solution of* (1)*, that is, for any initial condition* (S_0, I_0, R_0) *such that* $(S_0-S^*, I_0-I^*, R_0-R^*) \in A$ *we have*

$$\lim_{t\to\infty}(S(t)-S^*) = \lim_{t\to\infty}(I(t)-I^*) = \lim_{t\to\infty}(R(t)-R^*) = 0.$$

Proof. Let $v_1 = S - S^*$, $v_2 = I - I^*$, $v_3 = R - R^*$ and $v = (v_1, v_2, v_3)$. By (1) the t-derivatives of v_1, v_2 and v_3 are

$$\begin{cases} v_1' = -(d_1+\beta I^*)\ v_1 - \beta\, v_1v_2 - \beta S^* v_2 + \gamma_3 v_3 \\ v_2' = -(d_2+\gamma_1-\beta S^*)\, v_2 + \beta v_1 v_2 + \beta I^*\, v_1 + \gamma_2 v_3 \\ v_3 = \gamma_1 v_2 - (d_3+\gamma_2+\gamma_3)\, v_3. \end{cases} \tag{8}$$

Consider the functional

$$V_1(v) = \frac{1}{2}\left[w_1v_1^2 + w_2v_2^2 + w_3v_3^2 + w_4(v_1+v_2+v_3)^2\right].$$

The first derivative of V_1 along the trajectory of a solution of (8) is

$$\begin{aligned} \dot V_1 \;=\; & -[\beta\, w_1v_2 + (d_1+\beta I^*)\, w_1 + d_1w_4]\, v_1^2 \\ & -[d_2w_4 + (d_2+\gamma_1-\beta S^*)\, w_2 - \beta w_2v_1]\, v_2^2 \\ & -[(d_3+\gamma_2+\gamma_3)\, w_3 + d_3w_4]\, v_3^2 \\ & -[\beta S^* w_1 + (d_1+d_2)\, w_4 - \beta I^*\, w_2]\, v_1v_2 \\ & -[(d_1+d_3)\, w_4 - \gamma_3 w_1]\, v_1v_3 \\ & -[(d_2+d_3)\, w_4 - \gamma_2w_2 - \gamma_1w_3]\, v_2v_3. \end{aligned} \tag{4.9}$$

Choose $w_1,\ w_2$ and w_4 such that $\beta S^\star w_1 + (d_1+d_2)w_4 - \beta I^\star w_2 = 0$ and $(d_1+d_3)w_4 - \gamma_3 w_1 = 0$. Then $w_2 = kw_4$, where $k = \frac{1}{\beta I^*}\left(\frac{d_1+d_3}{\gamma_3}\beta S^* + d_1 + d_2\right)$. The relation in (9) can be expressed in terms of the previous variables as

$$\dot V_1 = -[\beta\, w_1v_2 + (d_1+\beta I^*)\, w_1 + d_1w_4]\, v_1^2 - [d_2w_4 - \beta w_2v_1]\, v_2^2 + P(v_2, v_3),$$

where

$$\begin{aligned} P(v_2,v_3) = -(d_2+\gamma_1-\beta S^*)\, w_2v_2^2 \;&-\; [(d_2+d_3)\, w_4 - \gamma_2w_2 - \gamma_1w_3]\, v_2v_3 \\ &-\; [(d_3+\gamma_2+\gamma_3)\, w_3 + d_3w_4]\, v_3^2. \end{aligned}$$

$P(v_2, v_3)$ is negative if the discriminant

$$\begin{aligned} \delta = [(d_2+d_3)\, w_4 - \gamma_2w_2 - \gamma_1w_3]^2 \;&-\; 4(d_2+\gamma_1-\beta S^*)\, w_2 \\ &\times [(d_3+\gamma_2+\gamma_3)\, w_3 + d_3w_4] \end{aligned}$$

is negative. Since $d_2 + \gamma_1 - \beta S^* = \frac{\gamma_1 \gamma_2}{d_3+\gamma_2+\gamma_3}$, we obviously have

