Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/020205

On Some Particular Solutions of the Chebyshev's Equation by Means of ∇^{α} Discrete Fractional Calculus Operator

Mustafa Inc and Resat Yilmazer*

Department of Mathematics,Science Faculty, Firat University, 23119 Elazig, Turkey

Received: 2 Oct. 2015, Revised: 3 Dec. 2015, Accepted: 18 Dec. 2015
Published online: 1 Apr. 2016

Abstract: In this work, we acquire some new particular solutions of the homogeneous and nonhomogeneous Chebyshev's equations (CE) by using discrete fractional nabla operator ∇^{α} $(0 < \alpha < 1)$.

Keywords: Discrete fractional calculus, Chebyshev equation, nabla operator.

1 Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders, and their applications appear in various fields in science, engineering, applied mathematics, economics, such as, viscoelasticity, diffusion, neurology, control theory, and statistics [1,2,3,4,5,6,7]. Therefore it has achieved significance during the past three decades. The similar theory for discrete fractional calculus was begun and features of the theory of fractional sums and differences were constituted. Many papers related to this topic have appeared recently [8]-[21].

In 1956 [8], differences of fractional order was first introduced by Kuttner. Difference of fractional order has attracted more attention in the last few years.. Diaz and Osler [9] investigated the fractional difference defined as an infinite series and they acquired a generalization of the binomial formula. Discrete fractional calculus is a generalization of ordinary difference and summation on arbitrary order that can be non-integer, and it has gained considerable popularity due mainly to its demonstrated applications in describing some real-world phenomena [20,21]. Among all the topics, the branch of discrete fractional boundary value problems is currently undergoing active investigation [8]-[21] and the references therein.

The aim of this paper is to acquire some new discrete fractional solutions of the homogeneous and nonhomogeneous CEs by means of the nabla discrete fractional operator.

The work is organized as follows. In Section 2, we present the basic definitions of the discrete fractional calculus. Our results are then given in Section 3. In the last Section, we give some conclusions.

2 Preliminaries

In this section, we present some essential information about discrete fractional calculus theory. We use the some notations: $\mathbb{N}$ is the set of natural numbers including zero and $\mathbb{Z}$ is the set of integers. $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$ for $a \in \mathbb{Z}$. Let $f(n)$ and $g(n)$ be a real-valued function defined on $\mathbb{N}_0^+$.

Definition 1 [14]. The rising factorial power is given by

$$t^{\overline{n}} = t(t+1)(t+2)\ldots(t+n-1), n \in \mathbb{N}, t^{\overline{0}} = 1. \quad (1)$$

* Corresponding author e-mail: minc@firat.edu.tr

Let α a real number. Then $t^{\overline{\alpha}}$ is defined to be

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \tag{2}$$

where $t \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$ and $0^{\overline{\alpha}} = 0$. Let us note that

$$\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}, \tag{3}$$

where $\nabla u(t) = u(t) - u(t-1)$. For $p = 2,3,\ldots$ define ∇^p indeductively by $\nabla^p = \nabla\nabla^{p-1}$.

Definition 2 [14]. The $\alpha - th$ order fractional sum of g is given by

$$\nabla_b^{-\alpha} g(t) = \sum_{s=b}^{t} \frac{(t-\delta(t))^{\overline{\alpha-1}}}{\Gamma(\alpha)} g(s), \tag{4}$$

where $t \in \mathbb{N}_b$, $\delta(t) = t-1$ is backward jump operator of the time scale calculus.

Theorem 1 [20]. Let $f(n)$ and $g(n) : \mathbb{N}_0^+ \longrightarrow \mathbb{R}$, $\gamma, \phi > 0$ and h, v are scalars. The following equality holds:

$$1.\ \nabla^{-\gamma}\nabla^{-\phi} f(n) = \nabla^{-(\gamma+\phi)} f(n) = \nabla^{-\phi}\nabla^{-\gamma} f(n). \tag{5}$$

$$2.\ \nabla^{\gamma}[hf(n) + vg(n)] = h\nabla^{\gamma} f(n) + v\nabla^{\gamma} g(n). \tag{6}$$

$$3.\ \nabla\nabla^{-\gamma} f(n) = \nabla^{-(\gamma-1)} f(n). \tag{7}$$

$$4.\ \nabla^{-\gamma}\nabla f(n) = \nabla^{(1-\gamma)} f(n) - \binom{n+\gamma-2}{n-1} f(0). \tag{8}$$

