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A Highly Accurate Numerical Method for Solving Time-Fractional Partial Differential Equation

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Abstract: An efficient numerical method is employed to approximate the numerical solutions of some very functional, time-fractional partial differential equations. Perturbation Iteration Algorithm applied on fractional PDEs can technically manipulate non-linear and fractional terms pretty well. Similarly, the precision of its results are even better then that of different techniques. Explanatory figures have been presented correlating the approximated and exact solutions and substantiating the precision of results.

Keywords: Perturbation iteration algorithm, advection diffusion equation, Fisher equation, hyperbolic partial differential equation.

1 Introduction

Fractional calculus is yet another subject of this era, attracting a large number of mathematical analysts who have been dealing with this subject [1–3]. Different methods have been presented dealing with partial and ordinary differential equations of fractional order. Some of them are proposed numerical solutions, e.g. Variational Iteration Method (VIM) [4, 5], Homotopy Analysis method (HAM) [6, 7], and Adomian Decomposition Method (ADM) [8, 9], whereas, some mathematical analysts even proposed analytical methods for time-fractional partial differential equation, such as Iteration Method [10], Fourier transform method [11], Sumudu Transform Method [12], Greens Function Method [13] and Laplace Transform Method [10–14].
Nonlinear fractional partial differential equations (FPDEs) are offshoots of established ordinary differential equations. If first order derivative be replaced by a fractional order (single or multiple fractions) derivative, in basic PDE, a fractional order PDE is acquired subsequently. Both, linear and non-linear FPDEs vitally contribute in the fields of social sciences, engineering and many physical phenomena such as [15, 16].
In this paper, the time-fractional advection, hyperbolic and Fisher partial differential equations have been numerically solved, that have been worked out by a good number of scholars.The time-fractional diffusion equation as considered by Wyss [17] through their solution in closed form in terms of H- function. In the work of Schneider and Wyss [18], consideration of wave and fractional diffusion is found. Srivastava et al, numerically solved time fractional hyperbolic telegraph equation by RDTM [19]. Fisher equation was initially designed by [20] for the breeding of a virile gene and can be confronted by many chemical reactions such as the Brownian motion [21], chemical kinetics [22], auto catalytic chemical reaction [23] etc. Neamaty [24] has analyzed time fractional PDEs by applying VIM, and also given comparison of their results with different results of other numerical techniques. The analytical area has been segmented as:

1.Employed definitions of fractional calculus.
2.Basic Theory of Perturbation Iteration Algorithm on FPDEs.
3.Numerical Examples
4.Conclusion

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2 Employed Definitions of Fractional Calculus

Most normally utilized definitions of fractional derivatives and integrals are Riemann-Liouville and Caputo sense, therefore here is a brief introduction to these concepts.

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $\in C_{\mu}$, $\mu > -1$, is defined as

$$J^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds; \quad \alpha > 0, \tag{1}$$

$$J^{\circ}y(t) = y(t)$$

Some properties of the operator J^{α}, used in this text are:

$$J^{\alpha}J^{\beta}y(t) = J^{\alpha+\beta}y(t); \quad \alpha, \beta \geq 0,$$

$$J^{\alpha}t^{m} = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}t^{\alpha+m}; \quad m \geq -1.$$

Definition 2.2 A real valued function $y(x)$, $x > 0$ is said to be in space C_{μ}, $\mu \in R$, if there exists a real number $p > \mu$, such that $y(t) = t^{p}y_1(t)$, where $y_1(t) \in C(0, \inf)$, and it is said to be in the space C_{μ}^{n} if and only if $y^{n} \in C_{\mu}$, $n \in N$.

