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The Unknown Set of Memory Constitutive Equations of Plastic Media

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Abstract: Plastic media have the particular property to exhibit non zero strain at a vanishing applied stress and are modeled by a variety of possible constitutive memory equations. In this note using laboratory data we compare the results based on the constitutive equations of Polycrystalline Halite (PH) which include the Caputo and Caputo-Fabrizio fractional derivatives to model the plastic properties of this medium. Based on this finding we suggest that a variety of different mathematical memory formalisms is needed to model all phenomena and media with memory. We suggest a method to obtain these mathematical formalisms with laboratory experiments which leads to determine the plastic properties of PH showing that its strain memory is much longer than that of stress.

Keywords: Strain, stress, plastic media, constitutive equations, fractional derivatives, memory formalism.

1 Introduction

Elastic properties of media have been modeled since long time using the classic Hooke's law, anelastic properties have been mathematically modeled by the following relations

$$\varepsilon(t) = \int_0^t r(t-u)\tau(u)du,$$

$$\tau(t) = \int_0^t \bar{r}(t-u)\varepsilon(u)du,$$

where $\varepsilon(t)$ is strain, $\tau(t)$ is stress, $r(t)$ and $\bar{r}(t)$ are called the causal functions and show the response of the medium to a unit impulse of stress and strain respectively. Later the anelastic properties have been modeled by several variations of Hooke's law obtained introducing the first order derivative the general case being that of the standard linear solid [1]. In order to model the dissipation of energy or the hysteretic and fatigue phenomena in different anelastic media is now usual to introduce in their constitutive equations the Caputo derivatives of fractional order, often considered as a memory operator A similar procedure is used for porous media in order to model the variable diffusivity due to the history of the previous flux [2] and also for other mathematical, physical, geophysical, biological and economic phenomena [3-12]. The number of papers written for modeling scientific phenomena using fractional calculus is so large that is would take a book to list all of them [13]. Is to be noted in the great variety of fields of science and in the great variety of phenomena studied with the use the fractional calculus the instrument used is the Caputo fractional derivative. It is then quite challenging that the same mathematical instrument may be used with success in the fitting of the experimental data in all the fields considered above. As we noted these fields range from biology and medical science to engineering and material properties, from all the variety of branches of geophysics to all the branches of economy and finance, from theoretical physics to chemistry, from geology to demography. Many interesting research articles have been written on the fractional derivative, on its physical and geometric meaning [14] and on its relation with fractals [15-16], few in particular on its definition [13,17-18], with common consensus on the its importance and the definition, although the latter not yet

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official. In the applications a frequent use is made of fractional derivatives for setting model of the constitutive equations for instance of anelastic media or diffusion. In few cases the validity of these constitutive equations have been studied experimentally [2]. Some models of constitutive equations obviously give better results than others in the approximation of the properties of the medium, most of them seem to give important information on the memory properties of the medium examined. However, as we already noted, is quite surprising how the single Caputo derivative may be helpful in modeling mathematically with the help of only two free parameters the properties of so many different phenomena and different media and in the fitting of the data resulting from many different experiments. Is then in order to investigate the possibility that the memory operators are in fact formally different in the different media or phenomena. The recent notes of Caputo and Fabrizio [19-20] are of help to the solution of the problem and allow to chose, among the available memory operators, which one would give an adequate fitting to the experimental data and a more appropriate model type. However the question remains open on which would be the memory formalism to introduce in the constitutive equations in order to obtain the best mathematical modeling of the properties of the medium examined. It is then reasonable to think that the answer could be imbedded in the experimental data and one may then seek the solution there. This approach has already been used by Caputo [21] concerning generic anelastic media giving the formulae identifying the memory operator, not necessarily a fractional derivative, more appropriate to fit the results of the experiments. Obviously there are different approaches to the solution of the problem depending on the variety of appropriate analytical models used to define the form of the sought memory operators. This note is addressed to the memory operators concerning the plastic media recently modeled by Caputo and Fabrizio [20] who used a set of constitutive equations, which is a particular case of that previously mentioned of Caputo [21,22], but used for the specific purpose of modeling plastic media. We will show that in the modeling the rheology of Polycristalline Halite the fractional derivative with exponential kernel of Caputo and Fabrizio [19] would be more adequate.