$$\begin{aligned} \delta < 0 \quad & \text{if and only if} \quad \left[(d_2+d_3)\,w_4 - \gamma_2 w_2 - \gamma_1 w_3\right]^2 < 4\,(d_2+\gamma_1-\beta S^*) \\ & \qquad\qquad\qquad \times w_2\left[(d_3+\gamma_2+\gamma_3)\,w_3 + d_3 w_4\right] \\ & \text{if and only if} \quad \left[\gamma_1 \tfrac{w_3}{w_4} - (d_2+d_3-\gamma_2 k)\right]^2 < 4k \\ & \qquad\qquad\qquad \times \left[\gamma_1\gamma_2 \tfrac{w_3}{w_4} + d_3\,(d_2+\gamma_1-\beta S^*)\right] \\ & \text{if and only if} \quad Q\left(\tfrac{w_3}{w_4}\right) < 0 \end{aligned}$$

with

$$\begin{aligned} Q(\frac{w_3}{w_4}) = \gamma_1^2 \left(\frac{w_3}{w_4}\right)^2 - 2\gamma_1\,(d_2+d_3+\gamma_2 k)\,\frac{w_3}{w_4} \quad &+ \quad (d_2+d_3-\gamma_2 k)^2 \\ & -4kd_3\,(d_2+\gamma_1-\beta S^*)\,. \end{aligned}$$

The discriminant of $Q\left(\frac{w_3}{w_4}\right)$ is

$$\begin{aligned} \delta' &= \gamma_1^2\,(d_2+d_3+\gamma_2 k)^2 - \gamma_1^2\,(d_2+d_3-\gamma_2 k)^2 + 4\gamma_1^2 k d_3\,(d_2+\gamma_1-\beta S^*) \\ &= 4\gamma_1^2 k\left[\gamma_2\,(d_2+d_3) + d_3\,(d_2+\gamma_1-\beta S^*)\right]. \end{aligned}$$

Let w and w' be the roots of $Q\left(\frac{w_3}{w_4}\right)$ such that $w' < w$ and $w > 0$. We choose w_3 and w_4 such that $\max(0, w') < \frac{w_3}{w_4} < w$. In this case, $\delta < 0$, we have also $P(v_2, v_3) < 0$. Set $\alpha_1 = \frac{d_2 w_4}{\beta w_2} = \frac{d_2}{\beta k}$, $\alpha_2 = \frac{(d_1+\beta I^*)w_1 + d_1 w_4}{\beta\ w_1} = d_1 + \beta I^* + \frac{d_1\gamma_3}{\beta(d_1+d_3)}$ and $\alpha = \min(\alpha_1, \alpha_2)$. If

$$v_1 < \alpha \text{ and } v_2 > -\alpha, \text{ then } \dot{V}_1 < 0. \tag{10}$$

Let $|v| = \max(|v_1|, |v_2|, |v_3|)$, $\theta = \min\limits_{|v|=\alpha} V_1(v)$ and $A = \left\{v \in \mathbb{R}^3 / \,|v| < \alpha, V_1(v) < \theta\right\}$. We claim that, if $v(0) \in A$, then $|\,v(t)| < \alpha$ for every t. If this be not the case, there exists $\tau > 0$ such that $|v(\tau)| = \alpha$ and

$$|v(t)| < \alpha \text{ for every } t \in [0, \tau)\,. \tag{11}$$

Then the definition of θ implies that $V_1(v(\tau)) \geq \theta$ since $|\,v(\tau)| = \alpha$. Now combining (10) with (11) we obtain $\frac{\mathrm{d}}{\mathrm{d}t}V_1(v(t)) < 0$. Then $V_1(v(t))$ is decreasing and $V_1(v(t)) < V_1(v(0))$ for all $t \in [0, \tau)$, but $V_1(v(t))$ is continuous and therefore $V_1(v(\tau)) \leq V_1(v(0))$. Hence $V_1(v(0)) \geq \theta$ since $v(0) \in A$. This is a contradiction. Finally, if $v(0) \in A$, we have $\dot{V}_1 < 0$. Therefore V_1 is a Lyapunov functional and this completes the proof of the Theorem. □

