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# Existence and Uniqueness of Positive Solutions for $(n-1,1)-$Type BVPs of Two-Term Fractional Differential Equations

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**Abstract:** In this article, we establish existence and uniqueness results for positive solutions of a boundary value problem of the nonlinear two-term fractional differential equation in which the lower fractional derivative order $\beta$ and the higher one $\alpha$ satisfy $\alpha-1<\beta<\alpha$. Our analysis is based on a fixed point theorem. An example is given to illustrate the efficiency of the main theorem.

**Keywords:** Positive solution, multi-term fractional differential equation, multi-point boundary value problems, fixed-point theorem.

## 1 Introduction, Motivation and Preliminaries

Fractional differential equations have many applications in modeling of physical and chemical processes. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [1,2], the survey paper [3] and papers [4,5,6,7,8,9,10,11,12] and the references therein.

In the literature, $D_{0^+}^{\alpha}u(t)+f(t,u(t))=0$ is known as a **single term** fractional differential equation. In certain cases, we find equations containing more than one fractional differential terms. These equations are called **multi-term fractional differential equations**. A classical example is the so-called **Bagley Torvik equation**

$$AD_{0^+}^{2}y(x)+BD_{0^+}^{\frac{3}{2}}y(x)+Cy(x)=f(x),$$

where $A,B,C$ are constants and $f$ is a given function. This equation arises from for example the modeling of motion of a rigid plate immersed in a Newtonian fluid. It was originally proposed in [13]. The Bagley Torvik equation is a linear two-term fractional differential equation. The general nonlinear form of a two term fractional differential equation is of the form $D_{0^+}^{\alpha}x(t)=f(t,x(t),D_{0^+}^{\beta}x(t))$, where $\alpha>\beta>0$.

There has been many papers discussed the solvability of boundary value problems of single term fractional differential equations. For example, E. R. Kaufmann and E. Mboumi [14] studied the following boundary value problem for the single term fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t)+a(t)f(u(t))=0, & 0<t<1, 1<\alpha<2,\\ u(0)=0, u'(1)=0, \end{cases} \tag{1}$$

by using the properties of the Green's function of the corresponding BVP, where $f:[0,1]\times[0,+\infty)\to[0,\infty)$ is continuous, $a\in L^{\infty}[0,1]$, there exists a constant $m>0$ such that $a(t)\geq m$ a.e. $t\in[0,1]$. By using the Leggett-Williams fixed point theorem, the Krasnoselskii fixed point theorem, the authors in [14] proved that BVP(1) has at least one or three positive solutions.

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In [15,16,17,18,19], authors studied the existence of positive solutions for the following singular nonlinear $(n-1,1)$ conjugate-type boundary value problem of single term fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0,\ 0<t<1, n-1<\alpha\leq n, \\ x^{(k)}(0)=0, 0\leq k\leq n-2, x(1)=\int_0^1 x(s)dA(s), \end{cases} \tag{2}$$

where $n\geq 2$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f:(0,1)\times(0,+\infty)\to[0,+\infty)$ is continuous and $f$ may be singular at $x=0$ and $t=0,1$.

Boundary value problems of two-term fractional differential equations were also studied extensively. For example, in recent paper [20], the authors studied the existence of positive solutions of the following boundary value problem of fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t),D_{0^+}^{\alpha-1}u(t))=0, t\in(0,1), 1<\alpha<2, \\ a\lim\limits_{t\to 0}t^{2-\alpha}u(t)-b\lim\limits_{t\to 0}D_{0^+}^{\alpha-1}u(t)=\int_0^1 g(t,u(t),D_{0^+}^{\alpha-1}u(t)), \\ cD_{0^+}^{\alpha-1}+du(1)=\int_0^1 h(t,u(t),D_{0^+}^{\alpha-1}u(t))dt, \end{cases} \tag{3}$$

where $a,b,c,d\geq 0$ with $\delta=ad+bd\Gamma(\alpha)+ac\Gamma(\alpha)>0$, $f,g,h$ defined on $(0,1)\times[0,\infty)\times\mathbf{R}$ are nonnegative Caratheodory functions that may be singular at $t=0$ and $t=1$, $f(t,0,0)\not\equiv 0$ on each subinterval of $[0,1]$, $D_{0^+}^{\alpha}$ ( or $D_{0^+}^{\alpha-1}$ ) is the Riemann-Liouville fractional derivative of order $\alpha$ (or $\alpha-1$).

In paper [21], the authors studied the existence of multiple positive solutions of the following boundary value problem of fractional differential equation

$$\begin{cases} {}^{c}D_{0^+}^{\alpha}u(t)=f(t,u(t),{}^{c}D_{0^+}^{\beta}u(t)), t\in(0,1), \\ u(0)+u'(0)=0, \\ u(1)+u'(1)=0 \end{cases} \tag{4}$$

where $f[0,1]\times[0,\infty)\times\mathbf{R}\to[0,\infty)$ is continuous, ${}^{c}D_{0^+}^{*}$ is the standard Caputo fractional derivative of order $*$, $1<\alpha<2$ and $0<\beta<\alpha-1$.

