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Common Coincidence Points for (α, ψ, ξ)-Contractive Multi-Valued and Single-Valued Mappings and Applications

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Abstract: The object of this paper is to extend the idea of (α, ψ, ξ)-contractive multi-valued mappings for a pair of weak compatible multi-valued as well as single-valued mappings and prove existence of coincidence points for such kind of mappings. We also provide suitable examples to support our results. At the end, applications of results illustrate usability of our results.

Keywords: α_*-admissible mappings; (α, ψ, ξ)-contractive mappings; weakly compatible mappings.

1 Introduction

Hybrid fixed point theory for single-valued and multi-valued mappings is an impressive development in nonlinear analysis (see, e.g., [8,9,10,11,14,16,18,21,24] and references therein). The concepts of commutativity and weak commutativity were extended to multi-valued mappings on metric spaces by Kaneko [9,10]. In [23], Singh and Mishra introduced the notion of (IT)-commutativity for hybrid pair of single-valued and multi-valued maps which need not be weakly compatible. Afterwards, Pathak [18] generalized the concept of compatibility by defining weak compatibility for hybrid pairs of mappings (including single-valued case) and utilized the same to prove common fixed point theorems. Naturally, compatible mappings are weakly compatible but not conversely.

On the other hand, Samet et al. [19] introduced the notions of α-ψ-contractive and α-admissible self-mappings and proved some fixed-point results for such mappings in complete metric spaces. Asl et al. [1] extended these notions to multi-valued by introducing the notions of α_*-ψ-contractive and α_*-admissible mappings and proved some fixed point results. Some results in this direction are also given in [1,2,3,7,12], and [15]. Salimi et al. [21] modified the notions of α-ψ-contractive and α-admissible self-mappings by introducing another function η and established some fixed-point theorems for single-valued mappings in complete metric spaces. Hussain et al. [7] extended these modified notions to multi-valued mappings. Ali et al. [2] introduced the notion of (α, ψ, ξ)-contractive multi-valued mappings and provide fixed point theorems for (α, ψ, ξ)-contractive multi-valued mappings in complete metric spaces.

In this paper, we will unify the (α, ψ, ξ)-contractive condition with a pair of weakly compatible mappings and examine the existence of common coincidence points for a weakly compatible hybrid pair f and T. In the sequel, we will establish the results for a pair of single-valued mappings also. A fixed point theorem in metric space endowed with an arbitrary binary relation validates the importance of obtained results.

Before proceeding towards our main result we will give some preliminaries:

We recollect the following definitions, for the sake of completeness. Let (X,d) be a metric space. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X. For every $A,B \in CL(X)$, let the functional $H : CL(X) \times CL(X) \to R^+ \cup \{\infty\}$ be defined by

$$H(A,B) = \max \begin{cases} \sup_{x\in A}(d(x,B)), \sup_{y\in B}(d(A,y)) & \text{if maximum exists} \\ \infty & \text{otherwise} \end{cases}$$

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for every $A,B \in CL(X)$, where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from a to $B \in X$.
In this paper, we denote by Ψ the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is a non-decreasing function;
(ψ_2) $\psi(t)^n < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as Bianchini-Grandolfi gauge functions in some sources (see [4]).

Remark 1.1. For each $\psi \in \Psi$ we see that the following assertions hold:

1. $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$;
2. $\psi(t) < t$ for each $t > 0$;
3. $\psi(0) = 0$.

In [19], Samet et al. introduced the concepts of an α-admissible mapping and an α-ψ-contractive mapping as follows:

Definition 1.2. [[19]] Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be another mapping. We say that T is α-admissible if the following condition holds:

$$x,y \in X, \alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1.$$

Definition 1.3. [[19]] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)) \quad \text{for all } x,y \in X.$$

Afterwards, Asl et al. [1] introduced the concept of an α_*-admissible mapping which is a multi-valued version of the α-admissible mapping provided in [19].

Definition 1.4. [[1]] Let X be a nonempty set, $T : X \to 2^X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is α_*-admissible if the following condition holds:

$$x,y \in X, \alpha(x,y) \ge 1 \Rightarrow \alpha_*(Tx,Ty) \ge 1,$$

where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) | a \in Tx, b \in Ty\}$.