2 The Modeling of Plastic Media

Most literature on applied fractional calculus shows that the successful use of the presently available fractional derivatives is a proof of the presence of memory in many scientific phenomena and is a first order approximation in taking account the memory phenomenon which is needed mostly, but not only, for taking into account the second law of thermodynamics. The literature on the constitutive equations for the mathematical modeling of plastic phenomena is vast beginning with Volterra [23] who modeled the phenomenon using hereditary mathematical tools and whose dislocation theory is the base of a new branch of plasticity studies [24] where the phenomenon of plasticity is considered due to the migration of dislocations. Is to be noted also the rich book of Argon [25], which appeared before the quick diffusion of fractional calculus, with the presentation and discussion of a variety cases. Concerning hereditary phenomena are also of interest the volume of Graffi [26] and the note of Fichera [27]. The applications of fractional calculus to plasticity was considered also in the notes of Caputo [21,22] who studied the rheological properties of polycrystalline halite, appearing in nature in large thick deposits, which were considered for the disposal of radioactive waste and whose experimental data will be used in the present note and discussed with the constitutive equations already used by Caputo and Fabrizio [19]. In order to find which could be the most appropriate model for the memory of a plastic medium let us consider the above mentioned set of constitutive equation for rheological media already studied by Caputo and Fabrizio [20].

$$h(u,t)*D\tau_{ij}+\mu(\tau_{ij}-\delta_{ij}\tau_{rr}/3)=(\lambda\delta_{ij}k(u,t)*D\varepsilon_{rr}+2\mu k(u,t)*D\varepsilon_{ij}), \tag{1}$$

whose LT, provided $\varepsilon_{ij}(0)=\tau_{ij}(0)=0$, may be written

$$pT_{ij}+\mu(T_{ij}-\delta_{ij}T_{rr}/3)/H(u,p)=(\lambda\delta_{ij}K(u,p)pE_{rr}+2\mu K(u,p)pE_{ij})/H(u,p), \tag{2}$$

where capital letters indicate LT of the function with similar lower case letter, D means classic derivative of first order, u is the order of the memory operator of $h(u,t)$, $k(u,t)$ which are L_1 , monotonically decreasing with $h(u,\infty)=0, k(u,\infty)=0, h(0,t)=k(0,t)=1$. The latter condition, would imply that the application of the operators $h(0,t)*D$ and $k(0,t)*D$ to a function reproduces the function itself, which however is not mandatory for the developments in this note but mostly matter of elegance and formality for similarity with the memory operators presently used. The range of u depends on the problem considered. All the functions concerning the physical conditions of the medium modeled by equations (1) and (2) are assumed to be initially zero that is the medium to be initially at rest. Examples of generic functions$h(u,t)$ and $k(u,t)$ are

$$h(t),k(t)=exp(-ut) \tag{1'}$$

or

$$h(t), k(t) = 1/log(e+ut),$$

where with $u = 0$ follows $h(0,t) = k(0,t) = 1$, which imply that the operator of order zero reproduces the function, that are monotonically decreasing and satisfy the conditions $h(\infty) = k(\infty) = 0$, moreover setting equations ($1'$) in the equation (1) would produce a general memory form of Standard Linear Solid. When $u = 1$ both operators are not representing the first order derivative of the function and $exp(-ut)$and $1/log(e+ut)$ are not kernels of a fractional derivatives of order u. Obviously the functions h(u,t) and k(u,t) would not define an operator with the all properties of the classical fractional derivative but are simple, hopefully useful, memory formalisms which reproduce the function when the operator has order $u = 0$. Classic examples of function $h(u,t) * D$ and $k(u,t) * D$ are the Caputo and the Caputo-Fabrizio fractional derivatives [19].

Setting

$$\begin{aligned} K/H &= N(u,p), \\ M &= N/H, \\ m &= n/h = (k/h)/h = k/h^2, \end{aligned} \tag{3}$$

we obtain from equation (2)

$$\begin{aligned} pT_{11} + \mu(T_{11} - T_{rr}/3)/H(u,p) &= (\lambda N(u,p)pE_{rr} + 2\mu N(u,p)pE_{11}), \\ pT_{22} + \mu(T_{11} - T_{rr}/3)/H(u,p) &= (\lambda N(u,p)pE_{rr} + 2\mu N(u,p)pE_{22})/H, \\ pT_{33} + \mu(T_{11} - T_{rr}/3)/H(u,p) &= (\lambda N(u,p)pE_{rr} + 2\mu N(u,p)pE_{33}). \end{aligned} \tag{4}$$

