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Estimation in Step-Stress Accelerated Life Testing for Lindely Distribution with Progressive First-Failure Censoring

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Abstract: In this paper, we obtained point estimation and interval estimation for Lindely distribution parameter and the acceleration factor under step-stress accelerated life test with progressive first failure sample. In addition, mean square errors (MSEs) of the maximum likelihood estimators (MLEs) are computed to asses their performance.

Keywords: Lindely distribution, maximum likelihood estimators, progressive first failure, step-stress partially accelerated life tests.

1 Introduction

Accelerated life tests (ALT) are used to estimate the lifetime of highly reliable products within a reasonable testing time. The test products are run at higher than usual levels of stress (which includes temperature, voltage, pressure, etc.) to induce early failures. The test data obtained at the accelerated conditions are analyzed in terms of a suitable physical model, and then extrapolated to stress to estimate the life distribution. The stress can be applied in different ways: commonly used methods are constant stress, progressive stress and step-stress (see [1]). In the constant-stress ALT, the stress is kept at a constant level throughout the life of test products, see for example, [2,3,4]. In the progressive-stress ALT, the stress applied to a test product is continuously increasing in time, see for example, [5,6], considered the estimation problem of the constant-stress accelerated life tests for extension of the exponential distribution under progressive censoring. [7] obtained the optimal plans of constant-stress accelerated life tests for the Lindley distribution. [8] estimated the parameters of Weibull distribution under step-stress acceleration, [9] estimated the parameters for power generalized Weibull under step-stress acceleration, see olso [10,11]. The step-stress ALT, in which the test condition changes at a given time or upon the occurrence of a specified number of failures, has been studied by several authors see for example, [12]. [13] obtained the optimal simple step-stress ALT plans for the case where test products have exponentially distributed lives and are observed continuously until all test products fail; [14] extended their results to the case of censoring.

Suppose that n independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $Y^R_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure occurred $Y^R_{2;m,n,k}$, and finally when the m-th failure $\mathrm{Y}^R_{m;m,n,k}$ is observed, the remaining groups $R_m,\ (m \le n)$ are removed from the test. Then $Y^R_{1;m,n,k} < ... < Y^R_{m;m,n,k}$ are called progressively first-failure censored order statistics with the progressive censored scheme $R = (R_1, R_2, ..., R_m)$, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally

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in the test are from a continuous population with distribution function $F(y)$ and probability density function $f(y)$, the joint probability density function for $Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}$ is given by [14] as follows:

$$f_{1,2,\ldots,m}(Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}) = A(n,m-1)k^m \prod_{i=1}^{m} f(Y^R_{i;m,n,k}) \left[1 - F(Y^R_{i;m,n,k})\right]^{k(R_i+1)-1}, \tag{1}$$

$$0 < x_1 < x_2 < \ldots < x_m < \infty,$$

where

$$A(n,m-1) = n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{m-1}-(m-1)). \tag{2}$$

The Lindley distribution was originally proposed by [15] in the context of Bayesian statistics, as a counter example of fudicial statistics.

Assume that the random variable X representing the lifetime of a product has Lindley distribution with parameters θ. Lindley distribution has the following probability density function

$$f(x) = \left[\frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}\right], \; x > 0, \; \theta > 0,$$

and cumulative distribution function

$$F(x) = \left[1 - \left(1 + \frac{\theta x}{1+\theta}\right) e^{-\theta x}\right], \; x > 0, \; \theta > 0.$$

Lindely distribution has many real life applications see for example, [16] have introduced real data represent the waiting times and fitting them. They proved that the Lindely distribution is better model than the exponential distribution. They also found that the maximum likelihood has a standard error reduced than the exponential distribution.

2 Assumptions and test procedure

The following assumptions are used in the paper in the framework of step-stress partially accelerated life test (SSPALT) :

1. Suppose that n identical and independent groups with k items within each group are put on a life test and the lifetime of each unit has Lindely distribution.
2. The test is terminated at the m-th failure, where m is prefixed ($m \leq n$).
3. Each of the n$\times$ k units is first run under normal use condition. If it does not fail or removed from the test by a precipiced time it is put under accelerated condition.
4. At the i-th failure a random number of the surviving groups R_i, $i = 1, 2, \ldots, m-1$, and the group in which the failure $Y^R_{i;m,n,k}$ has occurred are randomly selected and removed from the test. Finally, at the m-th failure the remaining surviving groups $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all removed from the test and the test is terminated.
5. Let n_1 be the number of failures before time at normal condition, and n_2 be the number of failures after time τ at stress condition, with these notations the observed progressive first-failure censored data are
$$Y^R_{1;m,n,k} < \ldots. < Y^R_{n_1;m,n,k} < \tau < Y^R_{n_1+1;m,n,k} < \ldots. < Y^R_{m;m,n,k},$$
where $n = m + \sum_{i=1}^{m} R_i$.
6. The tampered random variable (TRV) model holds. It was proposed by [17]. According to the tampered random variable model the lifetime of a unit under SSPALT can be written as :
$$Y = \begin{cases} T & if \; T \leq \tau, \\ \tau + \frac{T-\tau}{\beta} & if \; T > \tau \end{cases},$$
where T is the lifetime of the unit under normal condition, τ is the stress change time and β is the acceleration factor ($\beta > 1$).