Boundary value problems of multiple-term higher order fractional differential equations have been studied in known papers. In [22], Zhang studied the existence of positive solutions of the following singular boundary value problem of fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t)=-q(t)f(t,u(t),u'(t),\cdots,u^{(n-2)}(t)), t\in(0,1), \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, u(1)=\int_0^1 u(s)h(s)ds, \end{cases} \tag{5}$$

where $f$ may be singular at $u=0,u'=0,\cdots,u^{(n-2)}=0$, $q$ may be singular at $t=0$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $n-1<\alpha\leq n$.

In [23], authors studied the existence of positive solutions for the following boundary value problem of multi term fractional differential equations

$$\begin{cases} D_{0^+}^{\mu}x(t)+f(t,x(t),D_{0^+}^{\mu_1}x(t),\cdots,D_{0^+}^{\mu_{n-1}}x(t))=0,\ 0<t<1, \\ D_{0^+}^{\mu_i}x(0)=0, 1\leq i\leq n-2, D_{0^+}^{\mu_{n-1}+1}x(0)=0, D_{0^+}^{\mu_{n-1}}x(0)=0, \\ D_{0^+}^{\mu_{n-1}}x(1)=\sum\limits_{j=1}^{m-2}a_jD_{0^+}^{\mu_{n-1}}x(\xi_j), \end{cases} \tag{6}$$

where $n\geq 3$, $D_{0^+}^{*}$ is the standard Riemann-Liouville derivative of order $*$, $0<\mu_1<\cdots<\mu_{n-1}$ and $n-3<\mu_{n-1}<n-2$, $f:(0,1)\times\mathbf{R}^n\to\mathbf{R}$ is continuous, $a_j\in\mathbf{R}$, $0<\xi_1<\cdots<\xi_{m-2}<1$, $\sum\limits_{j=1}^{m-2}a_j\xi_j^{\mu-\mu_{n-1}-1}<1$.

It is easy to see that **Bagley Torvik equation** $AD_{0+}^2 y(x) + BD_{0+}^{\frac{3}{2}} y(x) + Cy(x) = f(x)$ can not be involved in equations discussed in [14,20,21] since $\alpha = 2$ and $\beta = \frac{3}{2}$ with $\alpha > \beta > \alpha - 1$. So it is interesting to study the solvability of the two-term fractional differential equation $D_{0+}^{\alpha} x(t) = f(t, x(t), D_{0+}^{\beta} x(t))$, where $\alpha > \beta > \alpha - 1$, i.e., to establish sufficient conditions for the solvability of boundary value problems of two-term fractional differential equations with the order of lower fractional derivative greater than $\alpha - 1$?

Motivated by mentioned problem, in this paper, we discuss the existence and uniqueness of positive solutions of the following multi-point boundary value problem of nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\beta} u(t)) = 0, a.e., t \in (0,1), \\ \lim\limits_{t \to 0} t^{n-\alpha} u(t) = 0, \ u(1) = \sum\limits_{j=1}^{m} a_j u(\xi_j), \\ D_{0^+}^{\alpha-(n-1)} u(0) = 0, \cdots, D_{0^+}^{\alpha-2} u(0) = 0, \end{cases} \tag{7}$$

where $\alpha \in (n-1, n)$, and $\alpha - 1 < \beta < \alpha$, $D_{0+}^{*}$ is the Riemann-Liouville fractional derivative of order $*$, and $f$ is defined on $(0,1) \times [0, +\infty) \times R$ is a sub-Carathéodory function ($f(t,x,y)$ may be singular at $t = 0, 1$), $a_i \geq 0 (i = 1, 2, \cdots, m)$, $0 = \xi_0 < \xi_1 < \cdots < \xi_m < \xi_{m+1} = 1$ with $\Delta = 1 - \sum\limits_{j=1}^{m} a_j \xi_j^{\alpha-1} > 0$.

A function $x$ defined on $(0,1]$ is called a positive solution if $x(t) > 0$ for all $t \in (0,1)$ and $x$ satisfies all equations in (7). We obtain the results on the existence and uniqueness of the positive solutions of BVP(7) by using a fixed point theorem. An example is given to illustrate the efficiency of the main theorems.

The remainder of this paper is as follows: in section 2, we present preliminary results. In section 3, the main theorems are proved. An example is given in section 4 to illustrate the main results.

## 2 Preliminary results

For the convenience of the readers, we present here the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [7,1,5,2,**?**].

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbf{R}$ is given by

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

provided that the right-hand side exists.

**Definition 2.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbf{R}$ is given by

$$D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha < n$, provided that the right-hand side exists.

**Definition 2.3.** $F : (0,1) \times \mathbf{R}^2 \times \mathbf{R}$ is a sub-Carathéodory function if $f$ satisfies the following items:

**(i)** $t \to F(t, t^{\alpha-n} x_0, t^{\alpha-\beta-n} x_1)$ is continuous on $(0,1)$ for all $(x_0, x_1) \in \mathbf{R}^2$;

**(ii)** $(x_0, x_1) \to F(t, t^{\alpha-n} x_0, t^{\alpha-\beta-n} x_1)$ is continuous on $\mathbf{R}^2$ for all $t \in (0,1)$;

**(iii)** there exist $k > -1$ and $l \in (\max\{\beta - \alpha, n-1-\alpha-k, -k-n\}, 0]$ such that for each $r > 0$ there exists constant $M_r \geq 0$ such that

$$|F(t, t^{\alpha-n} x_0, t^{\alpha-\beta-n} x_1)| \leq t^k (1-t)^l M_r, t \in (0,1), |x_0|, |x_1| \leq r.$$

**Lemma 2.1.** Let $n - 1 < \alpha < n$, $u \in C^0(0, \infty) \bigcap L^1(0, \infty)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots n$.