5 Stochastic Stability

In this Section we discuss the stochastic stability of the following model

$$dX = f(X)\,dt + g(X)\,dB, \tag{12}$$

where $g = (g_1, g_2, g_3)$, g_i , $i = 1, 2, 3$, are locally Lipchitz functions, B is three-dimensional brownian motion (see [4] or the references given therein) and

$$f(X) = \begin{pmatrix} b - \beta SI - d_1 S + \gamma_3 R \\ \beta SI - (d_2 + \gamma_1) I + \gamma_2 R \\ \gamma_1 I - (d_3 + \gamma_2 + \gamma_3) \end{pmatrix}.$$

We denote by L the differential operator associated with (12), defined for a nonnegative function, $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$, by

$$LV = \frac{\partial V}{\partial t} + f^T . \frac{\partial V}{\partial x} + \frac{1}{2} Tr \left[g^T . \frac{\partial^2 V}{\partial x^2} . g \right],$$

where $\frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3} \right)^T$ and $\frac{\partial^2 V}{\partial x^2} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{ij}$, $i, j = 1, 2, 3$, "T" and "Tr" mean, respectively, transposition and trace.
With the reference to the book by Afnas'ev *et al* [1] the following auxiliary results hold.

Theorem 5.1. *Suppose that there exist a nonnegative function $V(t, x) \in C^{1,2}(\mathbb{R}, \mathbb{R}^n)$ and two real positive continuous functions, a and b, and constant $K > 0$ such that, for $|x| < K$, $a(|x|) \leq V(t, x) \leq b(|x|)$.*

(i) *If $LV \leq 0$, $|x| \in]0, K[$, then the trivial solution of (12) is stable in probability.*

(ii) *If there exists a continuous function $\lambda : \mathbb{R}_+^0 \to \mathbb{R}_+^0$, positive on $\mathbb{R}_+$, such that*

$$LV \leq -\lambda(|x|),$$

then the trivial solution of (12) is asymptotically stable.

The best general reference to stability and related results can be found in [6] and [7].

Proposition 5.1. *Let g be a locally Lipchitz function such that* $\operatorname{supp} g \subset \overset{\circ}{\Omega}$. *Then the set Ω is stable by (12).*

Proof. Let $u(0) \in \Omega$. Suppose by contradiction that there exists t_0 such that $u(t_0) \notin \Omega$, then there exists τ_0 such that $u(t) \notin \overset{\circ}{\Omega}$ for every $t \in [\tau_0, t_0]$. Since $\operatorname{supp} g \subset \overset{\circ}{\Omega}$, it follows that $du = f(u)\, dt$ for every $t \in [\tau_0, t_0]$. This is a contradiction to the invariance of Ω by (1) . ☐

5.1 Stability of the disease-free point

Theorem 5.2. *Let $\frac{\beta b}{d_1} < d_2 + \gamma_1 - \frac{\gamma_1 \gamma_2}{d_3 + \gamma_2 + \gamma_3}$. For any locally Lipchitz function g such that* $\operatorname{supp} g \subset \overset{\circ}{\Omega}$ *and*

$$g_1^2(S, I, R) \leq \sigma_1 \left(S - \frac{b}{d_1} \right)^2, \text{ where } \frac{1}{2}\sigma_1 < d_1, \tag{13}$$

the disease-free point $\left(\frac{b}{d_1}, 0, 0 \right)$ is globally asymptotically stable.

Proof. Let $u_1 = S - \frac{b}{d_1},\ u_2 = I$ and $u_3 = R$. Consider the functional,

$$V_2(u) = \frac{1}{2}m_1u_1^2 + m_2u_2 + m_3u_3, u = (u_1, u_2, u_3) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

where the constants m_1, m_2 and m_3 are to be chosen in the course of the proof.