Lemma 1 [14]. (Leibniz Rule). For any $\alpha > 0$, $\alpha - th$ order fractional difference of the product fg is given by

$$\nabla_0^{\alpha}(fg)(t) = \sum_{n=0}^{t} \binom{\alpha}{n} \left[\nabla_0^{\alpha-n} f(t-n)\right]\left[\nabla^n g(t)\right], \tag{9}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}.$$

Lemma 2 (Index law). If the function $f(n)$ is single-valued and analytic then

$$\left(f_{\gamma}(n)\right)_{\mu} = f_{\gamma+\mu}(n) = \left(f_{\mu}(n)\right)_{\gamma}, \left(f_{\gamma}(n) \neq 0; f_{\mu}(n) \neq 0; \gamma, \mu \in \mathbb{R}; n \in \mathbb{N}\right). \tag{10}$$

3 Main Results

3.1 Discrete fractional solutions of nonhomogeneous CE

Theorem 2. Let $\psi = \psi(x) \in \{\psi : 0 \neq |\psi_{\alpha}| < \infty; \alpha \in \mathbb{R}\}$ and $g = g(x) \in \{g : 0 \neq |g_{\alpha}| < \infty\}$. Then the nonhomogeneous CE

$$\psi_2\left(x^2-1\right) + \psi_1 x - \psi v^2 = g \quad \left(v \in \mathbb{R}, x^2 - 1 \neq 0\right), \tag{11}$$

has particular solutions of the forms;

$$\psi \equiv \psi^{I} = \left\{\left[g_{E^{-1}v^2}\left(x^2-1\right)^{v^2-\frac{1}{2}}\right]_{-1}\left(x^2-1\right)^{-v^2-\frac{1}{2}}\right\}_{-\left(1+E^{-1}v^2\right)}, \tag{12}$$

$$\psi \equiv \psi^{II} = \left(x^2-1\right)^{1/2}\left(\left\{\left[g\left(x^2-1\right)^{-1/2}\right]_{\frac{E-1}{3}\left(v^2-1\right)}\left(x^2-1\right)^{-1+\left(\frac{2v^2+7}{6}\right)}\right\}_{-1} \times \left(x^2-1\right)^{-\left(\frac{2v^2+7}{6}\right)}\right)_{-1+\frac{E-1}{3}\left(1-v^2\right)}, \tag{13}$$

$$\psi \equiv \psi^{III} = (x-1)^{1/2} \left(\left\{ \left[g(x-1)^{-1/2} \right]_{\frac{E-1}{2}\left(v^2-\frac{1}{4}\right)} (x+1)^{\left(\frac{4v^2-5}{8}\right)} (x-1)^{\left(\frac{4v^2+3}{8}\right)} \right\}_{-1} \times (x+1)^{-\left(\frac{4v^2+3}{8}\right)} (x-1)^{-\left(\frac{4v^2+11}{8}\right)} \right)_{-1+\frac{E-1}{2}\left(\frac{1}{4}-v^2\right)}, \tag{14}$$

$$\psi \equiv \psi^{IV} = (x+1)^{1/2} \left(\left\{ \left[g(x+1)^{-1/2} \right]_{\frac{E-1}{2}\left(v^2-\frac{1}{4}\right)} (x+1)^{\left(\frac{4v^2+3}{8}\right)} (x-1)^{\left(\frac{4v^2-5}{8}\right)} \right\}_{-1} \times (x+1)^{-\left(\frac{4v^2+11}{8}\right)} (x-1)^{-\left(\frac{4v^2+3}{8}\right)} \right)_{-1+\frac{E-1}{2}\left(\frac{1}{4}-v^2\right)}, \tag{15}$$

Here $\psi_2 = d^2\psi/dx^2$, $\psi_0 = \psi = \psi(x)\,(x \in \mathbb{R})$.

Proof. *i*-) Operate ∇^α to both sides of (11), we have then

$$\nabla^\alpha \left[\psi_2 \left(x^2-1\right)\right] + \nabla^\alpha (\psi_1 x) - \nabla^\alpha (\psi) v^2 = \nabla^\alpha g,$$

$$\psi_{2+\alpha}\left(x^2-1\right) + \psi_{1+\alpha} x (2\alpha E+1) + \psi_\alpha \left(\alpha E - v^2\right) = g_\alpha. \tag{16}$$

Choose α such that

$$\alpha = \frac{v^2}{E} = E^{-1}v^2,$$

we have then

$$\psi_{2+E^{-1}v^2}\left(x^2-1\right) + \psi_{1+E^{-1}v^2} x \left(2v^2+1\right) + \psi_\alpha \left(\alpha E - v^2\right) = g_{E^{-1}v^2}, \tag{17}$$

from (16).