**Lemma 2.2.** Suppose that $h : (0,1) \to \mathbf{R}$ is continuous and satisfies that there exist $k > -1$ and $l \in (\max\{\beta - \alpha, n-1-\alpha-k, -k-n\}, 0]$ such that $|h(t)| \leq t^k (1-t)^l$ for all $t \in (0,1)$. Then the unique solution of

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, 0 < t < 1, \\ \lim\limits_{t \to 0} t^{n-\alpha} u(t) = 0, \ u(1) = \sum\limits_{j=1}^{m} a_j u(\xi_i), \\ D_{0^+}^{\alpha-(n-1)} u(0) = 0, \cdots, D_{0^+}^{\alpha-2} u(0) = 0, \end{cases} \tag{8}$$

is

$$u(t)=\int_0^1 G(t,s)h(s)ds, \tag{9}$$

where

$$G(t,s)=\frac{1}{\Delta}\begin{cases} -\Delta\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+t^{\alpha-1}\frac{(\xi_{m+1}-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -t^{\alpha-1}\sum\limits_{j=i+1}^{m}a_j\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad \xi_i\le s\le t, i=0,1,\cdots,m, \\ \\ t^{\alpha-1}\frac{(\xi_{m+1}-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -t^{\alpha-1}\sum\limits_{j=i+1}^{m}a_j\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad t\le s\le \xi_{i+1}, i=0,1,\cdots,m. \end{cases} \tag{10}$$

**Proof.** Suppose that $u$ is a solution of BVP(8). By Lemma 2.1, we have from $D_{0^+}^{\alpha}u(t)+h(t)=0$ that

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{j=1}^{n}c_jt^{\alpha-j}, t\in(0,1]$$

for some $c_j\in R,\ \ i=1,2,\cdots,n$. We get

$$D_{0^+}^{\alpha-v}u(t)=-\int_0^t\frac{(t-s)^{\alpha-v-1}}{\Gamma(\alpha-v)}h(s)ds+\sum_{j=1}^{v}c_j\frac{\Gamma(\alpha-j+1)}{\Gamma(v-j+1)}t^{v-j}, v=2,3,\cdots,n-1.$$

It is easy to see from $l\in(\max\{\beta-\alpha,n-1-\alpha-k,-k-2\},0]$ that

$$t^{n-\alpha}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right|\le t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds$$

$$\le t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(t-s)^lds$$

$$=t^{n-\alpha+\alpha+k+l}\int_0^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^kdw\to 0 \text{ as } t\to 0^+,$$

$$\left|\int_0^t\frac{(t-s)^{\alpha-v-1}}{\Gamma(\alpha-v)}h(s)ds\right|\le t^{\alpha+k+l-v}\int_0^1\frac{(1-w)^{\alpha+l-v-1}}{\Gamma(\alpha-v)}w^kdw\to 0 \text{ as } t\to 0^+, v=2,3,\cdots,n-1.$$

From $\lim\limits_{t\to 0}t^{n-\alpha}u(t)=0$, we get $c_n=0$. From $D_{0^+}^{\alpha-(n-1)}u(0)=0$, we get $c_v=0(v=2,3,\cdots,n-1)$. Then $u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1t^{\alpha-1}$. From $u(1)=\sum\limits_{j=1}^{m}a_ju(\xi_i)$, we get

$$-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1=\sum_{j=1}^{m}a_j\left(-\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1\xi_j^{\alpha-1}\right).$$

It follows that

$$c_1=\frac{1}{\Delta}\left(\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-\sum_{j=1}^{m}a_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

Therefore,

$$u(t)=\frac{-\Delta\int_{\xi_0}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+t^{\alpha-1}\int_{\xi_0}^{\xi_{m+1}}\frac{(\xi_{m+1}-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-t^{\alpha-1}\sum\limits_{j=1}^{m}a_j\int_{\xi_0}^{\xi_j}\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds}{\Delta}$$

$$=\int_0^1 G(t,s)h(s)ds.$$

Here the Green's function $G$ is defined by (10).

Reciprocally, let $u$ satisfy (9). We can prove that $u\in X$ and $u$ satisfies (8) by the assumption imposed on $h$. The proof is completed. □

**Lemma 2.3.** Suppose that $a_i \geq 0(i=1,2,\cdots,m)$ with $\sum_{j=1}^{m} a_j\xi_j^{\alpha-1} < 1$. Then

$$G(t,s) \leq \frac{1}{\Delta\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}, s,t \in [0,1],$$

$$G(t,s) \geq \begin{cases} \frac{1}{\Delta\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\sum_{j=1}^{i} a_j\xi_j^{\alpha-1} \geq 0, \xi_i \leq s \leq t \leq \xi_{i+1}(i=0,1,\cdots,m), \\ \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1} \geq 0, \xi_i \leq t \leq s \leq \xi_{i+1}(i=0,1,\cdots,m). \end{cases} \tag{11}$$

