Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/040310

Moment Properties of Generalized Order Statistics from Lindley Distribution

Jagdish Saran, Narinder Pushkarna and Rashmi Tiwari*

Department of Statistics, University of Delhi, Delhi-110007, India

Received: 10 May 2014, Revised: 31 Aug. 2015, Accepted: 17 Oct. 2015
Published online: 1 Nov. 2015

Abstract: In this paper, we have established several recurrence relations satisfied by the single and the product moments of generalized order statistics from Lindley distribution, to enable one to evaluate the single and product moments of all order in a recursive way.

Keywords: Order Statistics, Single moments, Product moments, Recurrence relations, Generalized order statistics, Lindley distribution.

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Let $X_{j:n}$ denote the j^{th} order statistic of a sample $(X_1, X_2, \ldots, X_n)$. Assume that $k > 0,\ n \in N,\ n \geq 2,\ \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + n - r + M_r > 0\ \forall\ r \in \{1, 2, \ldots, n-1\}$.
Then $X(r, n, \tilde{m}, k),\ r = 1, 2, \ldots, n$, are called generalized order statistics if their joint pdf is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n) = k\Big(\prod_{r=1}^{n-1}\gamma_r\Big)\Big(\prod_{i=1}^{n-1}\Big(1 - F(x_i)\Big)^{m_i} f(x_i)\Big) . \Big(1 - F(x_n)\Big)^{k-1} f(x_n), \tag{1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of R^n.
For convenience, let us define $X(0, n, \tilde{m}, k) = 0$. It can be seen that for $m_1 = \cdots = m_{n-1} = 0,\ k = 1, i.e., \gamma_i = n - i + 1; 1 \leq i \leq n-1$, we obtain the joint pdf of the ordinary order statistics. In a similar manner, choosing the parameters appropriately, some other models such as k^{th} upper record values $(m_1 = \ldots = m_{n-1} = -1,\ k \in N,\ i.e., \gamma_i = k, 1 \leq i \leq n-1)$, sequential order statistics $(m_r = (n-r+1)\alpha_r - (n-r)\alpha_{r+1} - 1;\ r = 1, \ldots, n-1,\ k = \alpha_n;\ \alpha_1, \alpha_2, \ldots, \alpha_n > 0,\ i.e., \gamma_i = (n-i+1)\alpha_i;\ 1 \leq i \leq n-1)$, order statistics with non-integral sample size $(m_1 = \ldots = m_{n-1} = 0,\ k = \alpha - n + 1\ with\ n - 1 < \alpha \in R,\ i.e., \gamma_i = \alpha - i + 1; 1 \leq i \leq n-1)$ [Rohatgi and Saleh (1988), Saleh, Scott and Junkins (1975)], Pfeifer's record values $(m_r = \beta_r - \beta_{r+1} - 1,\ r = 1, \ldots, n-1\ and\ k = \beta_n;\ \beta_1, \beta_2, \ldots, \beta_n > 0,\ i.e., \gamma_i = \beta_i; 1 \leq i \leq n-1)$ and progressively type-II right censored order statistics $(m_i \in N_0,\ k \in N)$ can be obtained [cf. Kamps (1995a,b), Kamps and Cramer (2001)].

We may now consider two cases:
Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.
Case II: $\gamma_i \neq \gamma_j,\ i \neq j,\ i, j = 1, 2, \ldots, n-1$.
For **Case I**, the r^{th} generalized order statistic will be denoted by $X(r, n, m, k)$. The pdf of $X(r, n, m, k)$ is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}\Big(1 - F(x)\Big)^{\gamma_r - 1} f(x) g_m^{r-1}\Big(F(x)\Big), \quad x \in R, \tag{2}$$

* Corresponding author e-mail: jagdish_saran52@yahoo.co.in

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by

$$f^{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\left[1-F(x)\right]^m f(x) g_m^{r-1}\Big(F(x)\Big)$$
$$.\Big[h_m(F(y)) - h_m(F(x))\Big]^{s-r-1}\Big[1-F(y)\Big]^{\gamma_s-1} f(y), \quad x<y, \tag{3}$$

where

$$c_{r-1} = \prod_{j=1}^{r}\gamma_j, \qquad \gamma_j = k+(n-j)(m+1), \qquad r=1,2,\ldots,n,$$
$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1),$$
$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & \text{if} \quad m \ne -1 \\ -\log(1-x), & \text{if} \quad m=-1 \end{cases}$$