Next writing

$$\psi_{1+E^{-1}v^2} = \varphi = \varphi(x) \quad \left(\psi = \varphi_{-\left(1+E^{-1}v^2\right)}\right), \tag{18}$$

we obtain

$$\varphi_1 \left(x^2-1\right) + \varphi x \left(2v^2+1\right) = g_{E^{-1}v^2}, \tag{19}$$

from (17). A particular solution to this linear first order equation is given by

$$\varphi = \left[g_{E^{-1}v^2} \left(x^2-1\right)^{v^2-\frac{1}{2}} \right]_{-1} \left(x^2-1\right)^{-v^2-\frac{1}{2}}. \tag{20}$$

Therefore, we obtain

$$\psi = \left\{ \left[g_{E^{-1}v^2} \left(x^2-1\right)^{v^2-\frac{1}{2}} \right]_{-1} \left(x^2-1\right)^{-v^2-\frac{1}{2}} \right\}_{-\left(1+E^{-1}v^2\right)}, \tag{21}$$

from (18) and (20).

***ii*-)** Set

$$\psi = \left(x^2-1\right)^\lambda \phi, \quad \phi = \phi(x), \tag{22}$$

we have then

$$\phi_2 \left(x^2-1\right) + \phi_1 x (4\lambda+1) + \phi \left[\left(4\lambda^2 - v^2\right) + \frac{2\lambda(2\lambda-1)}{\left(x^2-1\right)} \right] = g\left(x^2-1\right)^{-\lambda}, \tag{23}$$

from (11), applying (22).

When $\lambda = 0$, (23) is reduced to (11). When $\lambda = 1/2$, we have

$$\phi_2 \left(x^2-1\right) + \phi_1 3x + \phi \left(1-v^2\right) = g\left(x^2-1\right)^{-1/2}, \tag{24}$$

from (23).

Operate ∇^α to the both sides of (24), then yields

$$\phi_{2+\alpha}\left(x^2-1\right) + \phi_{1+\alpha} x (2\alpha E+3) + \phi_\alpha \left(3\alpha E + 1 - v^2\right) = \left[g\left(x^2-1\right)^{-1/2} \right]_\alpha. \tag{25}$$

Choose α such that

$$\alpha = \frac{E^{-1}}{3}\left(v^2-1\right),$$

we have then

$$\phi_{2+\frac{E^{-1}}{3}\left(v^2-1\right)}\left(x^2-1\right)+\phi_{1+\frac{E^{-1}}{3}\left(v^2-1\right)}x\left(\frac{2v^2+7}{3}\right)=\left[g\left(x^2-1\right)^{-1/2}\right]_{\frac{E^{-1}}{3}\left(v^2-1\right)}, \tag{26}$$

from (25).

In this case, letting

$$\phi_{1+\frac{E^{-1}}{3}\left(v^2-1\right)} = V = V(x) \quad \left(\phi = V_{-1+\frac{E^{-1}}{3}\left(1-v^2\right)}\right), \tag{27}$$

we obtain

$$V_1\left(x^2-1\right)+Vx\left(\frac{2v^2+7}{3}\right)=\left[g\left(x^2-1\right)^{-1/2}\right]_{\frac{E^{-1}}{3}\left(v^2-1\right)}, \tag{28}$$

from (26). A particular solution to this linear first order equation is given by

$$V=\left\{\left[g\left(x^2-1\right)^{-1/2}\right]_{\frac{E^{-1}}{3}\left(v^2-1\right)}\left(x^2-1\right)^{-1+\left(\frac{2v^2+7}{6}\right)}\right\}_{-1}\left(x^2-1\right)^{-\left(\frac{2v^2+7}{6}\right)}. \tag{29}$$

Therefore, we obtain

$$\psi=\left(x^2-1\right)^{1/2}\left(\left\{\left[g\left(x^2-1\right)^{-1/2}\right]_{\frac{E^{-1}}{3}\left(v^2-1\right)}\left(x^2-1\right)^{-1+\left(\frac{2v^2+7}{6}\right)}\right\}_{-1} \times\left(x^2-1\right)^{-\left(\frac{2v^2+7}{6}\right)}\right)_{-1+\frac{E^{-1}}{3}\left(1-v^2\right)} \tag{30}$$

from (22), applying (27) and (29), for $\lambda=1/2$.

iii-) Set

$$\psi=(x-1)^{\lambda}\phi, \quad \phi=\phi(x), \tag{31}$$

we have then

$$\phi_2\left(x^2-1\right)+\phi_1\left[x(2\lambda+1)+2\lambda\right]+\phi\left[\left(\lambda^2-v^2\right)+\frac{\lambda(2\lambda-1)}{(x-1)}\right]=g(x-1)^{-\lambda}, \tag{32}$$

from (11), applying (31).