They extended the α-ψ-contractive condition of Samet et al. [19] from a single-valued version to a multi-valued version as follows:

Definition 1.5. [[1]] Let (X,d) be a metric space, $T : X \to CL(X)$ be a multi-valued mapping and $\alpha : X \times X \to [0,\infty)$ be a given mapping. We say that T is an α-ψ-contractive multi-valued mapping if there exists $\psi \in \Psi$ such that

$$\alpha(x,y)H(Tx,Ty) \le \psi(d(x,y)) \quad \text{for all } x,y \in X.$$

Recently, Ali et al. [2] introduced the notion of (α,ψ,ξ)-contractive multi-valued mappings, where $\xi \in \Xi$ and Ξ is the family of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$(\xi 1)$ ξ is continuous;
$(\xi 2)$ ξ is non-decreasing on $[0,\infty)$;
$(\xi 3)$ $\xi(t) = 0$ if and only if $t = 0$;
$(\xi 4)$ ξ is sub-additive.

Remark 1.6. From $(\xi 2)$ and $(\xi 3)$, we have $\xi(t) > 0$, for all $t \in (0,\infty)$.

Lemma 1.7. *Let (X,d) be a metric space. If $\xi \in \Xi$ then $(X,\xi \circ d)$ is a metric space.*

Lemma 1.8. ([2]) *Let (X,d) be a metric space, $\xi \in \Xi$ and $B \in CL(X)$. If there exists $x \in X$ such that $\xi(d(x,B)) > 0$, then there exists $y \in B$ such that*

$$\xi(d(x,y)) < q\xi(d(x,B))$$

where $q > 1$.

Definition 1.9. [[2]] Let (X,d) be a metric space. A multi-valued mapping $T : X \to CL(X)$ is called an (α,ψ,ξ)-contractive mapping if there exist three functions $\psi \in \Psi$ and $\xi \in \Xi$

$$\alpha : X \times X \to [0,\infty) \text{ such that } x,y \in X, \alpha(x,y) \ge 1$$
$$\Rightarrow \quad \xi H(Tx,Ty) \le \psi\xi(M(x,y))$$

where
$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Definition 1.10. [[18]] Let (X,d) be a metric space. Then the pair $T : X \to X$ and $f : X \to X$ is called f-weak compatible iff $fTX \subseteq X$ for all $x \in X$ and the following limits exist and satisfy:

(i) $\lim_{n\to\infty} d(fTx_n, Tfx_n) \le \lim_{n\to\infty} d(Tfx_n, Tx_n)$, and
(ii) $\lim_{n\to\infty} d(fTx_n, fx_n) \le \lim_{n\to\infty} d(Tfx_n, Tx_n)$

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to t$ and $fx_n \to t$ for some $t \in X$.

Definition 1.11. [[18]] Let (X,d) be a metric space. Then the hybrid pair $T : X \to CL(X)$ and $f : X \to X$ is called f-weak compatible iff $fTX \subseteq CLX$ for all $x \in X$ and the following limits exist and satisfy:

(i) $\lim_{n\to\infty} H(fTx_n, Tfx_n) \le \lim_{n\to\infty} H(Tfx_n, Tx_n)$, and
(ii) $\lim_{n\to\infty} d(fTx_n, fx_n) \le \lim_{n\to\infty} H(Tfx_n, Tx_n)$

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to M$, $M \in CL(X)$ and $fx_n \to t \in M$.

It can be seen that two compatible maps f and T are weak compatible but the converse is not true.

Lemma 1.12. *([18]) Let $T : X \to CL(X)$ and $f : X \to X$ be f-weak compatible. If $fw \in Tw$ for some $w \in X$, then $fTw = Tfw$.*

Now, we will define our mappings:

Definition 1.13. Let X be a nonempty set, $T : X \to CL(X)$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is α_*-admissible with respect to f (a self-mapping on X) if the following condition holds:

$$x,y \in X, \alpha(fx,fy) \geq 1 \Rightarrow \alpha_*(Tx,Ty) \geq 1,$$

where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) | a \in Tx, b \in Ty\}$.