(cf. Kamps, 1995a,b).
For the sake of convenience, let us denote, under Case I,

$$E[X(r,n,m,k)]^i = \mu^{(i)}_{(r,n,m,k)}$$

and

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \mu^{(i,j)}_{(r,s,n,m,k)}\,.$$

For **Case II**, the r^{th} generalized order statistic will be denoted by $X(r,n,\tilde{m},k)$. The pdf of $X(r,n,\tilde{m},k)$ is given by

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x)\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1}, \quad x \in R, \tag{4}$$

and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$, is given by

$$f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = c_{s-1}\Big\{\sum_{i=r+1}^{s} a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}\Big\}\Big\{\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\Big\}$$
$$.\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}, \quad x<y, \tag{5}$$

where

$$c_{r-1} = \prod_{i=1}^{r}\gamma_i, \qquad \gamma_i = k+n-i+M_i, \qquad r=1,2,\ldots,n,$$
$$a_i(r) = \prod_{j(\ne i)=1}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \quad 1 \le i \le r \le n$$
$$\text{and} \quad a_i^{(r)}(s) = \prod_{j(\ne i)=r+1}^{s}\frac{1}{(\gamma_j-\gamma_i)}, \quad r+1 \le i \le s \le n,$$

(cf. Kamps and Cramer (2001)).
For the sake of convenience, let us denote, under Case II,

$$E[X(r,n,\tilde{m},k)]^i = \mu^{(i)}_{(r,n,\tilde{m},k)}$$

and

$$E[\{X(r,n,\tilde{m},k)\}^i\{X(s,n,\tilde{m},k)\}^j] = \mu^{(i,j)}_{(r,s,n,\tilde{m},k)}\,.$$

Further, it can be easily proved that

$$a_i(r) = (\gamma_{r+1} - \gamma_i)\, a_i(r+1),$$
$$c_{r-1} = \frac{c_r}{\gamma_{r+1}},$$
$$\text{and} \sum_{i=1}^{r+1} a_i(r+1) = 0. \tag{6}$$

Also, for $m_1 = m_2 = \ldots = m_{n-1} = m$, it can be shown that

$$\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} = \frac{(1-F(x))^{\gamma_r}}{(r-1)!} g_m^{r-1}(F(x)), \tag{7}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i} = \frac{1}{(s-r-1)!}\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_s}\Big(\frac{1}{1-F(x)}\Big)^{(m+1)(s-r-1)} .\Big[h_m(F(y)) - h_m(F(x))\Big]^{s-r-1}. \tag{8}$$

Several authors like Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001), Ahmed and Fawzy (2003), Athar and Islam (2004), Ahmed (2007), Khan et al. (2007), Khan et al. (2010) and Saran and Pandey (2004, 2009) have done some work on generalized order statistics. In this paper, we have established certain recurrence relations for single and product moments of generalized order statistics from Lindley distribution.

The Lindley distribution was first introduced in the literature by Lindley (1958) in connection with the Fiducial distribution and Bayes theorem. The probability density function (pdf) of Lindley distribution is given by

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \quad x>0, \;\; \theta>0 \tag{9}$$

and the cumulative distribution function (cdf) is given by

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}, \;\; x>0, \;\; \theta>0. \tag{10}$$

One can observe from eqs. (9) and (10) that the characterizing differential equation for Lindley distribution is given by

$$(1+\theta+\theta x)f(x) = \theta^2(1+x)[1-F(x)]. \tag{11}$$

2 Recurrence Relations for Single Moments

Theorem 1.*Let Case II be satisfied, i.e.,* $\gamma_i \neq \gamma_j,\; i \neq j,\; i,j = 1,2,\ldots,n-1$. *For Lindley distribution as given in* (9) *and* $k \geq 1,\; n \in N,\; 1 \leq r \leq n,\; p = 0,1,2,\ldots,$

$$(1+\theta)\mu^{(p)}_{(r,n,\tilde{m},k)} + \theta\mu^{(p+1)}_{(r,n,\tilde{m},k)} = \frac{\theta^2\,\gamma_r}{p+1}\Big[\mu^{(p+1)}_{(r,n,\tilde{m},k)} - \mu^{(p+1)}_{(r-1,n,\tilde{m},k)}\Big] + \frac{\theta^2\,\gamma_r}{p+2}\Big[\mu^{(p+2)}_{(r,n,\tilde{m},k)} - \mu^{(p+2)}_{(r-1,n,\tilde{m},k)}\Big]. \tag{12}$$