7.The probability density function of y after acceleration is give by

$$f(y)=\begin{cases} 0\,, & y<0 \\ f_1(y)=\left[\dfrac{\theta^2(1+y)e^{-\theta y}}{1+\theta}\right], & 0<y<\tau \\ f_2(y)=\dfrac{\theta^2\beta\,(1+(\beta(y_i-\tau)+\tau))e^{-\theta(\beta(y_i-\tau)+\tau)}}{1+\theta}, & \tau<y<\infty. \end{cases}$$

3 Parameters estimation

This section discusses the process of obtaining point and interval estimations of the parameters of Lindely distribution based on progressive first-failure censored data under SSPALT.

3.1 Point estimation

Let $y_i=Y^R_{i;m,n,k}$ be the observed values of the lifetime y obtained from a progressive first-failure censoring scheme under SSPALT, with censored scheme $R=(R_1,\ldots,R_m)$, then the maximum likelihood function of the observations is:

$$L(\theta,\beta)=Ak^m\prod_{i=1}^{n_1}f_1(y_i)\,[1-F_1(y_i)]^{k(R_i+1)-1}\prod_{i=n_1+1}^{m}f_2(y_i)\,[1-F_2(y_i)]^{k(R_i+1)-1} \tag{3}$$

The log likelihood function may have the form:

$$\begin{aligned}\ell(\theta,\beta)=&\log A+\log k^m+\sum_{i=1}^{m}\log\theta^2+\sum_{i=n_1+1}^{m}\log\beta-\sum_{i=1}^{m}\log(1+\theta)+\\ &\sum_{i=1}^{n_1}\log(1+y_i)+\sum_{i=1}^{n_1}(k(R_i+1)-1)\log(1+\frac{\theta y_i}{1+\theta})-\sum_{i=1}^{n_1}\theta y_i(k(R_i+1))+\\ &\sum_{i=n_1+1}^{m}\log(1+\beta(y_i-\tau)+\tau)+\sum_{i=n_1+1}^{m}(k(R_i+1)-1)\log(1+\frac{\theta(\beta(y_i-\tau)+\tau)}{1+\theta})-\\ &\sum_{i=n_1+1}^{m}\theta(\beta(y_i-\tau)+\tau))(k(R_i+1)),\end{aligned} \tag{4}$$

$$\begin{aligned}\ell(\theta,\beta)=&\log A+m\log k+2m\log\theta+(m-n_1)\log\beta-m\log(1+\theta)+\\ &\sum_{i=1}^{n_1}\log(1+y_i)+\sum_{i=1}^{n_1}(k(R_i+1)-1)\log(1+\frac{\theta y_i}{1+\theta})-\sum_{i=1}^{n_1}\theta y_i(k(R_i+1))+\\ &\sum_{i=n_1+1}^{m}\log(1+\beta(y_i-\tau)+\tau)+\sum_{i=n_1+1}^{m}(k(R_i+1)-1)\log(1+\frac{\theta(\beta(y_i-\tau)+\tau)}{1+\theta})-\\ &\sum_{i=n_1+1}^{m}\theta(\beta(y_i-\tau)+\tau))(k(R_i+1)).\end{aligned} \tag{5}$$

Obtaining the first derivatives w.r.t. θ and β as follows:

$$\begin{aligned}\frac{\partial\ell(\theta,\beta)}{\partial\beta}=&\frac{m-n_1}{\beta}+\sum_{i=n_1+1}^{m}\frac{y_i-\tau}{(1+\beta(y_i-\tau)+\tau)}+\\ &\sum_{i=n_1+1}^{m}(k(R_i+1)-1)\frac{\theta(y_i-\tau)}{((1+\theta)+\theta(\beta(y_i-\tau)+\tau))}-\sum_{i=n_1+1}^{m}\theta(y_i-\tau)(k(R_i+1)),\end{aligned} \tag{6}$$

$$\frac{\partial \ell(\theta,\beta)}{\partial \theta} = \frac{2m}{\theta} - \frac{m}{1+\theta} + \sum_{i=1}^{n_1}(k(R_i+1)-1)\left(\frac{y_i}{((1+\theta)^2+(1+\theta)\,\theta y_i)}\right) - \sum_{i=1}^{n_1} y_i k(R_i+1) + \sum_{i=n_1+1}^{m}(k(R_i+1)-1)\left(\frac{(\beta(y_i-\tau)+\tau))}{((1+\theta)^2+(1+\theta)\,\theta((\beta(y-\tau)+\tau))}\right) - (\beta(y_i-\tau)+\tau))k(R_i+1). \tag{7}$$

Solving this system of non linear equations for the unknowns θ,β numerically because they are very difficult to solve them algebraically.