**Proof.** For $\xi_i \leq s \leq t(i=0,1,\cdots,m)$, one sees from (10) that

$$\Gamma(\alpha)\left(1-\sum_{j=1}^{m} a_j\xi_j^{\alpha-1}\right)G(t,s) = -\left(1-\sum_{j=1}^{m} a_j\xi_j^{\alpha-1}t^{\alpha-1}\right)t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1}$$

$$+t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}\sum_{j=i+1}^{m} a_j\xi_j^{\alpha-1}\left(1-\frac{s}{\xi_j}\right)^{\alpha-1}$$

$$\geq t^{\alpha-1}(1-s)^{\alpha-1}\left[-\left(1-\sum_{j=1}^{m} a_j\xi_j^{\alpha-1}t^{\alpha-1}\right)+1-\sum_{j=i+1}^{m} a_j\xi_j^{\alpha-1}\right]$$

$$= t^{\alpha-1}(1-s)^{\alpha-1}\sum_{j=1}^{i} a_j\xi_j^{\alpha-1} \geq 0.$$

For $t \leq s \leq \xi_{i+1}(i=0,1,\cdots,m)$, we have from (10) that

$$\Gamma(\alpha)\left(1-\sum_{j=1}^{m} a_j\xi_j^{\alpha-1}\right)G(t,s) = t^{\alpha-1}\left[(1-s)^{\alpha-1} - \sum_{j=i+1}^{m} a_j\xi_j^{\alpha-1}\left(1-\frac{s}{\xi_j}\right)^{\alpha-1}\right]$$

$$\geq t^{\alpha-1}(1-s)^{\alpha-1}\left[1-\sum_{j=i+1}^{m} a_j\xi_j^{\alpha-1}\right] \geq 0.$$

It is easy to show from (10) that $G(t,s) \leq \frac{1}{\Delta\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}, s,t \in [0,1]$. The proof is completed. □

Choose

$$X = \left\{x : x,\ D_{0^+}^{\beta}x \in C(0,1], \lim_{t\to 0} t^{n-\alpha}x(t), \lim_{t\to 0} t^{1+\beta-\alpha}D_{0^+}^{\beta}x(t) \text{ exist}\right\}.$$

For $x \in X$, let $||x|| = \max\left\{\sup_{t\in(0,1]} t^{n-\alpha}|x(t)|,\ \sup_{t\in(0,1]} t^{1+\beta-\alpha}|D_{0^+}^{\beta}x(t)|\right\}$.

**Claim 2.1.** $X$ is a Banach space with $||\cdot||$ defined.

**Proof.** It is easy to see that $X$ is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in $X$. Then $||x_u - x_v|| \to 0,\ u,v \to +\infty$. It follows that

$$\lim_{t\to 0^+} t^{n-\alpha}x_u(t),\ \lim_{t\to 0^+} t^{1+\beta-\alpha}D_{0^+}^{\beta}x_u(t) \text{ exist},$$

$$\sup_{t\in(0,1]} t^{n-\alpha}|x_u(t)-x_v(t)| \to 0, u,v \to +\infty,$$

$$\sup_{t\in(0,1]} t^{1+\beta-\alpha}|D_{0^+}^{\beta}x_u(t) - D_{0^+}^{\beta}x_v(t)| \to 0, u,v \to +\infty.$$

Thus there exists two functions $x_0, y_0 \in C^0[0,1]$ such that

$$\lim_{u\to+\infty} t^{n-\alpha}x_u(t) = x_0(t),\ \lim_{u\to+\infty} t^{1+\beta-\alpha}D_{0^+}^{\beta}x_u(t) = y_0(t).$$

It follows that

$$\sup_{t\in(0,1]} |t^{n-\alpha}x_u(t)-x_0(t)| \to 0, u\to+\infty,$$

$$\sup_{t\in(0,1]} \left|t^{1+\beta-\alpha}D^{\beta}_{0^+}x_u(t)-y_0(t)\right|, u\to+\infty.$$

We have from $\beta\in(\alpha-1,\alpha)$ that

$$\left|I^{\beta}_{0^+}D^{\beta}_{0^+}x_u(t)-I^{\beta}_{0^+}(t^{\alpha-\beta-1}y_0(t))\right| = \left|I^{\beta}_{0^+}(D^{\beta}_{0^+}x_u(t)-t^{\alpha-\beta-1}y_0(t)|\right|$$

$$= \left|I^{\beta}_{0^+}t^{\alpha-\beta-1}(t^{1+\beta-\alpha}D^{\beta}_{0^+}x_u(t)-y_0(t)|\right|$$

$$\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{\alpha-\beta-1}|s^{1+\beta-\alpha}D^{\beta}_{0^+}x_u(s)-y_0(s)|ds$$

$$\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{\alpha-\beta-1}ds \sup_{t\in(0,1]}\left|t^{1+\beta-\alpha}D^{\beta}_{0^+}x_u(t)-y_0(t)\right|$$

$$= t^{\alpha-1}\int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)}w^{\alpha-\beta-1}dw \sup_{t\in(0,1]}\left|t^{1+\beta-\alpha}D^{\beta}_{0^+}x_u(t)-y_0(t)\right|$$