*Proof.*In view of (4), we have

$$(1+\theta)\mu^{(p)}_{(r,n,\tilde{m},k)} + \theta\mu^{(p+1)}_{(r,n,\tilde{m},k)} = c_{r-1}\int_0^\infty x^p\Big(\sum_{i=1}^{r} a_i(r)\Big(1-F(x)\Big)^{\gamma_i-1}\Big)(1+\theta+\theta x)f(x)dx.$$

Now, on application of (11), we get

$$(1+\theta)\mu^{(p)}_{(r,n,\tilde{m},k)} + \theta\mu^{(p+1)}_{(r,n,\tilde{m},k)} = \theta^2[L_0(x) + L_1(x)], \tag{13}$$

where

$$L_b(x) = c_{r-1}\int_0^{\infty} x^{p+b}\sum_{i=1}^{r} a_i(r)\Big(1-F(x)\Big)^{\gamma_i}dx, \qquad b=0,1.$$

Integrating by parts treating x^{p+b} for integration and rest of the integrand for differentiation, we get

$$L_b(x) = \frac{c_{r-1}}{(p+b+1)}\int_0^{\infty} x^{p+b+1}\sum_{i=1}^{r} a_i(r)\gamma_i\Big(1-F(x)\Big)^{\gamma_i-1} f(x)dx. \tag{14}$$

Further, on using (6) and simplifying, the above equation yields:

$$L_b(x) = \frac{c_{r-1}\ \gamma_r}{(p+b+1)}\Big[\mu^{(p+b+1)}_{(r,n,\tilde{m},k)} - \mu^{(p+b+1)}_{(r-1,n,\tilde{m},k)}\Big]. \tag{15}$$

For $b=0,1$, substituting the expression of $L_b(x)$ obtained in (15) into the equation (13) and then simplifying the resultant expression, we obtain the desired relation in (12).

*Remark.*One can observe that by putting $m_1 = m_2 = \ldots = m_{n-1} = m$ in (4) and using (7), the recurrence relation for single moments of generalized order statistics from Lindley distribution, for Case I, can easily be deduced from Theorem 1, and is given in the following corollary.

Corollary 1.*Let Case I be satisfied, i.e.,* $m_1 = m_2 = \ldots = m_{n-1} = m$. *For Lindley distribution as given in* (9) *and* $k \geq 1,\ n \in N,\ m \in R,\ 1 \leq r \leq n,\ \theta > 0,\ i = 0,1,2,\ldots,$

$$(1+\theta)\mu^{(i)}_{(r,n,m,k)} + \theta\mu^{(i+1)}_{(r,n,m,k)} = \frac{\gamma_r\theta^2}{i+1}\Big[\mu^{(i+1)}_{(r,n,m,k)} - \mu^{(i+1)}_{(r-1,n,m,k)}\Big] + \frac{\gamma_r\theta^2}{i+2}\Big[\mu^{(i+2)}_{(r,n,m,k)} - \mu^{(i+2)}_{(r-1,n,m,k)}\Big]. \tag{16}$$

*Remark.*Under the assumptions of Corollary 1, with $k=1, m=0$, we shall deduce the recurrence relation for single moments of ordinary order statistics from Lindley distribution, which is in agreement with the corresponding result obtained by Athar et al. (2014, Remark 2.1, p.4).

*Remark.*Putting $k=0, m=-1$ in Corollary 1, we obtain the recurrence relation for single moments of upper record values from Lindley distribution.