Definition 1.14. Let (X,d) be a metric space. Then the hybrid pair $T : X \to CL(X)$ and $f : X \to X$ are called an (α,ψ,ξ)-contractive mapping if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ such that

$$x,y \in X, \alpha(fx,fy) \geq 1 \Rightarrow \xi H(Tx,Ty) \leq \psi\xi(M(x,y))$$

where $M(x,y)$
$=\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}$.

2 Main results

This section consists two parts. In the first one, we have established existence of coincidence point for a hybrid pair of weak compatible mappings, and in the next one we have proved the results for a pair of self mappings.
Recently, Ali et al. [2] proved the following theorem for (α,ψ,ξ)-contractive mapping:

Theorem 2.1. **([2])** *Let (X,d) be a complete metric space and let $G : X \to CL(X)$ be a strictly (α,ψ,ξ)-contractive mapping satisfying the following assumptions:*

(i) *G is an α_*-admissible mapping;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0,x_1) \geq 1$;*
(iii) *G is continuous.*

Then G has a fixed point.

Now, we will extend the above said theorem for a pair of weakly compatible mappings. Our theorem is as follows:

Theorem 2.2. *Let (X,d) be a complete metric space and $f : X \to X$ and $T : X \to CL(X)$ be the continuous f-weak compatible hybrid pair such that $TX \subseteq fX$. Suppose that the following conditions hold:*

S_1. *T is an α_*-admissible multi-valued mapping wrt. f;*
S_2. *T and f are (α,ψ,ξ)-contractive mappings;*
S_3. *There exists $fx_0 \in X$ and $fx_1 \in Tx_0$ such that*

$$\alpha(fx_0,fx_1) \geq 1;$$

Then there exists a point $t \in X$ such that $ft \in Tt$ or f and T have a common coincidence point.

*Proof.*Let $x_0 \in X$ be arbitrary and choose, $x_1 \in X$ such that $fx_1 \in Tx_0$. This is possible since

$$Tx_0 \subseteq f(X).$$

If $x_0 = x_1$ then we see that x_0 is common fixed point of f and T. Assume that $x_0 \neq x_1$.
Since $T(X) \subseteq f(X)$, let $x_2 \in X$ be such $y_1 = fx_2$ and $fx_2 \in Tx_1$.
In general, if x_n has been selected, choose $x_{n+1} \in X$ so that $y_n = fx_{n+1} \in Tx_n$.
From (α,ψ,ξ)-contractive condition, we get

$$\begin{aligned}
&\xi(H(Tx_0,Tx_1)) \\
&\leq \left(\xi\left(\max\left\{d(fx_0,fx_1), d(fx_0,Tx_0), d(fx_1,Tx_1), \frac{d(fx_0,Tx_1)+d(fx_1,Tx_0)}{2}\right\}\right)\right) \\
&\leq \left(\xi\left(\max\{d(fx_0,fx_1), d(fx_1,Tx_1), d(fx_0,Tx_1)\}\right)\right) \\
&\leq (\xi(\max\{d(fx_0,fx_1), d(fx_1,Tx_1), \\
&\quad d(fx_0,Tx_1)+d(fx_1,Tx_1)\})) \\
&\leq (\xi(\max\{d(fx_0,fx_1), d(fx_1,Tx_1)\})) \qquad (2.1)
\end{aligned}$$

If $\max\{d(fx_0,fx_1), d(fx_1,Tx_1)\} = d(fx_1,Tx_1)$ then, we get

$$\begin{aligned}
0 &< \xi(d(fx_1,Tx_1)) \\
&< \xi(H(Tx_0,Tx_1)) \\
&< \psi(\xi(\max\{(d(fx_0,fx_1), d(fx_1,Tx_1))\})) \\
&\leq \psi(\xi(\max\{(d(fx_0,fx_1), d(fx_1,Tx_1))\})). \\
&\leq \psi(\xi(d(fx_1,Tx_1))), \qquad (2.2)
\end{aligned}$$

which is a contradiction.
Therefore, $\max\{d(fx_0,fx_1), d(fx_1,Tx_1)\} = d(fx_0,fx_1)$.
From (2.1) we get

$$\begin{aligned}
0 &< \xi(d(fx_1,Tx_1)) \\
&\leq \xi(H(Tx_0,Tx_1)) \\
&\leq \psi(\xi(\max\{d(fx_0,fx_1), d(fx_1,Tx_1)\})) \\
&\leq \psi\xi(d(fx_0,fx_1)). \qquad (2.3)
\end{aligned}$$