Definition 2.3 The fractional derivative of $y(t)$ in the Caputo sense is defined as

$$D^{\alpha}y(t) = J^{m-\alpha}D^{m}y(t); \quad for\ m-1 < \alpha \leq m,\ m \in N,\ t > 0,\ and\ y \in C_{-1}^{m}. \tag{2}$$

Firstly, Caputo fractional derivative evaluates only an ordinary derivative then through fractional integral obtains the required fractional derivative. This approach Riemann- Liouville fractional integral operator resembles very much the integer order integration so is a linear operation.

$$J^{\alpha}\left(\Sigma_{i=1}^{n}c_i y_i(t)\right) = \Sigma_{i=1}^{n}c_i J^{\alpha}y_i(t); \quad where\ \{c_i\}_{i=1}^{n}\ are\ constants. \tag{3}$$

3 Perturbation Iteration Algorithm (PIA)

Step I

Consider an initial value problem such as

$$D_t^{\alpha}y + M(y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y) + H(y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y) = g(x,t); \quad 0 < \alpha \leq 1,\ t > 0,\ x \in R \tag{4}$$

with initial condition $\frac{\partial^k}{\partial t^k}y(x,0) = y_k(x); \quad k = 0,1,2,...,m-1$. Where $y = y(x,t)$, H is the linear operator, M is the nonlinear operator and $g(x,t)$ is the known analytic function.

Step II

Introducing ε with nonlinear term yield

$$D^{\alpha}y + \varepsilon M + H - g(x,t) = 0 \tag{5}$$

Here $PIA(1,1)$ will be considered, which means only one correction term in this expansion will be obtained by taking $n = 1$, $m = 1$.

Step III

Consider the following fractional order differential equation.

$$F(D_t^{\alpha}y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, \varepsilon) = 0. \tag{6}$$

By applying PIA on Eq.7 as [25, 26]. Only n correction terms in this perturbation expansion will be acknowledged

$$y_{n+1} = y_n + \varepsilon(y_n)_c, \tag{7}$$

where ε is the perturbation parameter. The developed Perturbation Iteration Algorithm is given here as $PIA(n,m)$; here n are the terms involved in the expansion , m is the m^{th} order derivative in the Taylor's Series expansion provided $n \leq m$, w.r.t ε then comparing the coefficient of same power of ε renders the unknown correction terms. Back substitution of these results in Eq. (7)thus yields an algorithm for the solution of Eq. (4). Substituting Eq. (8) in Eq. (7), expanding in a Taylor's Series with first derivative only yields

$$\begin{aligned}&F(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0) + F_{D^\alpha y}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((D^\alpha y)_n\Big)_c +\\ &F_{y_{xx}}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y_{xx})_n\Big)_c + F_{y_{tt}}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y_{tt})_n\Big)_c +\\ &F_{y_{xt}}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y_{xt})_n\Big)_c + F_{y_t}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y_t)_n\Big)_c +\\ &F_{y_x}(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y_x)_n\Big)_c + F_y(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon\Big((y)_n\Big)_c +\\ &F_\varepsilon(D_t^\alpha y, y_{xx}, y_{tt}, y_{xt}, y_x, y_t, y, 0)\varepsilon = 0.\end{aligned} \tag{8}$$

All above derivatives will be taken at $\varepsilon = 0$. First $(y_\circ)_c$ has been calculated by using initial condition $y_\circ(x,t)$ and Eq.(8). Then we substitute $(y_\circ)_c$ into Eq. (7) to find y_1. Iteration process is repeated using Eq.(7) and Eq.(8) until we obtain a satisfactory result.

4 Numerical Examples

4.1 Example:

Table 1: Result Comparison of PIA and other numerical methods provided in [24] for Eq. (9)

x	t	VIM	ADM	HPM	VHPIM	PIA	Exact
0.2	0.25	0.0503090	0.0500000	0.0499876	0.0499876	0.0500001	0.0500000
	0.50	0.1006190	0.1000000	0.0999780	0.0999746	0.1000002	0.1000000
	0.75	0.1509280	0.1500010	0.1499680	0.1499620	0.1500004	0.1500000
	1.00	0.2012370	0.2000010	0.1999570	0.1999510	0.2000005	0.2000000
0.4	0.25	0.1018940	0.1000230	0.0995290	0.0996450	0.1000158	0.1000000
	0.50	0.2037870	0.2000460	0.1990590	0.1992900	0.2000316	0.2000000
	0.75	0.3056810	0.3000690	0.2985880	0.2989350	0.3000475	0.3000000
	1.00	0.4075750	0.4000920	0.3981180	0.3985800	0.4000633	0.4000000
0.6	0.25	0.1530940	0.1504110	0.1471580	0.1456900	0.1502739	0.1500000
	0.50	0.3061880	0.3008230	0.2943170	0.2913800	0.3005478	0.3000000
	0.75	0.4592820	0.4512340	0.4414750	0.4370700	0.4508218	0.4500000
	1.00	0.6123760	0.6016460	0.5886340	0.5827590	0.6010957	0.6000000