Summing equations (4) we find

$$T_{rr} = (3\lambda + 2\mu)N(u,p)E_{ii}, \tag{5}$$

which implies that the function N(u,p) is determined when, besides the elastic parameters also and ε_{ii} and τ_{rr} are experimentally determined. Substituting T_{rr} in equations (4) we obtain

$$\begin{aligned} T_{11}(p + \mu/H) &= \mu(\lambda + 2\mu/3)E_{rr}(N(u,p)/H(u,p)) + N(u,p)(\lambda pE_{rr} + 2\mu pE_{11}), \\ T_{22}(p + \mu/H) &= \mu(\lambda + 2\mu/3)E_{rr}(N(u,p)/H(u,p)) + N(u,p)(\lambda pE_{rr} + 2\mu pE_{22}), \\ T_{33}(p + \mu/H) &= \mu(\lambda + 2\mu/3)E_{rr}(N(u,p)/H(u,p)) + N(u,p)(\lambda pE_{rr} + 2\mu pE_{33}). \end{aligned} \tag{6}$$

Knowing the values of the elastic parameters and the experimentally obtained values of τ_{ii} and ε_{ii} one may consider to determine from the equations (6) the unknown memory operators $M = N/H$ and H. In order to simplify the formulae without losing in generality we assume $\tau_{22} = \tau_{33}$ which implies $\varepsilon_{22} = \varepsilon_{33}$ then third and second equations (6) are identical, therefore we use only the first and the second equation of the system (6) that is the system

$$\begin{aligned} T_{11}(p + \mu/H) &= \mu(\lambda + 2\mu/3)E_{rr}(N(u,p)/H(u,p) + N(u,p)(\lambda pE_{rr} + 2\mu pE_{11}), \\ T_{22}(p + \mu/H) &= \mu(\lambda + 2\mu/3)E_{rr}(N(u,p)/H(u,p)) + N(u,p)(\lambda pE_{rr} + 2\mu pE_{22}). \end{aligned} \tag{7}$$

Subtracting we find

$$(T_{11} - T_{22})(p + \mu/H) = N(u,p)2\mu p(E_{11} - E_{22}), \tag{8}$$

while equation (5) gives

$$N(u,p) = T_{rr}/(3\lambda + 2\mu)E_{ii}. \tag{9}$$

Where is readily verified that if $h(u,t)$ and $k(v,t)$ are kernels of Caputo fractional derivative with $1>v>u>0$, then a constant volumetric change implies a nil asymptotic stress, in other words if the stress has shorter memory than the strain then the stress is asymptotically vanishing as required by plastic media. The same results are obtained if instead of the Caputo fractional derivative is used that of Caputo- Fabrizio [19] with same assumption $1>v>u>0$ on the orders of fractional derivation. If $h(u,t) = k(u,t)$ equation (9) is simply

$$(3\lambda + 2\mu)E_{ii} = T_{rr}, \tag{10}$$

which implies that, when the stress field is constant the dilatation is constant and depends only on the sum of the principal components of the stress; the strain components may change and the shape of the body changes keeping the volume constant. Moreover when the strain τ_{11} is observed in a laboratory experiment with τ_{11} constant and $\tau_{22} = \tau_{33}$ constant then the strains $\varepsilon_{22} = \varepsilon_{33}$ may be computed. In order to proceed in the study of the system (1) we substitute equation (8) in equation (9) obtaining

$$(T_{11} - T_{22})(p + \mu/H) = 2\mu p(E_{11} - E_{22})T_{rr}/(3\lambda + 2\mu)E_{rr}, \tag{11}$$

or

$$\begin{aligned} &(T_{11} - T_{22})p - 2\mu p(E_{11} - E_{22})T_{rr}/(3\lambda + 2\mu)E_{rr} = -\mu(T_{11} - T_{22})/H) \\ &H = \mu(T_{11} - T_{22})(3\lambda + 2\mu)E_{rr}/2p\mu(E_{11} - E_{22})T_{rr}. \end{aligned} \tag{12}$$

Since K = N/H we find

$$K = \mu(T_{11} - T_{22})/2\mu p(E_{11} - E_{22}). \tag{13}$$

When τ_{11} ,and τ_{22} are constant with $\tau_{11} \neq \tau_{22}$, which implies that $\varepsilon_{11} \neq \varepsilon_{22}$, and E_{11}, E_{22} are finite , the necessary condition for the existence of LT^{-1} of (11) and (12) is satisfied. Also the EVT applied to the function $H(u,p)$ and $K(u,p)$ shows that $h(u,\infty) = k(u,\infty)$ while $h(u,0) = k(u,0) = 0$. Concerning the modeling of constitutive equations with fractional derivatives the literature proves the presence of memory in many scientific phenomena but this is often only a first order approximation in taking account of the memory phenomenon which is needed mostly, but not only, for accounting for the second law of thermodynamics. In order to distinguish the different types of memory, we note now that when ε_{11}, $\tau_{11} = \tau_{22}$ are measured and the elastic parameters are known the analytical expressions of memory operator of the medium $H(u,p)$ and $K(u,p)$ may be obtained experimentally from equations (3), (11) and (12). If their LT-1 exists then we may infer the memory properties of the medium directly from the experimental data. Rewriting equations (5) and (8) giving evidence to the single component of the trace of the deformation