Since $f : X \to X$ and $T : X \to CL(X)$, therefore $fx_1, fx_2 \in X$ also $T(X) \subseteq f(X)$.
Thus from the inspiration from Lemma 1.8 for fixed $q > 1$ there exists $fx_2 \in Tx_1$ such that

$$0 < \xi(d(fx_1,fx_2) < q(d(fx_1,Tx_1)) \qquad (2.4)$$

From (2.3) and (2.4) we get

$$0 < \xi(d(fx_1,fx_2) < q\psi((d(fx_0,fx_1))) \qquad (2.5)$$

Since ψ is strictly increasing function, we have,

$$0 < \psi(\xi(d(fx_1,fx_2))) < \psi(q(\xi(d(fx_0,fx_1)))) \qquad (2.6)$$

put $q_1 = \dfrac{\psi(q\psi(\xi(d(fx_0,fx_1))))}{\psi(\xi(d(fx_1,fx_2)))}$ and then $q_1 > 1$.
If $x_1 = x_2$ or $fx_2 \in Tx_2$ we can find x_2 is common fixed point of f and T. Therefore $x_1 \neq x_2$ since $fx_1 \in Tx_0$, $fx_2 \in$

Tx_1 and $(fx_0, fx_1) > 1$, and T is an α_*-admissible multi-valued mapping with respect to f, we have

$$(fx_1, fx_2) > 1$$

Applying from $(\alpha,\psi\xi)$-contractive condition

$$\xi(H(Tx_1,Tx_2)) < \psi(\xi(\max\{d(fx_1,fx_2),d(fx_2,Tx_2)\})). \quad (2.7)$$

Suppose that $\max\{d(fx_1,fx_2),d(fx_2,Tx_2)\}$
$= d(fx_2,Tx_2)$.
From (2.7) we get

$$\begin{aligned} 0 &< \xi(d(fx_2,Tx_2)) \\ &\le \xi(H(Tx_1,Tx_2)) \\ &\le \psi(\xi(\max\{d(fx_1,fx_2),d(fx_2,Tx_2)\})) \\ &\le \psi(\xi(d(fx_2,Tx_2)), \end{aligned} \quad (2.8)$$

which is a contradiction. Therefore, we may let

$$\max\{d(fx_1,fx_2),d(fx_2,Tx_2)\} = d(fx_1,fx_2).$$

From (2.7), we have

$$0 < \xi(d(fx_2,Tx_2)) \le \xi(H(Tx_1,Tx_2)) \le \psi(\xi(d(fx_1,fx_2))). \quad (2.9)$$

By using Lemma 1.8 with q_1, there exists $fx_3 \in x_2$ such that

$$0 < \xi(d(fx_2,fx_3) < q_1(d(fx_2,Tx_2)). \quad (2.10)$$

From (2.9) and (2.10), we get

$$\begin{aligned} 0 &< \xi(d(fx_2,fx_3)) < q_1\psi(\xi(d(fx_1,fx_2))) \\ &< \psi(q_1\psi(\xi(d(fx_0,fx_1)))). \end{aligned} \quad (2.11)$$

It follows from ψ being a strictly increasing function that

$$0 < \psi(\xi(d(fx_2,fx_3))) < \psi^2(q\psi(\xi(d(fx_0,fx_1)))). \quad (2.12)$$

Continuing this process, we can construct a sequence $\{fx_n\}$ in X such that

$$f(x_n) = fx_{n+1} \in Tx_n \quad \text{and} \quad \alpha(fx_n,fx_{n+1}) > 1 \quad (2.13)$$

and

$$0 < \xi(d(fx_{n+1},fx_{n+2})) < \psi^n(q\psi(\xi(d(fx_0,fx_1)))), \text{ for all } n \in N \cup \{0\}. \quad (2.14)$$