3 Recurrence Relations for Product Moments

Theorem 2.*Let Case II be satisfied, i.e.,* $\gamma_i \neq \gamma_j, i \neq j,\ i,j = 1,2,\ldots,n-1$. *For Lindley distribution as given in* (9) *and* $k \geq 1,\ n \in N,\ 1 \leq r < s \leq n,\ s-r \geq 2,\ p,q = 0,1,2,\ldots,$

$$(1+\theta)\mu^{(p,q)}_{(r,s,n,\tilde{m},k)} + \theta\mu^{(p,q+1)}_{(r,s,n,\tilde{m},k)} = \frac{\theta^2\ \gamma_s}{q+1}\Big[\mu^{(p,q+1)}_{(r,s,n,\tilde{m},k)} - \mu^{(p,q+1)}_{(r,s-1,n,\tilde{m},k)}\Big] + \frac{\theta^2\ \gamma_s}{q+2}\Big[\mu^{(p,q+2)}_{(r,s,n,\tilde{m},k)} - \mu^{(p,q+2)}_{(r,s-1,n,\tilde{m},k)}\Big], \tag{17}$$

and, for $1 \leq r \leq n-1$,

$$(1+\theta)\mu^{(p,q)}_{(r,r+1,n,\tilde{m},k)} + \theta\mu^{(p,q+1)}_{(r,r+1,n,\tilde{m},k)} = \frac{\theta^2\ \gamma_{r+1}}{q+1}\Big[\mu^{(p,q+1)}_{(r,r+1,n,\tilde{m},k)} - \mu^{(p+q+1)}_{(r,n,\tilde{m},k)}\Big] + \frac{\theta^2\ \gamma_{r+1}}{q+2}\Big[\mu^{(p,q+2)}_{(r,r+1,n,\tilde{m},k)} - \mu^{(p+q+2)}_{(r,n,\tilde{m},k)}\Big]. \tag{18}$$

*Proof.*In view of (5), we have for $1 \leq r < s \leq n,\ s-r \geq 2$ and $p,q = 0,1,2,\dots,$

$$(1+\theta)\mu^{(p,q)}_{(r,s,n,\tilde{m},k)} + \theta\mu^{(p+1,q)}_{(r,s,n,\tilde{m},k)} = c_{s-1}\int_0^\infty x^p\Big\{\sum_{i=1}^r a_i(r)\Big(1-F(x)\Big)^{\gamma_i}\Big\}\frac{f(x)}{1-F(x)}I(x)dx, \tag{19}$$

where

$$I(x) = \int_x^\infty y^q\Big\{\sum_{i=r+1}^s a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}\Big\}(1+\theta+\theta y)f(y)dy.$$

Now, on employing (11), we get

$$I(x) = \theta^2[E_0(x)+E_1(x)], \tag{20}$$

where

$$E_d(x) = \int_x^\infty y^{q+d}\Big\{\sum_{i=r+1}^s a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}\Big\}dy, \qquad d = 0,1. \tag{21}$$

Integrating by parts treating y^{q+d} for integration and rest of the integrand for differentiation, we obtain

$$E_d(x) = \frac{1}{q+d+1}\int_x^\infty y^{q+d+1}\Big\{\sum_{i=r+1}^s \frac{a_i^{(r)}(s)}{[1-F(x)]^{\gamma_i}}\ \gamma_i[1-F(y)]^{\gamma_i-1}\Big\}f(y)dy.$$

Substituting the expressions for $E_0(x)$ and $E_1(x)$ in (20), and then putting the resulting value of $I(x)$ in (19) and simplifying, it leads to (17). Likewise, (18) can be easily established.

*Remark.*On putting $m_i = m_j = m$ in (5) and using (7) and (8), the recurrence relations for product moments of generalized order statistics from Lindley distribution, for Case I, can be deduced from Theorem 2, and is given in the following corollary.

Corollary 2.*Let Case I be satisfied, i.e., $m_1 = m_2 = \ldots = m_{n-1} = m$. For Lindley distribution as given in (9) and $k \geq 1,\ n \in N,\ m \in R,\ 1 \leq r < s \leq n,\ s-r \geq 2,\ i,j = 0,1,2,\dots,$*

$$(1+\theta)\mu^{(i,j)}_{(r,s,n,m,k)} + \theta\mu^{(i,j+1)}_{(r,s,n,m,k)} = \frac{\theta^2\ \gamma_s}{j+1}\Big[\mu^{(i,j+1)}_{(r,s,,n,m,k)} - \mu^{(i,j+1)}_{(r,s-1,n,m,k)}\Big] + \frac{\theta^2\ \gamma_s}{j+2}\Big[\mu^{(i,j+2)}_{(r,s,n,m,k)} - \mu^{(i,j+2)}_{(r,s-1,n,m,k)}\Big], \tag{22}$$

and, for $1 \leq r \leq n-1,$

$$(1+\theta)\mu^{(i,j)}_{(r,r+1,n,m,k)} + \theta\mu^{(i,j+1)}_{(r,r+1,n,m,k)} = \frac{\theta^2\ \gamma_{r+1}}{j+1}\Big[\mu^{(i,j+1)}_{(r,r+1,n,m,k)} - \mu^{(i+j+1)}_{(r,n,m,k)}\Big] + \frac{\theta^2\ \gamma_{r+1}}{j+2}\Big[\mu^{(i,j+2)}_{(r,r+1,n,m,k)} - \mu^{(i+j+2)}_{(r,n,m,k)}\Big]. \tag{23}$$