$$\begin{aligned} &T_{rr} = (3\lambda + 2\mu)N(u,p)E_{ii} = (3\lambda + 2\mu)N(u,p)(E_{11} + 2E_{22}), \\ &(T_{11} - T_{22})(p + \mu/H) = N(u,p)2\mu p(E_{11} - E_{22}) \end{aligned}$$

and expressing E_{22} from (8) we find

$$\begin{aligned} &(T_{11} - T_{22})(p + \mu/H) = N(u,p)2\mu pE_{11} - N(u,p)2\mu pE_{22}, \\ &E_{22} = E_{11} - (T_{11} - T_{22})(p + \mu/H)/N(u,p)2\mu p \qquad , \end{aligned}$$

and substituting E_{22} in (5) we obtain

$$T_{rr} = (3\lambda + 2\mu)N(u,p)[3E_{11} - (T_{11} - T_{22})\{p + \mu/H(u,p)\}/N(u,p)\mu p], \tag{14}$$

which is a relation between H(u,p) and N(u,p) provided by the values of E_{11} experimentally determined.

3 Experimental Checks

In order to study further the properties of the constitutive equations (1) we confront the results of the present study with those of the experimental studies on Polycrystalline Halite (PH) made by Waversick [29] and discussed in a note of Caputo [21]. The experiments were made on PH cylindrical samples subject to lateral confining pressure $\sigma_2 = \tau_{22} = \tau_{33}$ and to a pressure $\sigma_1 = \tau_{11}$ parallel to the axis of the cylinder; using the data resulting from these experiments Caputo [21] found that the creep curves are accurately fitting curves of the following type.

$$\varepsilon_{11} = A + Bt + C\exp(-Dt), \tag{15}$$

whose parameters A, B, C, D are shown in the table and resulting from 4 laboratory experiments. The LT of equation (15) to use in the comparison with the theoretical results obtained here is

$$E_{11} = A/p + B/p^2 + C/(p+D) \tag{16}$$

and substituting it in equation (14) we obtain the relation between H and N of the PH

$$\begin{aligned} T_{rr} = (3\lambda+2\mu)N(u,p)\{(3[A/p+B/p^2+C/(p+D)]+ \\ -(T_{11}-T_{22})(p+\mu/H)/N(u,p)2\mu p\}, \end{aligned} \tag{17}$$

where we note that the Caputo derivative of fractional order assumed for H and K may seem to not fit equation (17). In order to compare rigorously the expression of E_{11} as it results from our model with that resulting from the laboratory experiments we may explicit E_{11} from equation (17) and use equation (16) finding

$$\{T_{rr} + (3\lambda+2\mu)(T_{11}-T_{22})(p+\mu/H)/2\mu p\}/3(3\lambda+2\mu)N(u,p) = A/p + B/p^2 + C/(p+D), \tag{18}$$

where the equality sign is only symbolic in the sense that we mean to compare the two expressions where

$$E_{11} = T_{rr}H(u,p)/3(3\lambda+2\mu)K(u,p) + (T_{11}-T_{22})H/2\mu 3K(u,p) + (T_{11}-T_{22})/6pK(u,p).$$

Assuming now h(t) and k(t) as kernels of the Caputo derivative, which gives

$$\begin{aligned} LT(h(u,t)*Df(t)) = ap^uF(p), \\ LT(k(v,t)*Df(t)) = bp^vF(p), \end{aligned}$$

and inverting the LT we obtain

$$\begin{aligned} S(t-g)\{(\tau_{rr}/3(3\lambda+2\mu)+(\tau_{11}-\tau_{22})/6\mu)(a/b)t^{-u+v}/\Gamma(1-u+v)+ \\ ((\tau_{11}-\tau_{22})/6b)t^{1+v}/\Gamma(2+v)\} = A+Bt+Cexp(-Dt), \end{aligned} \tag{19}$$

where S(t-g) is the unit step function beginning at t = g, the first term in the left side is constant, the second term is monotonically decreasing if $u<v$, as we expect that the stress has a shorter memory than the strain, and the third term is monotonically increasing. The inspection of equation (19) shows that the lack of the linear term in the term on the left side of the equation, rigorously, only a poor approximation would be obtained and in a limited range. However, the Caputo-Fabrizio derivative with exponential kernel, gives an acceptable approximation of the memory of PH which, in turn, shows that each medium may need to be studied without any preconception regarding the use of any derivatives: the laboratory experiments will suggest the correct form of the medium memory. The kernels of the Caputo Fabrizio fractional derivatives of orders u and v respectively are