Let $m,n \in N$ such that $m > n$, by triangle inequality, we have

$$\begin{aligned} \xi(d(fx_m,fx_n)) &\le \sum_{i=n}^{m-1} \xi(d(fx_i,fx_{i+1})) \\ &\le \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(\xi(d(fx_0,fx_1))). \end{aligned}$$

Since, $\psi \in \Psi$, we have $\lim_{n,m\to\infty} \xi(d(fx_m,fx_n)) = 0$.
Using (ξ_1), we get, $\lim_{n,m\to\infty} d(fx_m,fx_n)) = 0$.
This implies that $\{fx_n\}$ is a Cauchy sequence in (X,d).
From (2.13) and the completeness of (X,d), there exists $t \in X$ such that

$$fx_n \to t \text{ as } n \to \infty$$

Furthermore, above inequalities show that

$$\xi(H(Tx_n,Tx_{n-1})) < \psi(\xi(fx_n,fx_{n-1}))$$

Since, $\{fx_n\}$ is a Cauchy sequence, therefore, this must imply that $\{Tx_n\}$ is a Cauchy sequence, in the complete metric space $(CL(X),H)$ (refer to [23]).
Now let $Tx_n \to M \in CL(X)$, thus,

$$\begin{aligned} d(t,M) &\le d(t,fx_n) + d(fx_n,M) \\ &\le d(t,fx_n) + d(Tx_{n-1},M) \\ &\le d(t,fx_n) + H(Tx_{n-1},M) \to 0 \text{ as } n \to \infty \end{aligned}$$

Since M is closed, $t \in M$ and the f-weak compatibility of f and T implies that

(i) $\lim_{n\to\infty} H(fTx_n,Tfx_n) \le \lim_{n\to\infty} H(Tfx_n,Tx_n)$,and
(ii) $\lim_{n\to\infty} d(fTx_n,fx_n) \le \lim_{n\to\infty} H(Tfx_n,Tx_n)$.

This along with the continuity of f and T imply that

$$H(fM,Tt) \le H(Tt,M) \text{ and } d(ft,t) \le H(Tt,M).$$

Now

$$\begin{aligned} d(ft,Tt) &\le d(ft,ffx_{n+1}) + d(ffx_{n+1},Tt) \\ &\le d(ft,ffx_{n+1}) + H(fTx_n,Tt) \end{aligned}$$

that is,

$$\begin{aligned} d(ft,Tt) &\le d(ft,ffx_{n+1}) + H(fTx_n,Tfx_n) + H(Tfx_n,Tt) \\ &\le d(ft,ffx_{n+1}) + H(Tfx_n,Tt), \text{ as } n \to \infty \end{aligned}$$

that is, $d(ft,Tt) \le H(Tt,M)$. Now, using contractive condition,

$$\begin{aligned} &(H(Tx_n,Tt)) \\ &\le \psi\left(\xi\left(\max\left\{d(fx_n,ft),d(fx_n,Tx_n),d(ft,Tt),\frac{d(fx_n,Tt)+d(ft,Tx_n)}{2}\right\}\right)\right) \\ &\le \psi\left(\xi\left(\max\left\{d(fx_n,ft),d(fx_n,Tx_n),d(ft,Tt),\frac{d(fx_n,Tt)+d(ft,fx_n)+d(fx_n,Tx_n)}{2}\right\}\right)\right). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}&\xi(H(M,Tt))\\ &\le \psi\left(\xi\left(\max\left\{d(t,ft),d(t,M),d(ft,Tt),\right.\right.\right.\\ &\qquad\left.\left.\left.\frac{d(t,Tt)+d(ft,t)+d(t,M)}{2}\right\}\right)\right)\\ &\le \psi\left(\xi\left(\max\left\{H(Tt,M),0,H(Tt,M),\right.\right.\right.\\ &\qquad\left.\left.\left.\frac{H(M,Tt)+H(Tt,M)}{2}\right\}\right)\right)\end{aligned}$$

that is,

$\xi(H(M,Tt)) \le \psi(\xi(H(Tt,M))$, a contradiction and so $H(M,Tt)=0$.

Therefore, $d(ft,Tt)=0$; that is $ft \in Tt$, since Tt is closed. Thus f and T have a coincidence point.