3.2 Interval estimation

In this subsection we obtain the confidence intervals of the parameters based on asymptotic distribution of the MLEs of the unknown parameters $\Theta = (\beta,\theta)$. The asymptotic distribution of the MLEs is given by [18]:

$\left((\hat{\theta}-\theta),\left(\hat{\beta}-\beta\right)\right) \to N\left(0, I^{-1}(\theta,\beta\right)$, where I^{-1} is the variance covariance matrix of the unknown parameters (θ,β).

Where

$$I_{ij}(\Theta) = \frac{-\partial^2 \ell(\Theta)}{\partial\theta\,\partial\beta} \quad \text{at} \quad \theta = \hat{\theta}. \tag{8}$$

$$\frac{\partial^2 \ell(\Theta)}{\partial \theta^2} = -\frac{2m}{\theta^2} + \frac{m}{(1+\theta)^2} + \sum_{i=1}^{n_1}(k(R_i+1)-1)\left(\frac{-y_i(2(1+\theta)+y_i(1+2\theta)}{((1+\theta)^2+(1+\theta)\,\theta y_i)}\right) + \sum_{i=n_1+1}^{m}(k(R_i+1)-1)\left(\frac{-(\beta(y_i-\tau)+\tau))(2(1+\theta)+(1+2\theta)(\beta(y_i-\tau)+\tau)}{((1+\theta)^2+(1+\theta)\,\theta(\beta(y_i-\tau)+\tau)}\right). \tag{9}$$

$$\frac{\partial^2 \ell(\Theta)}{\partial \beta^2} = -\left(\frac{m-n_1}{\beta^2}\right) - \sum_{i=n_1+1}^{m}\frac{(y_i-\tau)^2}{(1+\beta(y_i-\tau)+\tau)^2} - \sum_{i=n_1+1}^{m}(k(R_i+1)-1)\left(\frac{\theta^2(y-\tau)^2}{((1+\theta)+\theta(\beta(y_i-\tau)+\tau))^2}\right). \tag{10}$$

$$\frac{\partial^2 \ell(\Theta)}{\partial\theta\,\partial\beta} = \sum_{i=n_1+1}^{m}(k(R_i+1)-1)\frac{((1+\theta)+\theta(\beta(y_i-\tau)+\tau)(y_i-\tau)-\theta(y_i-\tau)(1+\beta(y_i-\tau)+\tau))}{((1+\theta)+\theta(\beta(y_i-\tau)+\tau)^2} - \sum_{i=n_1+1}^{m}(y_i-\tau)(k(R_i+1)). \tag{11}$$

3.3 Approximate confidence intervals

When the sample size is small, the normal approximation may be poor. However, a different transformation of the MLEs can be used to correct the inadequate performance of the normal approximation. Based on the normal approximation of the log-transformed MLEs ([19]) and the approximate $100(1-\gamma)\%$ confidence interval for θ and β, are respectively given by:

$$\left[\frac{\hat{\theta}}{\exp\left(\frac{z_{1-\frac{\gamma}{2}}\sqrt{I_{11}^{-1}(\hat{\theta})}}{\hat{\theta}}\right)},\ \hat{\theta}\exp\left(\frac{z_{1-\frac{\gamma}{2}}\sqrt{I_{11}^{-1}(\hat{\theta})}}{\hat{\theta}}\right)\right], \tag{12}$$

$$\left[\frac{\hat{\beta}}{\exp\left(\frac{z_{1-\frac{\gamma}{2}}\sqrt{I_{11}^{-1}(\hat{\beta})}}{\hat{\beta}}\right)}, \hat{\beta}\exp\left(\frac{z_{1-\frac{\gamma}{2}}\sqrt{I_{11}^{-1}(\hat{\beta})}}{\hat{\beta}}\right)\right]. \tag{13}$$

4 Algorithm for simulation studies

In this section, simulation studies are conducted to investigate the performances of the MLEs in terms of their biases and mean square errors (MSEs) for different values of n, m and k . Also, 95% asymptotic confidence intervals based on the asymptotic distribution of the MLEs are computed. Two progressive censoring schemes are considered:

scheme I: $R_1 = n - m, R_r = 0, r = 2, 3, ..., m-1$

scheme II: $R_1 = 0, \ R_2 = n - m, \ R_r = 0, \ r = 3, ..., m-1$

The estimation procedure is performed according to the following algorithm:

1. Specify the values of n, m, k and τ.
2. Specify the values of the parameters θ and β.
3. Generate a random sample of size $n \times k$ from the random variable Y by using mathematica because it is hard to generate data manually.
4. Use the tampered random variable (TRV) model to generate progressively first-failure censored data for given n and m
5. Use the progressive first failure censored data to compute the MLEs of the model parameters. The Newton Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.
6. Replicate the steps (3-5) N times.
7. Compute the average values of the parameters and the mean square errors (MSEs) of the parameters.
8. Estimate the asymptotic variances of the estimators of model parameters.
9. Compute the approximate confidence bounds with confidence level 95% for the two parameters of the model.
10. Steps 1-9 are done with different values of n, m and k.

5 Numerical results

Average values of MLEs of the parameter, the associated MSEs and the associated approximate confidence intervals based on 1000 simulations, when population parameters values θ =0.2 , β =1.1 and stress change time τ =0.5, N=1000.

Table 1: Values of MLEs of the parameter, MSEs and the confidence intervals

k	n	m	c.s	$\hat{\theta}$	mseof $\hat{\theta}$	$\hat{\beta}$	mseof $\hat{\beta}$	95% CI length of $\hat{\theta}$	95% CIof length of $\hat{\beta}$
1	25	15	I	0.01163	0.03549	1.69911	0.47295	0.00851	0.49999
			II	0.01042	0.03595	1.85151	0.78613	0.00762	0.52199
1	30	15	I	0.01165	0.03548	1.68662	0.48463	0.00853	0.49675
			II	0.01025	0.03609	1.94816	0.89754	0.00749	0.5286
1	40	15	I	0.01179	0.03543	1.69187	0.46703	0.00862	0.50216
			II	0.01014	0.04142	1.63068	0.47419	0.03383	0.50372
1	50	15	I	0.01168	0.03547	1.69525	0.48165	0.00854	0.49929
			II	0.01446	0.03468	1.61327	0.45664	0.01056	0.49995
1	50	25	I	0.00712	0.03721	1.59506	0.41958	0.004	0.36635
			II	0.00831	0.03676	1.43945	0.23919	0.00467	0.35349
1	60	15	I	0.01177	0.03544	1.6835	0.48442	0.0086	0.49597
			II	0.0142	0.03453	1.6096	0.4427	0.01039	0.50299
1	60	30	I	0.00649	0.03885	1.2926	0.07771	0.00402	0.29344
			II	0.00715	0.03723	1.35265	0.10685	0.00366	0.30662

k	n	m	c.s	$\hat{\theta}$	mseof $\hat{\theta}$	$\hat{\beta}$	mseof $\hat{\beta}$	95% CI length of $\hat{\theta}$	95% CIof length of $\hat{\beta}$
2	30	10	I	0.01023	0.03686	1.52751	0.34378	0.01004	0.60108
			II	0.04453	0.00356	1.34308	0.39516	0.71822	0.68313
2	30	15	I	0.01881	0.11074	1.49999	0.25547	0.01421	0.4677
			II	0.16135	0.13846	1.3264	0.6304	0.23747	0.44604
2	35	15	I	0.00849	0.03898	1.49921	0.24322	0.00666	0.46834
			II	0.06889	0.06566	1.48206	0.51619	0.14009	0.47317
2	40	15	I	0.00986	0.0425	1.5019	0.25718	0.0083	0.46823
			II	0.02779	0.04411	1.58064	0.4371	0.03244	0.49543
2	40	18	I	0.08332	0.53572	1.21296	0.13282	0.07289	0.36595
			II	0.06779	0.14879	1.38538	0.27628	0.08194	0.41104
2	50	15	I	0.00831	0.0392	1.51422	0.26255	0.00733	0.47057
			II	0.00727	0.03716	1.64046	0.4043	0.00531	0.50736

6 Conclusion

We estimated the parameter of lindely distribution $\hat{\theta}$ and the accelerating factor $\hat{\beta}$ by using maximum likelihood technique under step stress acceleration with progressive first failure data. we used two different schemes (I and II) and we conclouded the following:

1.The MSEs of $\hat{\theta}$ are less than that of $\hat{\beta}$.
2.For $\tau = 0.5$ as n increase the MSEs $\hat{\theta}$ and $\hat{\beta}$ decrease for fixed θ and the fixed censoring scheme.
3.The confidence interval length also decreases when n increases for the fixed censoring scheme.

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