$$\begin{aligned} f^T.\frac{\partial V_2}{\partial u} &= m_1u_1\left[-\beta\left(u_1 + \frac{b}{d_1}\right)u_2 - d_1u_1 + \gamma_3u_3\right] \\ &\quad +m_2\left[\beta\left(u_1 + \frac{b}{d_1}\right)u_2 - (d_2+\gamma_1)u_2 + \gamma_2u_3\right] \\ &\quad +m_3\left[\gamma_1u_2 - (d_3+\gamma_2+\gamma_3)u_3\right] \\ &= -d_1m_1u_1^2 - \left[(d_2+\gamma_1)m_2 - \gamma_1m_3\right]u_2 \\ &\quad -\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2m_2\right]u_3 - \beta m_1\left(u_1 + \frac{b}{d_1}\right)u_1\,u_2 \\ &\quad +\beta m_2\left(u_1 + \frac{b}{d_1}\right)u_2 + \gamma_3m_1u_1\,u_3 \\ &= -d_1m_1u_1^2 - \beta m_1u_1^2\,u_2 - \beta\left(\frac{b}{d_1}m_1 - m_2\right)u_1\,u_2 + \gamma_3m_1u_1\,u_3 \\ &\quad -\left[\left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2 - \gamma_1m_3\right]u_2 \\ &\quad -\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2m_2\right]u_3. \end{aligned}$$

We choose m_1 such that $\frac{b}{d_1}m_1 - m_2 = 0$. We obtain

$$\begin{aligned} f^T.\frac{\partial V_2}{\partial u} &= -d_1m_1u_1^2 - \beta m_1u_1^2\,u_2 - \left[\left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2 - \gamma_1w_3\right]u_2 \\ &\quad -\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2m_2\right]u_3 + \frac{\gamma_3d_1}{b}m_2u_1\,u_3. \end{aligned}$$

Since $S + I + R \le \frac{b}{d_1}$ in Ω, we have $u_1 \le 0,\ 0 < u_2 \le \frac{b}{d_1}$ and $0 < u_3 \le \frac{b}{d_1}$. Hence

$$\begin{aligned} f^T.\frac{\partial V_2}{\partial u} &\le -d_1m_1u_1^2 - \left[\left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2 - \gamma_1m_3\right]u_2 \\ &\quad -\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2m_2\right]u_3. \end{aligned}$$

We choose m_2 and m_3 such that

$$\left(d_2+\gamma_1-\beta\tfrac{b}{d_1}\right)m_2 - \gamma_1m_3 > 0 \quad \text{and} \quad (d_3+\gamma_2+\gamma_3)m_3 - \gamma_2m_2 > 0$$

which are equivalent to $\frac{\gamma_1\gamma_2}{(d_3+\gamma_2+\gamma_3)}m_2 < \gamma_1m_3 < \left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2$. Hence the choice of w_2 and m_3 is possible since $\frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3} < d_2+\gamma_1-\beta\frac{b}{d_1}$. Using (13) we have

$$\begin{aligned}
LV_2 &= \frac{\partial V_2}{\partial t} + f^T.\frac{\partial V_2}{\partial u} + \frac{1}{2}Tr\left[g^T.\frac{\partial^2 V_2}{\partial u^2}.g\right] \\
&= f^T.\frac{\partial V_2}{\partial u} + \frac{1}{2}m_1 g_1^2 \\
&\leq -d_1 m_1 u_1^2 - \left[\left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2 - \gamma_1 m_3\right]u_2 \\
&\quad -\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2 m_2\right]u_3 + \frac{1}{2}\sigma_1 m_1 u_1^2 \\
&\leq -\left(d_1 - \frac{1}{2}\sigma_1\right)m_1 u_1^2 - \frac{b}{d_1}\left[\left(d_2+\gamma_1-\beta\frac{b}{d_1}\right)m_2 - \gamma_1 m_3\right]u_2^2 \\
&\quad -\frac{b}{d_1}\left[(d_3+\gamma_2+\gamma_3)m_3 - \gamma_2 m_2\right]u_3^2.
\end{aligned}$$

By Theorem 5 the proof is complete. □

5.2 Stability of the endemic point

In this Section we use the notation of Theorems 4 and 6 and their proofs.