Example 2.3. Let $X=[0,\infty)$ be endowed with the Euclidean metric d. Let $fx=\frac{3}{2}(x^2+x)$ and $Tx=[0,x^2+2]$ for each $x\ge 0$. T and f are clearly continuous and $T(X)=f(X)=X$. Since $fx_n \to 3$ and $Tx_n \to [0,3]$ iff $x_n \to 1$.
Also, $d(fTx_n,fx_n)\to\infty$ and

$$H(fTx_n,Tfx_n)=\left|\frac{3}{4}x_n^4+\frac{9}{2}x_n^3+\frac{21}{4}x_n^2-7\right|\to 7$$
$$H(Tfx_n,Tx_n)=\left|\frac{3}{4}x_n^4+\frac{13}{2}x_n^2+7\right|\to 15 \text{ if } x_n\to 1.$$

Therefore, f and T are f-weak compatible, but they are not compatible, and $\alpha : X\times X \to [0,\infty)$ by $\alpha(x,y)=\begin{cases}1, & \text{when } x,y\ge 0\\ 0, & \text{otherwise}\end{cases}$.
Take $\psi(t)=\frac{t}{2}$ and $\xi(t)=\sqrt{t}$ for each $t\ge 0$. Then T and f are (α,ψ,ξ)-contractive mapping.
Moreover T is α_*-admissible multi-valued mapping with respect to f. Thus all the conditions of theorem are satisfied. Therefore, T and f have coincidence point such as 1 is coincidence point of T and f.

Remark 2.4. By weakening the inequality (2.1) we can construct single-valued version of Theorem 2.2 which is more generalized in the sense that it requires continuity of only one of the two mappings T and f.

First, we will define α-admissibility for a pair of mappings.

Definition 2.5. Let T and f be self-mappings on a nonempty set X and $\alpha : X\times X\to[0,\infty)$ be another mapping. We say that T and f are α-admissible if the following condition holds:

$$x,y\in X,\ \alpha(fx,fy)\ge 1 \Rightarrow \alpha(Tx,Ty)\ge 1.$$

Now, we will state and prove our result for single-valued f-weak compatible mappings.

Theorem 2.6. *Let (X,d) be a complete metric space and $f:X\to X$ and $T:X\to X$ be the f-weak compatible pair such that $TX\subseteq fX$. Suppose that the following conditions hold:*

S_1. *T and f are α-admissible mappings;*
S_2. $\alpha(fx,fy)\ge 1 \Rightarrow \xi d(Tx,Ty)\le \psi\xi(M(x,y))$ (2.15)
where $M(x,y)=$
$\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\}$
and ξ and ψ are as defined earlier.
S_3. *There exists $fx_0\in X$ and $fx_1\in Tx_0$ such that*

$$\alpha(fx_0,fx_1)\ge 1;$$

If one of the mappings T and f is continuous, then there exists a point $t\in X$ such that $ft=Tt=t$.

Proof. It is seen that the sequence $\{Tx_n\}$, where $Tx_n=fx_{n+1}$ for each n, is a Cauchy sequence (as proved in Theorem 2.2). Hence it converges to some point $z\in X$. Suppose that T is continuous.
Then $T^2x_n \to Tz$ and $Tfx_n\to Tz$. By f-weak compatibility of f and T, we have

(i) $\lim_{n\to\infty} d(fTx_n,Tfx_n)\le \lim_{n\to\infty} d(Tfx_n,Tx_n)$, and
(ii) $\lim_{n\to\infty} d(fTx_n,fx_n)\le \lim_{n\to\infty} d(Tfx_n,Tx_n)$. (2.16)

Now, using (2.15), (2.16) and the continuity of T, we get

$$\begin{aligned}&\xi d(T^2x_n,Tx_n)\\ &\le \psi\xi(M(x,y))\\ &\le \psi\xi\left(\max\left\{d(fTx_n,fx_n),d(fTx_n,T^2x_n),d(fx_n,Tx_n),\right.\right.\\ &\qquad\left.\left.d(fTx_n,Tx_n),d(fx_n,T^2x_n)\right\}\right)\\ &\le \psi\xi\left(\max\left\{d(fTx_n,fx_n),d(fTx_n,Tfx_n)+d(Tfx_n,T^2x_n),\right.\right.\\ &\qquad\left.\left.d(fTx_n,Tx_n)+d(fx_n,Tx_n),d(fTx_n,Tx_n),d(fx_n,T^2x_n)\right\}\right)\end{aligned}$$