$$a(1/(1-u))exp(-ut/(1-u)) \qquad ; \qquad b(1/(1-v))exp(-vt/(1-v)),$$

and the LT gives

$$ap/[(1-u)(p+u/(1-u))] \qquad ; \qquad bp/[(1-v)(p+v/(1-v))].$$

By substituting in equation (18) we find

$$\begin{aligned} \exp(-gp)\{[\tau_{rr}/3(3\lambda+2\mu)+(\tau_{11}-\tau_{22})/2\mu 3]/p \\ +a\left[(1-v)\left(p+v/\left(1-v\right)\right)\right]/\left[b\left(1-u\right)\left(p+u/\left(1-u\right)\right)\right] \\ +(\tau_{11}-\tau_{22})\left[(1-v)\left(p+v/\left(1-v\right)\right)\right]/6bp^2\} = A/p+B/p^2+C/(p+D), \end{aligned} \tag{20}$$

whose LT^{-1} gives in the first line of equations (20) a constant, the second line gives an exponential and a delta function at t = 0, while the third line gives a constant and a linear term all beginning at the time g which, because of the convolutions with the step function beginning at t = g, excludes the presence of the delta function. Finally, we find

Table 1 Values of the coefficients A, B, C and D concerning the experiments on PH appearing in equation (16) and their 95% confidence levels for the data presented in the table. The columns under A, B, C and D give the 95% confidence level of the data in the columns A, B, C and D respectively; time is in seconds A and C are strain, B is strain/sec and D is sec^{-1}.

	A	**A'**	**B**	**B'**
CA121	1,25 10-3	1,22;1,28	1,43 10-8	1,41;1,48
CA 7	1,40 10-2	1,15;1,65	1,59 10-8	1,54;2,65
CA107	7,77 10-3	7,44;8,18	8,70 10-7	8,50;8-91
CA120	3,48 10-2	3,14;3,83	2,04 10-7	1,97;2,11
	C	**C'**	**D**	**D'**
CA121	-6,30 10-2	-6,70;-5,90	5,34 10-6	4,61;6,10
CA 7	-9,27 10-3	-9,07;-9,48	1,55 10-5	1,20;1,91
CA107	-6,70 10-3	-7,01;-6,42	3,11 10-6	2,81;3,51
CA120	-3,03 10-2	-3,34;-2,72	9,31 10-6	7,6 ;11,4

$$S(t-g)\{\,[\tau_{rr}/3(3\lambda+2\mu)+(\tau_{11}-\tau_{22})/2\mu 3]+[(a\,(1-v)\,/b(1-u))(v(1-v)-u/(1-u))exp(-vt(1-v)] \\ +\{(\tau_{11}-\tau_{22})\,[(1-v)+vt/6b]]\,\}\,. \tag{21}$$

Equation (21) seem then appropriate to approximate the experimental results on PH given by equation (16).

Using the above mentioned memory formalism with simple exponential kernels $exp(-ut)$ and $exp(-vt)$ one would obtain for ε_{11} an expression similar, but somehow formally simpler, than formula (21).

The comparison of formulae (21) and (15) indicates that

$$C=u/(1-u),\quad D=v/(1-v).$$

From which follows

$$u=C/(1+C),\quad v=D/(1+D),$$

or since $C\ll 1,\ D\gg 1$, $u=C,\ v=D$ and we may see directly in the Table 1 that, in all the 4 cases, the memory acting on the strain is orders of magnitude longer that that acting on the stress, that is $u\ll v$. The variety of the temperature in the 4 different experiments ranging from 22^0C to 200^0C, of the confining pressures ranging from 35 bar to 430 bar and of the limited number of cases considered does not allow any conclusion on the effect of the temperature and of the difference $\tau_{11}-\tau_{22}$, however we may say that their effects are relevant.

4 Conclusion

The use of presently available fractional derivatives is a proof of the presence of memory in many scientific phenomena but, generally only a first order The memory is generally inserted in the constitutive equations by including a memory formalism represented by fractional derivatives however we have seen that some fractional derivatives may not be always appropriate for a good fitting of the experimental data in the theory while others would allow a good fitting. We have seen in this note that the plastic media may need to be studied without any preconception regarding the use of any type of derivatives: the laboratory experiments will suggest the correct form of the medium memory. This may be true for many other media and phenomena; as we have shown here laboratory experiments may be the way to identify the most appropriate model for the memory of the media or the phenomena .

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