Lemma 5.1. *Let $a \in \mathbb{R}^*_-$, $b, c \in \mathbb{R}$ and set $\Delta = b^2 - 4ac$. For all $x \in \mathbb{R}$*

$$ax^2 + bx + c < \frac{-\Delta}{4a}.$$

Theorem 5.3. *If $\frac{\beta b}{d_1} > d_2+\gamma_1-\frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}$ for any locally Lipchitz function g such that $g_1^2(S,I,R) \leq \lambda_1(S-S^*)^2$, $g_2^2(S,I,R) \leq \lambda_2(I-I^*)^2$ and $g_3^2(S,I,R) \leq \lambda_3(R-R^*)^2$, where $\frac{1}{2}\lambda_1 < d_1$, $\frac{1}{2}\lambda_2 < \frac{d_2}{k+1}$ and $\frac{1}{2}\lambda_3 < \frac{-w_4 Q\left(\frac{w_3}{w_4}\right)}{4k(d_2+\gamma_1-\beta S^*)(w_3+w_4)}$ for any $\frac{w_3}{w_4}$ such that $\max(0,w') < \frac{w_3}{w_4} < w$, then the endemic point is asymptotically stable.*

Proof. We have

$$\begin{aligned}
LV_1 &= \frac{\partial V_1}{\partial t} + \frac{1}{2}(w_1+w_4)g_1^2 + \frac{1}{2}(w_2+w_4)g_2^2 + \frac{1}{2}(w_3+w_4)g_3^2 \\
&= -\left[\beta\, w_1 v_2 + (d_1+\beta I^*)w_1 + d_1 w_4\right]v_1^2 - \left[d_2 w_4 - \beta w_2 v_1\right]v_2^2 \\
&\quad + P(v_2,v_3) + \frac{1}{2}(w_1+w_4)g_1^2 + \frac{1}{2}(w_2+w_4)g_2^2 + \frac{1}{2}(w_3+w_4)g_3^2.
\end{aligned}$$

Using the estimations satisfied by the functions g_1, g_2 and g_3 we obtain

$$\begin{aligned} LV_1 &\leq -\left[\beta\, w_1 v_2 + \left(d_1 + \beta I^*\right) w_1 + d_1 w_4\right] v_1^2 - \left[d_2 w_4 - \beta w_2 v_1\right] v_2^2 \\ &\quad + P\left(v_2, v_3\right) + \frac{1}{2}\lambda_1 \left(w_1 + w_4\right) v_1^2 + \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) v_2^2 \\ &\quad + \frac{1}{2}\lambda_3 \left(w_3 + w_4\right) v_3^2 \\ &= -\left[\beta\, w_1 v_2 + \left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4\right] v_1^2 \\ &\quad - \left[d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) - \beta w_2 v_1\right] v_2^2 + P\left(v_2, v_3\right) \\ &\quad + \frac{1}{2}\lambda_3 \left(w_3 + w_4\right) v_3^2. \end{aligned}$$

Applying Lemma 3 to the polynomial $P\left(v_2, v_3\right)$ like the function of v_2 we give, namely $P\left(v_2, v_3\right) < \frac{w_4 Q\left(\frac{w_3}{w_4}\right)}{4k\left(d_2+\gamma_1-\beta S^*\right)\left(w_3+w_4\right)} v_3^2$, it follows that

$$\begin{aligned} LV_1 &< -\left[\beta\, w_1 v_2 + \left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4\right] v_1^2 \\ &\quad - \left[d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) - \beta w_2 v_1\right] v_2^2 \\ &\quad + \left[\frac{1}{2}\lambda_3 \left(w_3 + w_4\right) + \frac{w_4 Q\left(\frac{w_3}{w_4}\right)}{4k\left(d_2 + \gamma_1 - \beta S^*\right)\left(w_3 + w_4\right)}\right] v_3^2. \end{aligned}$$

The conditions satisfied by λ_1 and λ_2 lead to

$$k_1 = \frac{\left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4}{\beta\, w_1} > 0$$

and

$$k_2 = \frac{d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right)}{\beta w_2} > 0.$$