that is,

$$\xi d(Tz,z)\le \psi\xi(\max\{d(Tz,z),d(Tz,z),0,d(Tz,z),d(z,Tz)\}) \quad \text{as } n\to\infty,$$

that is, $Tz=z$. Since $TX\subseteq fX$, there exists a point z', such that $z=Tz=fz'$ and using (2.15) again,

$$\begin{aligned}&\xi d(T^2x_n,Tx_n)\\ &\le \psi\xi(\max\{d(fTx_n,z),d(fTx_n,T^2x_n),\\ &\qquad d(z,Tz'),d(fTx_n,Tz'),d(z,T^2x_n)\}).\end{aligned}$$

As $n\to\infty$ we deduce that $\xi d(z,Tz')\le\psi\xi d(z,Tz')$; that is, $z=Tz'=fz'$ and by the Lemma 1.12, we get

$$fz=fTz'=Tfz'=Tz=z.$$

Now, suppose that f is continuous. Then, $f^2x_n\to fz$ and $fTx_n\to fz$. By f-weak compatibility of f and T and continuity of f, we have

$$\text{(i)}\lim_{n\to\infty} d(fz,Tfx_n) \leq \lim_{n\to\infty} d(Tfx_n,z), \text{ and}$$
$$\text{(ii)}\lim_{n\to\infty} d(fz,z) \leq \lim_{n\to\infty} d(Tfx_n,z). \tag{2.17}$$

Now, using (2.15), (2.17) and continuity of f, we get

$$\begin{aligned}&\xi d(Tfx_n,Tx_n)\\ &\leq \psi\xi(M(x,y))\\ &\leq \psi\xi(\max\{d(f^2x_n,fx_n),d(f^2x_n,Tfx_n),d(fx_n,Tx_n),\\ &\qquad d(f^2x_n,Tx_n),d(fx_n,Tfx_n)\})\end{aligned}$$

that is,

$$\begin{aligned}&\xi d(fz,z)\\ &\leq \xi d(Tfx_n,z)\\ &\leq \psi\xi(\max\{d(fz,z),d(fz,Tfx_n),0,d(fz,z),d(z,Tfx_n)\})\\ &\qquad \text{as } n\to\infty\\ &\xi d(fz,z) \leq \xi d(Tfx_n,z)\\ &\leq \psi\xi(\max\{d(fz,z),d(Tfx_n,z),0,d(Tfx_n,z),d(z,Tfx_n)\})\\ &\qquad \text{as } n\to\infty\end{aligned}$$

that is, $Tfx_n \to z$ as $n\to\infty$ and $fz = z$. Again using (2.15) and (2.17), we have

$$\begin{aligned}\xi d(Tz,Tfx_n) \leq \psi\xi(\max\{&d(fz,f^2x_n),d(fz,Tz),\\ &d(f^2x_n,Tfx_n),d(fz,Tfx_n),d(f^2x_n,Tz)\})\end{aligned}$$

that is,

$$\xi d(Tz,z) \leq \psi\xi(\max\{d(0,d(z,Tz),0,0,d(z,Tz)\}) \quad \text{as } n\to\infty,$$

a contradiction. Therefore, z is a common fixed point of f and T.

Finally, we furnish an example to discuss the validity of Theorem 2.6.

Example 2.7. Let $X = [0,\infty)$ be endowed with the Euclidean metric d. Let $fx = \frac{1}{2}(x^2+x)$ and $Tx = \frac{1}{3}(x^2+2)$ for each $x \geq 0$. T and f are clearly continuous and $T(X) = f(X) = X$. Since $fx = Tx$ iff $x_n \to 1$, Also we can show that f and T are f-weak compatible.

Let $\alpha : X\times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases}1, & \text{when } x,y\geq 0\\ 0, & \text{otherwise}\end{cases}$.