$$\to 0 \text{ as } u\to+\infty.$$

So $\lim_{u\to+\infty} I^{\beta}_{0^+}D^{\beta}_{0^+}x_u(t) = I^{\beta}_{0^+}(t^{\alpha-\beta-1}y_0(t))$. Note $\alpha-1<\beta<\alpha$. Then there exists an integer $m$ such that $m-1<\beta\le m$. Then Lemma 2.1 implies that there exist some numbers $c_{iu}\in\mathbf{R}$ such that $\lim_{u\to+\infty}\left[x_u(t)+\sum_{j=1}^{m}c_{ju}t^{\beta-j}\right] = I^{\beta}_{0^+}(t^{\alpha-\beta-1}y_0(t))$. It follows that there exist numbers $c_{j0}\in\mathbf{R}$ such that $t^{\alpha-n}x_0(t)+\sum_{j=1}^{m}c_{j0}t^{\beta-j} = I^{\beta}_{0^+}(t^{\alpha-\beta-1}y_0(t))$. It follows that $t^{\alpha-\beta-1}y_0(t) = D^{\beta}_{0^+}(t^{\alpha-n}x_0(t))$. So $t\to t^{\alpha-n}x_0(t)$ is an element in $X$ with $x_u\to x_0$ as $u\to+\infty$. It follows that $X$ is a Banach space. □

**Claim 2.2.** Let $M$ be a subset of $X$. Then $M$ is relatively compact if and only if the following conditions are satisfied:

**(i)** both $\{t\to t^{n-\alpha}x(t): x\in M\}$ and $\{t\to t^{1+\beta-\alpha}D^{\beta}_{0^+}x(t): x\in M\}$ are uniformly bounded,

**(ii)** both $\{t\to t^{n-\alpha}x(t): x\in M\}$ and $\{t\to t^{1+\beta-\alpha}D^{\beta}_{0^+}x(t): x\in M\}$ are equicontinuous in $(0,1]$.

**Proof.** Let

$$t^{n-\alpha}\overline{x}(t) = \begin{cases} t^{n-\alpha}x(t), t\in(0,1], \\ \lim_{t\to0^+} t^{n-\alpha}x(t), t=0, \end{cases} \quad t^{1+\beta-\alpha}D^{\beta}_{0^+}\overline{x}(t) = \begin{cases} t^{1+\beta-\alpha}D^{\beta}_{0^+}x(t), t\in(0,1], \\ \lim_{t\to0^+} t^{1+\beta-\alpha}D^{\beta}_{0^+}x(t), t=0. \end{cases}$$

By Ascoli-Arzela theorem, $M$ is relatively compact if and only if (i) and (ii) hold. Consequently, the Claim is proved. □

We seek solutions of BVP(7) that lie in the cone $P=\{u\in X: u(t)\ge0,\ 0<t\le1\}$. Define the operator $T:P\to P$, by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s),D^{\beta}_{0^+}u(s))ds, u\in P.$$

**Lemma 2.4.** Suppose that $f$ is a sub-Carathéodory function. Then $T:P\to P$ is completely continuous.

**Proof.** For $x\in P$, we firstly prove that $Tx\in X$. Since $f$ is a sub-Carathéodory function and $||x||=r$, then there exists $M_r\ge0$ such that

$$|f(t,x(t),D^{\beta}_{0^+}x(t))| = |f(t,t^{\alpha-n}t^{n-\alpha}x(s),t^{\alpha-\beta-1}t^{1+\beta-\alpha}D^{\beta}_{0^+}x(s))| \le M_r t^k(1-t)^l, t\in(t,1).$$

By the definition of $T$, we have

$$(Tx)(t) = -\int_{\xi_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),D_{0^+}^\beta x(s))ds$$

$$+t^{\alpha-1}\frac{\int_0^1(1-s)^{\alpha-1}f(s,x(s),D_{0^+}^\beta x(s))ds-\sum_{j=1}^m a_j\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}f(s,x(s),D_{0^+}^\beta x(s))ds}{\Gamma(\alpha)\Delta},$$

$$D_{0^+}^\beta(Tx)(t) = -\int_{\xi_0}^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s,x(s),D_{0^+}^\beta x(s))ds$$

$$+t^{\alpha-\beta-1}\frac{\int_0^1(1-s)^{\alpha-1}f(s,x(s),D_{0^+}^\beta x(s))ds-\sum_{j=1}^m a_j\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}f(s,x(s),D_{0^+}^\beta x(s))ds}{\Gamma(\alpha-\beta)\Delta}$$

One sees that $Tx, D_{0^+}^\beta(Tx)\in C^0(0,1]$. Furthermore,

$$t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),D_{0^+}^\beta x(s))ds\right| \le M_r t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds$$

$$\le M_r t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(t-s)^l ds$$

$$= M_r t^{n-\alpha+\alpha+k+l}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw \to 0 \text{ as } t\to 0^+,$$

$$\left|\int_0^t \frac{(t-s)^{\alpha-v-1}}{\Gamma(\alpha-v)} f(s,x(s),D_{0^+}^\beta x(s))ds\right| \le M_r t^{\alpha+k+l-v}\int_0^1 \frac{(1-w)^{\alpha+l-v-1}}{\Gamma(\alpha-v)} w^k dw$$

$$\to 0 \text{ as } t\to 0^+, v=2,3,\cdots,n-1.$$

Then $\lim_{t\to 0^+} t^{n-\alpha}(Tx)(t)$ and $\lim_{t\to 0^+} t^{1+\beta-\alpha}D_{0^+}^\beta(Tx)(t)$ exist. Hence $Tx\in X$. So $T:P\to P$ is well defined comes from $G$ and $f$ are non-negative and Lemma 2.2. Let

$$t^{n-\alpha}\overline{(Tx)}(t) = \begin{cases} t^{n-\alpha}(Tx)(t), t\in(0,1], \\ \lim_{t\to 0^+} t^{n-\alpha}(Tx)(t), t=0, \end{cases}$$

$$t^{1+\beta-\alpha}D_{0^+}^\beta\overline{(Tx)}(t) = \begin{cases} t^{1+\beta-\alpha}D_{0^+}^\beta(Tx)(t), t\in(0,1], \\ \lim_{t\to 0^+} t^{1+\beta-\alpha}D_{0^+}^\beta(Tx)(t), t=0. \end{cases}$$