Let $k < \min\left(k_1, k_2\right)$. If $|v| = \max\left(|v_1|, |v_2|, |v_3|\right) < k$, we have

$$\begin{aligned} \beta w_1 v_2 &+ \left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4 \\ &> -\beta w_1 k + \left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4 > 0 \end{aligned}$$

and $d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) - \beta w_2 v_1 > d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) - \beta w_2 k > 0$. It follows that

$$\begin{aligned} LV_1 &< -\left[-\beta\, w_1 k + \left(d_1 - \frac{1}{2}\lambda_1 + \beta I^*\right) w_1 + \left(d_1 - \frac{1}{2}\lambda_1\right) w_4\right] v_1^2 \\ &\quad - \left[d_2 w_4 - \frac{1}{2}\lambda_2 \left(w_2 + w_4\right) - \beta w_2 k\right] v_2^2 \\ &\quad + \left[\frac{1}{2}\lambda_3 \left(w_3 + w_4\right) + \frac{w_4 Q\left(\frac{w_3}{w_4}\right)}{4k\left(d_2 + \gamma_1 - \beta S^*\right)\left(w_3 + w_4\right)}\right] v_3^2. \end{aligned}$$

The condition satisfied by λ_3 implies that

$$\frac{1}{2}\lambda_3 \left(w_3 + w_4\right) + \frac{w_4 Q\left(\frac{w_3}{w_4}\right)}{4k\left(d_2 + \gamma_1 - \beta S^*\right)\left(w_3 + w_4\right)} < 0.$$

Finally, when we apply Theorem 5, Theorem 7 follows. □

6 Numerical Examples

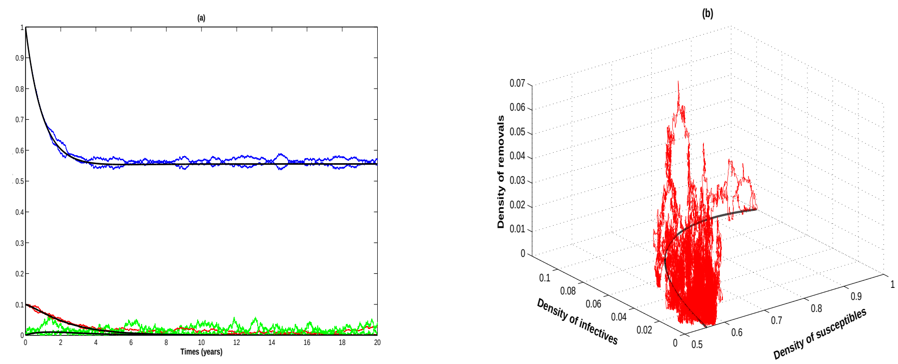

Figure 6.1: The stochastic model and its deterministic model (black). (a) the density of the three classes of individuals (S: blue, I: red,R: green) versus time, (b) the dynamic behaviour of $(S(t), I(t), R(t))$(red). Here $b = 0.5,\ \beta = 0.7,\ d_1 = 0.9,\ d_2 = 0.7,\ d_3 = 0.5,\ \lambda = 0.2,\ \lambda = 0.1,\ \lambda = 0.6$ and we have $R_0 = 0.5$ and $E = (0.55, 0, 0)$. (For interpretation of the references to colour in the legend of this figure the reader is referred to the electronic version of this article.)