Take $\psi(t) = \frac{t}{2}$ and $\xi(t) = \sqrt{t}$ for each $t \geq 0$. Then T and f satisfy condition (2.15).
Moreover T and f are α-admissible mappings. Thus all the conditions of Theorem 2.6 are satisfied. Therefore, T and f have coincidence point such as 1 is coincidence point of T and f.

Remark 2.8. We can construct coupled fixed point theorems for multi-valued as well as single-valued mappings by taking T defined as $T : X\times X \to CL(X)$ and $T : X\times X \to X$ in the above proved theorems respectively. In order to deduce the results for coupled fixed point we have to take defined as $\alpha : X^2\times X^2 \to [0,\infty)$.

3 Remarks

1. It can be seen, by taking $\psi(t) = \xi(t) = t$ for each $t \geq 1$ and $\alpha(x,y) = 1$ in Theorem 2.2 we get Theorem 2 of Pathak et al. [18] is a special case of Theorem 2.2;
2. If we assume $f(x) = x$ in Theorem 2.2 then we observe that Theorem 2.5 of Ali et al. [2] is a special case of Theorem 2.2;
3. By restricting $T : X \to X$ and taking $\xi(t) = t$ and $f(x) = x$ we get that results of Samet et al. [19] are the special case of above proved results.

4 Applications

Fixed point results in metric spaces endowed with an arbitrary binary relation

It has been pointed out in some studies that some results in metric spaces endowed with an arbitrary binary relation can be concluded from the fixed point results related with α-admissible mappings on metric spaces. In this section, we give some fixed point results on metric spaces endowed with an arbitrary binary relation which can be regarded as applications of results presented in the previous section. The following definitions and notions are needed.
Let (X,d) be a metric space and let R be a binary relation over X. Denote by $S = R\cup R^1$ the symmetric relation attached to R; that is,

$$x,y\in X,\ xSy \Leftrightarrow xRy \text{ or } yRx.$$

Definition 4.1. [5] Let $g : X \to X$ be a mapping. We say that a subset D of X is S-g-directed if for every $x,y \in D$, there exists $z \in X$ such that $gxSgz$ and $gySgz$.

Definition 4.2. (see [20]) We say that (X,d,S) is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that

$$x_nSx_{n+1} \text{ for all } n \text{ and } \lim_{n\to\infty} d(x_n,x) = 0,$$

then there exists a subsequence $\{x_n(p)\}$ of $\{x_n\}$ such that $x_{n(p)}Sx$ for all p.

Definition 4.3. [5] Let $T : X \to X$ and $f : X \to X$ be two mappings. We say that T is f-comparative mapping if T maps f-comparable elements into comparable elements; that is,

$$x,y\in X,\ fxSfy \Rightarrow TxSTy.$$

Definition 4.4. Let $T : X \to CL(X)$ and $f : X \to X$ be two mappings. We say that T is f-comparative mapping if T maps f-comparable elements into comparable elements; that is,

$$x,y\in X,\ fxSfy \Rightarrow TxSTy.$$

Definition 4.5. Let (X,d) be a metric space and let R be a binary relation over X. Then the hybrid pair $T : X \to CL(X)$ and $f : X \to X$ are called an (S,ψ,ξ)-contractive mapping if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that

$$x,y \in X,\ xSy \Rightarrow \xi H(Tx,Ty) \leq \psi\xi(M(x,y)),$$

where $M(x,y) =$
$\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}$.

Theorem 4.6. *Let (X,d) be a complete metric space and $f : X \to X$ and $T : X \to CL(X)$ be the continuous f-weak compatible hybrid pair such that $TX \subseteq fX$. Suppose that the following conditions hold:*

S_1. *T is an f-comparative mapping;*
S_2. *T and f are (S,ψ,ξ)-contractive mappings;*
S_3. *There exists $fx_0 \in X$ and $fx_1 \in Tx_0$ such that $fx_0 S fx_1$;*

Then there exists a point $t \in X$ such that $ft \in Tt$.
or f and T have a common coincidence point.

*Proof.*This result can be obtained from Theorem 2.2 by defining a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in xSy, \\ 0, & \text{otherwise} \end{cases}$$

This completes the proof.

*Remark.*We can obtain the single-valued version of above said results by using Theorem 2.6 and restricting T as a single-valued mapping.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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