Thus, both $t\to t^{n-\alpha}\overline{(Tx)}(t)$ and $t\to t^{1+\beta-\alpha}D_{0^+}^\beta\overline{(Tx)}(t)$ are continuous on $[0,1]$, the completely continuous property of $T:P\to P$ can be derived by a similar method see Lemma 2.3 in [**?**]. □

## 3 Main Result

In this section, we prove the main result. The result is based upon the assumption that $f$ is a sub-Carathéodory function.

**Theorem 3.1.** Suppose that $\Delta>0$, and $f:(0,1)\times[0,+\infty)\times R\to[0,\infty)$ is a Carathéodory function and there exist numbers $k_i>-1$ and $l_i>\max\{\beta-\alpha, n-1-\alpha-k_i, -k_i-n, 0\}(i=1,2)$, $M_1,M_2\ge 0$ such that $f$ satisfies $f(t,t^{\alpha-n}u,t^{\alpha-\beta-1}v)\not\equiv 0$ on $[a,b]\times[0,r]\times[-r,r](0<a<b<1)$ and

$$|f(t,t^{\alpha-n}u_1,t^{\alpha-\beta-1}v_1)-f(t,t^{\alpha-n}u_1,t^{\alpha-\beta-1}v_2)|$$

$$\le M_1t^{k_1}(1-t)^{l_1}|u_1-u_2|+M_2t^{k_2}(1-t)^{l_2}|v_1-v_2|,$$

holds for all $t \in (0,1), u_1, u_2 \in [0,\infty)$ and $v_1, v_2 \in \mathbf{R}$. Then BVP(7) has a unique positive solution if

$$M = \max\left\{ \sum_{s=1}^{2} \frac{M_s}{\Delta} \left( \Delta + 1 + \sum_{j=1}^{m} a_j \xi_j^{\alpha+k_s+l_s} \right) \frac{\mathbf{B}(\alpha+l_s,k_s+1)}{\Gamma(\alpha)}, \right.$$

$$\left. \sum_{s=1}^{2} \frac{M_s}{\Delta} \left[ \Delta \frac{\mathbf{B}(\alpha+l_s-\beta,k_s+1)}{\Gamma(\alpha-\beta)} + \left(1 + \sum_{j=1}^{m} a_j \xi_j^{\alpha+k_s+l_s}\right) \frac{\mathbf{B}(\alpha+l_s,k_s+1)}{\Gamma(\alpha-\beta)} \right] \right\} < 1.$$

**Proof.** We shall prove that under the assumptions supposed, $T$ is a contraction operator. Indeed, one gets that

$$\begin{aligned}
&|f(t,u(t),D_{0^+}^{\beta}u(t)) - f(t,v(t),D_{0^+}^{\beta}v(t))| \\
&= |f(t,t^{\alpha-n}t^{n-\alpha}u(t),t^{\alpha-\beta-1}t^{1+\beta-\alpha}D_{0^+}^{\beta}u(t)) - f(t,t^{\alpha-n}t^{n-\alpha}v(t),t^{\alpha-\beta-1}t^{1+\beta-\alpha}D_{0^+}^{\beta}v(t))| \\
&\leq M_1 t^{k_1}(1-t)^{l_1}t^{n-\alpha}|u(t)-v(t)| + M_2 t^{k_2}(1-t)^{l_2}t^{1+\beta-\alpha}|D_{0^+}^{\beta}u(t) - D_{0^+}^{\beta}v(t)| \\
&\leq M_1 t^{k_1}(1-t)^{l_1}||u-v|| + M_2 t^{k_2}(1-t)^{l_2}||u-v||.
\end{aligned}$$

We have that

$$\begin{aligned}
&t^{n-\alpha}|(Tu)(t)-(Tv)(t)| \\
&\leq t^{n-\alpha}\int_0^1 G(t,s)|f(s,u(s),D_{0^+}^{\beta}u(s)) - f(s,v(s),D_{0^+}^{\beta}v(s)|ds \\
&\leq t^{n-\alpha}\int_0^1 G(t,s)[M_1 s^{k_1}(1-s)^{l_1} + M_2 s^{k_2}(1-s)^{l_2}]ds||u-v|| \\
&= \frac{M_1 t^{n-\alpha}||u-v||}{\Delta}\left[ -\Delta \int_{\xi_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}ds + t^{\alpha-1}\int_{\xi_0}^{\xi_{m+1}} \frac{(\xi_{m+1}-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}ds \right. \\
&\left. -t^{\alpha-1}\sum_{j=1}^{m} a_j \int_{\xi_0}^{\xi_j} \frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}ds \right] \\
&+\frac{M_2 t^{n-\alpha}||u-v||}{\Delta}\left[ -\Delta \int_{\xi_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2}ds + t^{\alpha-1}\int_{\xi_0}^{\xi_{m+1}} \frac{(\xi_{m+1}-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2}ds \right. \\
&\left. -t^{\alpha-1}\sum_{j=1}^{m} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2}ds \right] \\
&\leq \frac{M_1 t^{n-\alpha}||u-v||}{\Delta}\left[ \Delta \int_0^{t} \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds \right. \\
&\left. +t^{\alpha-1}\sum_{j=1}^{m} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds \right] \\
&+\frac{M_2 t^{n-\alpha}||u-v||}{\Delta}\left[ \Delta \int_0^{t} \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds \right. \\
&\left. +t^{\alpha-1}\sum_{j=1}^{m} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds \right] \\
&= \frac{M_1 t^{n-\alpha}||u-v||}{\Delta}\left[ \Delta t^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1}ds \right. \\
&\left. +t^{\alpha-1}\sum_{j=1}^{m} a_j \xi_j^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1}dw \right]
\end{aligned}$$