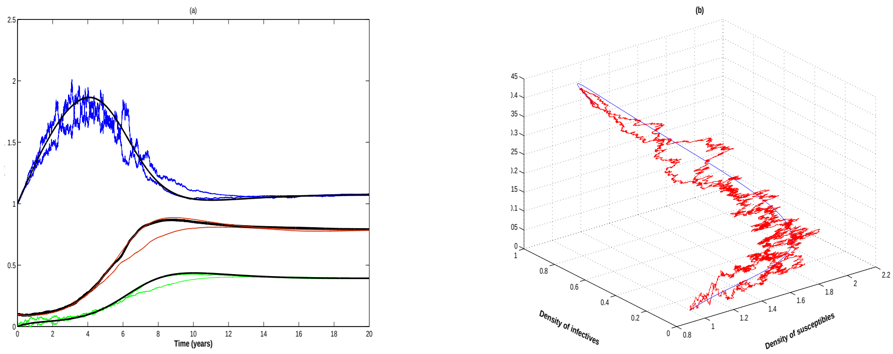

Figure 6.2: The stochastic model and its deterministic model (black). (a) the density of the three classes of individuals (S: blue,I: red,R: green) versus time, (b) the dynamic behavior of $(S(t), I(t), R(t))$ (red). Here $b = 0.5,\ \beta = 0.7,\ d_1 = 0.1,\ d_2 = 0.4,\ d_3 = 0.2,\ \lambda = 0.6,\ \lambda = 0.5,\ \lambda = 0.5$ and we have $R_0 = 4.6$ and $E = (1.07, 0.78, 0.39)$. (For interpretation of the references to colour in the legend of this figure the reader is referred to the electronic version of this article.)

In this Section as an example of random perturbation we adopt the idea of Mukherje in [9]. We allow the random perturbations of the variables (S, I, R) around the disease-free point $E_0 = (b/d_1, 0, 0)$ if the constant $R_0 = \frac{\frac{\beta b}{d_1}}{d_2+\gamma_1-\frac{\gamma_1\gamma_2}{d_3+\gamma_2+\gamma_3}} < 1$ and otherwise around the endemic positive equilibrium point $E^* = (S^*, I^*, R^*)$ in the case when it is asymptotically stable. We assume that the random perturbations are a type of white noise proportional to the distance of S, I and R from values of the equilibria. So the system (12) becomes

$$\begin{cases} \mathrm{d}S = b - \beta SI - d_1 S + \gamma_3 R + \sigma_1(S - E_1)\mathrm{d}W_t^1 \\ \mathrm{d}I = \beta SI - (d_2 + \gamma_1)\, I + \gamma_2 R + \sigma_2(I - E_2)\mathrm{d}W_t^2 \\ \mathrm{d}R = \gamma_1 I - (d_3 + \gamma_2 + \gamma_3)\, R + \sigma_3(R - E_3)\mathrm{d}W_t^3, \end{cases} \tag{12}$$

where (E_1, E_2, E_3) is equal to E_0 or E^*, σ_i, $i = 1, 2, 3$, are real positives constants, W_t^1, W_t^2 and W_t^3 are standard Wiener processes independent from each other (Stroock and Varadham [10]).

In the following we present some numerical simulations of two examples which validate the theoretical results obtained in this paper. For simplicity we choose the initial conditions: $S_0 = 1$, $I_0 = 0.01$ and $\sigma_1 = 0.04$, $\sigma_2 = 0.01$, $\sigma_3 = 0.1$ are supposed to satisfy the conditions of Theorem 6 and Theorem 7. The values of the other parameters are explained in each example.

It can be seen from Figure 6(a) and Figure 6(b) that, when $R_0 < 1$, it increases away from the disease-free point. In this case the endemic equilibrium E^* is asymptotically stable. We can also see that the trajectory of the stochastic process remains close to the trajectory of its deterministic analogue during a finite time interval. We should note that the path of stochastic process eventually leaves the trajectory and is absorbed in the equilibrium point (Figure 1(b) and Figure 2(b)).

7 Conclusion

The dynamic behaviour of deterministic as well as the stochastic model for the spread of an SIRS epidemic are presented in this paper. We established the same properties of stability. The numerical results also indicate that there exist positive-stable disease-free and endemic equilibria. Moreover for future research it should be feasible to use the stochastic differential equation, (12), with a general diffusion term g and to find a suitable Lyapunov functional for unconditional stability of the positive equilibrium of the model, (1), of the SIRS epidemic. Another possible direction for future research is to consider how control strategies may be devised. Finally, taking into account the available statistical data, we can use the stochastic diffusion inference to estimate the parameters of the model.

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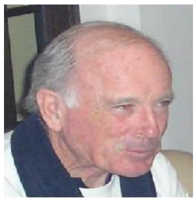

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