Consider the time-fractional advection partial differential equation

$$D_t^\alpha y(x,t) + y(x,t)y_x(x,t) = x(1+t^2); \quad t > 0,\ x \in R,\ 0 < \alpha \leq 1 \tag{9}$$

with initial condition; $y(x,0) = 0$. Applying perturbation parameter ε on non-linear and fractional terms, time-fractional advection equation becomes

$$D_t^\alpha y(x,t) + \varepsilon y(x,t)y_x(x,t) = \varepsilon x(\frac{1}{\varepsilon} + t^2).$$

By applying PIA, the obtained corrected term is

$$y_c(x,t) = J^\alpha\Big(-D_t^\alpha y(x,t) - y(x,t)y_x(x,t) + x + xt^2\Big)$$

following iterations have been obtained

$$y_\circ(x,t) = 0,$$

$$y_1(x,t) = xt^{\alpha}\Big(\frac{1}{\Gamma(1+\alpha)} + \frac{2t^2}{\Gamma(3+\alpha)}\Big),$$

$$y_2(x,t) = xt^{\alpha}\Big(\frac{1}{\Gamma(1+\alpha)} + \frac{2t^2}{\Gamma(3+\alpha)} - \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{4t^{2+2\alpha}\Gamma(3+2\alpha)}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)}\Big).$$

4.2 Example:

Table 2: Result Comparison of PIA and other numerical methods provided in [24] for Eq. (10)

t	x	VIM	ADM	HPM	VHPIM	PIA	Exact
0.2	0.25	0.0434000	0.0433951	0.0434000	0.0432049	0.0434000	0.0434030
	0.50	0.1736000	0.1735800	0.1736000	0.1728200	0.1735999	0.1736110
	0.75	0.3906000	0.3905560	0.3906000	0.3888440	0.3905998	0.3906250
	1.00	0.6944000	0.6943210	0.6944000	0.6912780	0.6943997	0.6944440
0.4	0.25	0.0317790	0.0315670	0.0317790	0.0299125	0.0317795	0.0318880
	0.50	0.1271180	0.1262680	0.1271180	0.1196500	0.1271179	0.1275510
	0.75	0.2860150	0.2841030	0.2860150	0.2692120	0.2860152	0.2869900
	1.00	0.5084710	0.5050720	0.5084710	0.4786000	0.5084715	0.5084710
0.6	0.25	0.0236650	0.0220050	0.0236650	0.0188604	0.0236649	0.0244140
	0.50	0.0946600	0.0880180	0.0946600	0.0754415	0.0946595	0.0976560
	0.75	0.2129840	0.1980400	0.2129840	0.1697430	0.2129839	0.2197270
	1.00	0.3786380	0.3520710	0.3786380	0.3017660	0.3786380	0.3906250

Consider the time fractional hyperbolic equations.

$$D_t^{\alpha} y(x,t) = \Big(y(x,t)y_x(x,t)\Big)_x; \quad t>0,\ x\in R,\ 1<\alpha\le 2 \tag{10}$$

with initial condition; $y(x,0) = x^2$, $y_t(x,0) = -2x^2$. Applying perturbation parameter ε on non-linear and fractional terms, time-fractional hyperbolic equation becomes

$$D_t^{\alpha} y(x,t) - \varepsilon\Big(y(x,t)y_x(x,t)\Big)_x = 0.$$

By applying *PIA*, the obtained corrected term is

$$y_c(x,t) = J^{\alpha}\Big(-D_t^{\alpha} y(x,t) + \Big(y(x,t)y_x(x,t)\Big)_x\Big) - 2x^2t + x^2.$$