$$+\frac{M_2t^{n-\alpha}||u-v||}{\Delta}\left[\Delta t^{\alpha+k_2+l_2}\int_0^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw+t^{\alpha-1}\int_0^1\frac{(1-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds\right.$$

$$\left.+t^{\alpha-1}\sum_{j=1}^m a_j\xi_j^{\alpha+k_2+l_2}\int_0^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\right]$$

$$=\frac{M_1||u-v||}{\Delta}\left[\Delta t^{n+k_1+l_1}+t^{n-1}+t^{n-1}\sum_{j=1}^m a_j\xi_j^{\alpha+k_1+l_1}\right]\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$

$$+\frac{M_2||u-v||}{\Delta}\left[\Delta t^{n+k_2+l_2}+t^{n-1}+t^{n-1}\sum_{j=1}^m a_j\xi_j^{\alpha+k_2+l_2}\right]\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}$$

$$\le ||u-v||\sum_{s=1}^2\frac{M_s}{\Delta}\left(\Delta+1+\sum_{j=1}^m a_j\xi_j^{\alpha+k_s+l_s}\right)\frac{\mathbf{B}(\alpha+l_s,k_s+1)}{\Gamma(\alpha)}.$$

On the other hand, we have

$$D_{0^+}^{\beta}(Tu)(t)=\frac{1}{\Gamma(\alpha-\beta)\Delta}\left[-\Delta\int_{\xi_0}^t(t-s)^{\alpha-\beta-1}f(s,u(s),D_{0^+}^{\beta}u(s))ds\right.$$

$$+t^{\alpha-\beta-1}\int_{\xi_0}^{\xi_{m+1}}(\xi_{m+1}-s)^{\alpha-1}f(s,u(s),D_{0^+}^{\beta}u(s))ds$$

$$\left.-t^{\alpha-\beta-1}\sum_{j=1}^m a_j\int_{\xi_0}^{\xi_j}(\xi_j-s)^{\alpha-1}f(s,u(s),D_{0^+}^{\beta}u(s))ds\right].$$

Note $\alpha-\beta+l_i>0,\alpha-(n-1)+k_i+l_i>0$ implies that $1+k_i+l_i>0$. Similarly we get

$$t^{1+\beta-\alpha}|D_{0^+}^{\beta}(Tu)(t)-D_{0^+}^{\beta}(Tv)(t)|$$

$$\le\frac{||u-v||t^{1+\beta-\alpha}}{\Gamma(\alpha-\beta)\Delta}\left[\Delta\int_{\xi_0}^t(t-s)^{\alpha-\beta-1}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds\right.$$

$$+t^{\alpha-\beta-1}\int_{\xi_0}^{\xi_{m+1}}(\xi_{m+1}-s)^{\alpha-1}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds$$

$$\left.t^{\alpha-\beta-1}\sum_{j=1}^m a_j\int_{\xi_0}^{\xi_j}(\xi_j-s)^{\alpha-1}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds\right]$$

$$\le\frac{M_1||u-v||}{\Delta}\left[\Delta t^{1+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1-\beta,k_1+1)}{\Gamma(\alpha-\beta)}+\left(1+\sum_{j=1}^m a_j\xi_j^{\alpha+k_1+l_1}\right)\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha-\beta)}\right]$$

$$+\frac{M_2||u-v||}{\Delta}\left[\Delta t^{1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2-\beta,k_2+1)}{\Gamma(\alpha-\beta)}+\left(1+\sum_{j=1}^m a_j\xi_j^{\alpha+k_2+l_2}\right)\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha-\beta)}\right]$$

$$\le||u-v||\sum_{s=1}^2\frac{M_s}{\Delta}\left[\Delta\frac{\mathbf{B}(\alpha+l_i-\beta,k_i+1)}{\Gamma(\alpha-\beta)}+\left(1+\sum_{j=1}^m a_j\xi_j^{\alpha+k_i+l_i}\right)\frac{\mathbf{B}(\alpha+l_i,k_i+1)}{\Gamma(\alpha-\beta)}\right].$$

Therefor, we get $||Tu-T||\le M||u-v||$. Hence the contraction map principle implies that $T$ has a fixed point $x\in P$ by $M<1$. Since $T:P\to P$, then we have $x(t)\ge 0$ for all $t\in(0,1]$. Furthermore, for $t\in(0,1)$, there exists $i$ such that