When $\lambda=0$, (32) is reduced to (11). When $\lambda=1/2$, we have

$$\phi_2\left(x^2-1\right)+\phi_1(2x+1)+\phi\left(\frac{1}{4}-v^2\right)=g(x-1)^{-1/2}, \tag{33}$$

from (32).

Operate ∇^{α} to the both sides of (33), then yields

$$\phi_{2+\alpha}\left(x^2-1\right)+\phi_{1+\alpha}\left[x(2\alpha E+2)+1\right]+\phi_{\alpha}\left(2\alpha E+\frac{1}{4}-v^2\right)=\left[g(x-1)^{-1/2}\right]_{\alpha}. \tag{34}$$

Choose α such that

$$\alpha=\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right),$$

we have then

$$\phi_{2+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}\left(x^2-1\right)+\phi_{1+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}\left[x\left(v^2+\frac{7}{4}\right)+1\right]=\left[g(x-1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}, \tag{35}$$

from (34).

In this case, letting

$$\phi_{1+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)} = W = W(x) \quad \left(\phi = W_{-1+\frac{E^{-1}}{2}\left(\frac{1}{4}-v^2\right)}\right), \tag{36}$$

we obtain

$$W_1\left(x^2-1\right) + W\left[x\left(v^2+\frac{7}{4}\right)+1\right] = \left[g(x-1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}, \tag{37}$$

from (35). A particular solution to this equation is given by

$$W = \left\{\left[g(x-1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}\left(x^2-1\right)^{\left(\frac{4v^2-5}{8}\right)}(x-1)\right\}_{-1}\frac{\left(x^2-1\right)^{-\left(\frac{4v^2+3}{8}\right)}}{x-1}. \tag{38}$$

Therefore, we obtain

$$\begin{aligned}\psi = (x-1)^{1/2}\Bigg(\Bigg\{\left[g(x-1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}(x+1)^{\left(\frac{4v^2-5}{8}\right)}(x-1)^{\left(\frac{4v^2+3}{8}\right)}\Bigg\}_{-1} \\ \times (x+1)^{-\left(\frac{4v^2+3}{8}\right)}(x-1)^{-\left(\frac{4v^2+11}{8}\right)}\Bigg)_{-1+\frac{E^{-1}}{2}\left(\frac{1}{4}-v^2\right)}\end{aligned} \tag{39}$$

from (31), and (36), applying (38), for $\lambda = 1/2$.

iv-) Set

$$\psi = (x+1)^{\lambda}\phi, \quad \phi = \phi(x), \tag{40}$$

we have then

$$\phi_2\left(x^2-1\right) + \phi_1\left[x(2\lambda+1)-2\lambda\right] + \phi\left[\left(\lambda^2-v^2\right) - \frac{\lambda(2\lambda-1)}{(x+1)}\right] = g(x+1)^{-\lambda}, \tag{41}$$

from (11), applying (30).

When $\lambda = 1/2$, we have

$$\phi_2\left(x^2-1\right) + \phi_1(2x-1) + \phi\left(\frac{1}{4}-v^2\right) = g(x+1)^{-1/2}, \tag{42}$$

from (41).

Operate ∇^{α} to the both sides of (42), then yields

$$\phi_{2+\alpha}\left(x^2-1\right) + \phi_{1+\alpha}\left[x(2\alpha E+2)-1\right] + \phi_{\alpha}\left(2\alpha E+\frac{1}{4}-v^2\right) = \left[g(x+1)^{-1/2}\right]_{\alpha}. \tag{43}$$

Choose α such that

$$\alpha = \frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right),$$

we have then

$$\phi_{2+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}\left(x^2-1\right) + \phi_{1+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}\left[x\left(v^2+\frac{7}{4}\right)-1\right] = \left[g(x+1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}, \tag{44}$$

from (43).