Iterations obtained by adding initial condition in corrected term are as follows

$$y_\circ(x,t) = x^2(1-2t),$$

$$y_1(x,t) = x^2\Big(1-2t+\frac{6t}{\Gamma(1+\alpha)} - \frac{24t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{48t^{2+\alpha}}{\Gamma(3+\alpha)}\Big),$$

$$y_2(x,t) = x^2\Big(1-2t+\frac{6t}{\Gamma(1+\alpha)} - \frac{24t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{48t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{72t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{288t^{1+2\alpha}}{\Gamma(2+2\alpha)}\Big).$$

4.3 Example:

Consider the time-fractional Fisher's equation

$$D_t^{\alpha} y(x,t) = y_{xx}(x,t) + 6y(x,t)\Big(1-y(x,t)\Big); \quad t>0,\ x\in R,\ 0<\alpha\le 1, \tag{11}$$

Table 3: Result Comparison of PIA and other numerical methods provided in [24] for Eq. (11)

x	t	VIM	ADM	HPM	VHPIM	PIA	Exact
0.1	0.25	0.3159400	0.3179480	0.3159400	0.3280190	0.3159398	0.3160420
	0.50	0.2499260	0.2505000	0.2499260	0.2565130	0.2499257	0.2500000
	0.75	0.1916060	0.1909640	0.1916060	0.1943030	0.1916059	0.1916890
	1.00	0.1424110	0.1409790	0.1424110	0.1427150	0.1424105	0.1425370
0.2	0.25	0.4593200	0.4811990	0.4593200	0.5121930	0.4593203	0.4612840
	0.50	0.3864500	0.3969410	0.3864500	0.4146970	0.3864505	0.3874560
	0.75	0.3154780	0.3152660	0.3154780	0.3247160	0.3154775	0.3160420
	1.00	0.2490920	0.2411750	0.2490920	0.2458810	0.2490923	0.2500000
0.3	0.25	0.5911790	0.6814400	0.5911790	0.6302750	0.5911793	0.6041950
	0.50	0.5276350	0.5818610	0.5276350	0.5076430	0.5276353	0.5344470
	0.75	0.4597190	0.4758330	0.4597190	0.4882980	0.4597193	0.4612840
	1.00	0.3870250	0.3729170	0.3870250	0.3784720	0.3870253	0.3874560

with initial condition; $y(x,0) = \frac{1}{(1+e^x)^2}$. Applying perturbation parameter ε on non-linear and fractional terms, then time-fractional Fisher's equation becomes

$$D_t^{\alpha} y(x,t) - \varepsilon y_{xx}(x,t) - 6\varepsilon y(x,t)\Big(1 - y(x,t)\Big) = 0.$$

By applying *PIA*, the obtained corrected term is

$$y_c(x,t) = J^{\alpha}\Big(- D_t^{\alpha} y(x,t) + y_{xx}(x,t) + 6y(x,t) - 6y(x,t)^2\Big).$$

Iterations obtained by adding initial condition in corrected term are as follows

$$y_{\circ}(x,t) = \frac{1}{(1+e^x)^2},$$

$$y_1(x,t) = \frac{1}{(1+e^x)^2} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big(\frac{6}{(1+e^x)^4} + \frac{6e^{2x}}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3} + \frac{6}{(1+e^x)^2}\Big),$$

$$y_2(x,t) = \frac{1}{(1+e^x)^2} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big(\frac{6}{(1+e^x)^4} + \frac{6e^{2x}}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3} + \frac{6}{(1+e^x)^2}\Big) - \frac{t^{2\alpha}}{(1+e^x)^6\Gamma(1+2\alpha)}\Big(50e^x + 150e^{3x} + 100e^{4x}\Big) - \frac{600t^{\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}.$$

5 Conclusions

In this work a powerful and easily manageable numerical method PIA has been applied on three different time space fractional partial differential equations. This method uses Riemann-Liouville and Caputo definitions for fractional integration and differentiation. Results obtained by PIA in this work has been compared by the approximated results given in [24] by different methods such as VIM, HPM , ADM and VHPIM. Also it can easily be observed that results of this numerical method is more accurate and convergent than other numerical techniques especially comparison by VIM shows the efficiency of PIA. It is recommended that this satisfactory method ought to be utilized vivaciously for other complex dynamical systems.

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