$t \in [\xi_i, \xi_{i+1})$. Then

$$x(t) = \int_0^1 G(t,s) f(s,x(s),D_{0^+}^{\beta}x(s))ds = \sum_{s=0}^{m} \int_{\xi_i}^{\xi_{i+1}} G(t,s) f(s,x(s),D_{0^+}^{\beta}x(s))ds$$

$$\geq \int_{\xi_i}^{t} G(t,s) f(s,x(s),D_{0^+}^{\beta}x(s))ds + \int_t^{\xi_{i+1}} G(t,s) f(s,x(s),D_{0^+}^{\beta}x(s))ds$$

$$\geq \frac{t^{\alpha-1}}{\Delta}\left(\sum_{j=1}^{i} a_j \xi_j^{\alpha-1} \frac{\int_{\xi_i}^{t}(1-s)^{\alpha-1} f(s,x(s),D_{0^+}^{\beta}x(s))ds}{\Gamma(\alpha)} + \Delta \frac{\int_t^{\xi_{i+1}}(1-s)^{\alpha-1} f(s,x(s),D_{0^+}^{\beta}x(s))ds}{\Gamma(\alpha)}\right)$$

$$\geq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_t^{\xi_{i+1}} (1-s)^{\alpha-1} f(s,x(s),D_{0^+}^{\beta}x(s))ds.$$

Since $f(t,t^{\alpha-n}u,t^{\alpha-\beta-1}v) \not\equiv 0$ on $[a,b]\times[0,r]\times[-r,r](0<a<b<1)$, then there exists $(t_0,u_0,v_0) \in [t,\xi_{i+1}]\times[0,||x||]\times[-||x||,||x||]$ such that $f(t_0,t_0^{\alpha-n}u_0,t_0^{\alpha-\beta-1}v_0) > 0$. So $f(t_0,x(t_0),D_{0^+}^{\beta}x(t_0)) > 0$. Then

$$x(t) \geq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_t^{\xi_{i+1}} (1-s)^{\alpha-1} f(s,x(s),D_{0^+}^{\beta}x(s))ds > 0$$

for every $t \in (0,1)$. Then $x$ is positive. So BVP(7) has an unique positive solution. The proof is completed. □

## 4 An example

In this section, we give an example to illustrate the main theorem.

**Example 4.1.** Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) + t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}\left(A + B_1 t^{\frac{1}{2}}u(t) + B_2 t^{\frac{1}{4}}|D_{0^+}^{\frac{7}{4}}u(t)|\right) = 0, a.e., t \in (0,1), \\ \lim_{t\to 0} t^{\frac{1}{2}}u(t) = 0, \quad u(1) = \frac{1}{2}u\left(\frac{1}{2}\right) + \frac{1}{3}u\left(\frac{3}{4}\right), \\ D_{0^+}^{\frac{1}{2}}u(0) = 0. \end{cases} \tag{12}$$

where $A > 0, B_1, B_2 \geq 0$.

Corresponding to BVP(7), we find that $n-1 = 2 < \alpha = \frac{5}{2} < 3 = n$, $\beta = \frac{7}{4}$ and

$$f(t,x,y) = t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}\left(A + Bt^{\frac{1}{2}}x + Ct^{\frac{1}{4}}y\right).$$

One sees that $\Delta = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}} > 0$, $f\left(t,t^{-\frac{1}{2}}u,t^{-\frac{1}{4}}v\right) \not\equiv 0$ on $[a,b]\times[0,r]\times[-r,r](0<a<b<1)$, $f(t,u)$ satisfies

$$\left|f\left(t,t^{-\frac{1}{2}}u_1,t^{-\frac{1}{4}}v_1\right) - f\left(t,t^{-\frac{1}{2}}u_1,t^{-\frac{1}{4}}v_1\right)\right|$$

$$\leq t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}[B_1|u_1-u_2| + B_2|v_1-v_2|], t \in (0,1), u_1,u_2 \in [0,\infty), v_1,v_2 \in R.$$

Let $M_1 = B_1, M_2 = B_2$ and $k_1 = k_2 = 2$ and $l_1 = l_2 = -\frac{1}{8}$. One finds that $k_i > -1$ and $l_i > \max\{\beta-\alpha, n-1-\alpha-k_i, -k_i-n, 0\}(i=1,2)$. Hence Theorem 3.1 implies that BVP(12) has a unique positive solution if

$$M = [B_1+B_2]\max\left\{\frac{2-\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}-\frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}}+\frac{1}{2}\left(\frac{1}{2}\right)^{9/4}+\frac{1}{3}\left(\frac{3}{4}\right)^{9/4}}{1-\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}-\frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}}}\frac{\mathbf{B}(5/4,3)}{\Gamma(5/2)},\right.$$
$$\left.\frac{\left(1-\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}-\frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}}\right)\frac{\mathbf{B}(5/8,3)}{\Gamma(3/4)}+\left(1+\frac{1}{2}\left(\frac{1}{2}\right)^{9/4}+\frac{1}{3}\left(\frac{3}{4}\right)^{9/4}\right)\frac{\mathbf{B}(5/2,3)}{\Gamma(3/4)}}{1-\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}-\frac{1}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}}}\right\} < 1. \tag{13}$$

By Mathlab 7.0, (13) holds if $0.6997[B_1+B_2] < 1$. It is easy to see that BVP(12) has a unique positive solution if $B_1 + B_2 < \frac{1}{0.6997}$. □

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