In this case, letting

$$\phi_{1+\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)} = U = U(x) \quad \left(\phi = U_{-1+\frac{E^{-1}}{2}\left(\frac{1}{4}-v^2\right)}\right), \tag{45}$$

we obtain

$$U_1\left(x^2-1\right) + U\left[x\left(v^2+\frac{7}{4}\right)-1\right] = \left[g(x+1)^{-1/2}\right]_{\frac{E^{-1}}{2}\left(v^2-\frac{1}{4}\right)}, \tag{46}$$

from (44). A particular solution to this equation is given by

$$U=\left\{\left[g\left(x+1\right)^{-1/2}\right]_{\frac{E-1}{2}\left(v^2-\frac{1}{4}\right)}\left(x^2-1\right)^{\left(\frac{4v^2+3}{8}\right)}\frac{1}{(x-1)}\right\}_{-1}\left(x^2-1\right)^{-\left(\frac{4v^2+11}{8}\right)}\left(x-1\right). \tag{47}$$

Therefore, we obtain

$$\psi=(x+1)^{1/2}\left(\left\{\left[g\left(x+1\right)^{-1/2}\right]_{\frac{E-1}{2}\left(v^2-\frac{1}{4}\right)}\left(x+1\right)^{\left(\frac{4v^2+3}{8}\right)}\left(x-1\right)^{\left(\frac{4v^2-5}{8}\right)}\right\}_{-1}\right.$$
$$\left.\times\left(x+1\right)^{-\left(\frac{4v^2+11}{8}\right)}\left(x-1\right)^{-\left(\frac{4v^2+3}{8}\right)}\right)_{-1+\frac{E-1}{2}\left(\frac{1}{4}-v^2\right)}, \tag{48}$$

from (40) and (45), applying (47), for $\lambda=1/2$.

3.2 Discrete fractional solutions of homogeneous CE

Theorem 3. Let $\psi=\psi(x)\in\{\psi:0\neq|\psi_\alpha|<\infty;\alpha\in\mathbb{R}\}$. Then the homogeneous Chebyshev's equation

$$\psi_2\left(x^2-1\right)+\psi_1 x-\psi v^2=0\quad\left(v\in\mathbb{R},x^2-1\neq0\right), \tag{49}$$

has particular solutions of the forms

$$\psi=k\left[\left(x^2-1\right)^{-v^2-\frac{1}{2}}\right]_{-\left(1+E^{-1}v^2\right)}\equiv\psi^{(\imath)}, \tag{50}$$

$$\psi=k\left(x^2-1\right)^{1/2}\left[\left(x^2-1\right)^{-\left(\frac{2v^2+7}{6}\right)}\right]_{-1+\frac{E-1}{3}\left(1-v^2\right)}\equiv\psi^{(\imath\imath)}, \tag{51}$$

$$\psi=k\left(x-1\right)^{1/2}\left[\left(x+1\right)^{-\left(\frac{4v^2+3}{8}\right)}\left(x-1\right)^{-\left(\frac{4v^2+11}{8}\right)}\right]_{-1+\frac{E-1}{2}\left(\frac{1}{4}-v^2\right)}\equiv\psi^{(\imath\imath\imath)}, \tag{52}$$

$$\psi=k\left(x+1\right)^{1/2}\left[\left(x+1\right)^{-\left(\frac{4v^2+11}{8}\right)}\left(x-1\right)^{-\left(\frac{4v^2+3}{8}\right)}\right]_{-1+\frac{E-1}{2}\left(\frac{1}{4}-v^2\right)}\equiv\psi^{(\imath v)}, \tag{53}$$

where k is an arbitrary constant.

Proof. If we take $g=0$ in Theorem 2, we have the following homogeneous CEs

$$\varphi_1\left(x^2-1\right)+\varphi x\left(2v^2+1\right)=0, \tag{54}$$

$$V_1\left(x^2-1\right)+Vx\left(\frac{2v^2+7}{3}\right)=0, \tag{55}$$

$$W_1\left(x^2-1\right)+W\left[x\left(v^2+\frac{7}{4}\right)+1\right]=0 \tag{56}$$

and

$$U_1\left(x^2-1\right)+U\left[x\left(v^2+\frac{7}{4}\right)-1\right]=0. \tag{57}$$

If we apply the nabla discrete fractional operator to both sides of Eqs. $(54)-(57)$ and we use similar process in Theorem 2, then we get discrete fractional solutions $(50)-(53)$ for Eqs. $(54)-(57)$, respectively.

4 Conclusions

In this work, we use the nabla discrete fractional operator for the homogeneous and nonhomogeneous CEs. We acquire many different discrete fractional solutions for these equations. Previously, no one obtains solutions for these equations. Miyakoda and Nishomoto [22] gave some fractional solutions of the nonhomogeneous Chebyshev's equation using $N-$fractional calculus operator. We will obtain discrete fractional solutions of the same equations by using the combined delta-nabla sum operator in discrete fractional calculus [23] in our future work.

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