*Remark.*Under the assumptions of Corollary 2, with $k = 1, m = 0$, we shall deduce the recurrence relations for product moments of ordinary order statistics from Lindley distribution, which are in agreement with the corresponding results obtained by Athar et al. (2014, Remark 2.1, p.4) .

*Remark.*Putting $k = 0, m = -1$ in Corollary 2, we obtain the recurrence relations for product moments of upper record values from Lindley distribution.

Acknowledgement

The authors are grateful to the referees for giving valuable comments which led to significant improvement in the presentation of the paper.

References

[1] Ahmad, A. A. (2007) Relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution, *Journal of the Egyptian Mathematical Society*, **15**, 117-128.

[2] Ahmad, A. A. and Fawzy, M. (2003). Recurrence relations for single moments of generalized order statistics from doubly truncated distributions, *Journal of Statistical Planning and Inference*, **117**, 241-249.

[3] Ahsanullah, M. (2000). Generalized order statistics from exponential distribution, *Journal of Statistical Planning and Inference*, **85**, 85-91.

[4] Athar, H., Akhter, Z. and Saran, J. (2014). Moments of Progressive Type-II Right Censored Order Statistics from Lindley Distribution, *Statistics Research Letters*, **3**, No.1, 1-6.

[5] Athar, H. and Islam, H.M. (2004). Recurrence relations between single and product moments of generalized order statistics from a general class of distribution, *Metron*, **LXII**, 327-337.

[6] Cramer, E. and Kamps, U. (2000). Relations for expectations of functions of generalized order statistics, *Journal of Statistical Planning and Inference*, **89**, 79-89.

[7] Kamps, U. (1995a). A concept of generalized order statistics, *B.G. Teubner, Stuttgart*.

[8] Kamps, U. (1995b). A concept of generalized order statistics, *Journal of Statistical Planning and Inference*, **48**, 1-23.

[9] Kamps, U. and Cramer, E. (2001). On distributions of generalized order statistics, *Statistics*, **35**, 269-280.

[10] Kamps, U. and Gather, U. (1997). Characteristic property of generalized order statistics for exponential distribution, *Appl. Math. (Warsaw)*, **24**, 383-391.

[11] Keseling, C. (1999). Conditional distributions of generalized order statistics and some characterizations, *Metrika*, **49**, 27-40.

[12] Khan, R. U., Anwar, Z. and Athar, H. (2007). Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution, *Aligarh Journal of Statistics*, **27**, 69-79.

[13] Khan, R. U., Kumar, D. and Athar, H. (2010). Moments of generalized order statistics from Erlang-truncated exponential distribution and its characterization, *International Journal of Statistics and System*, **5**, 455- 464.

[14] Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem, *J. R. Statist. Soc.*, **B20**, 102-107.

[15] Pawlas, P. and Szynal, D. (2001). Recurrence Relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions,*Commun. Statist.-Theory Meth.*, **30(4)**, 739-746.

[16] Rohatgi, V.K. and Saleh, A.K. Md.E. (1988). A class of distributions connected to order statistics with non integral sample size. *Commun. Statist.-Theory Meth.*, **17(6)**, 2005-2012.

[17] Saleh, A.K. Md.E., Scott, C. and Junkins, D.B. (1975). Exact first and second order moments of order statistics from truncated exponential distribution. *Naval Research Logistics Quarterly*, **22**, 65-77.

[18] Saran, J. and Pandey, A. (2004). Recurrence relations for single and product moments of generalized order statistics from linear exponential distribution, *Journal of Applied Statistical Science*, **13(4)**, 323-333.

[19] Saran, J. and Pandey, A. (2009). Recurrence relations for single and product moments of generalized order statistics from linear exponential and Burr distributions, *Journal of Statistical Theory and Applications*, **8**, 383-391.