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The New Modified Generalized Linear Failure Rate Distribution

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Abstract: A New Modified Generalized Linear Failure Rate Distribution (NMGLFRD) with five parameters which generalizes the exponential-Weibull distribution, generalized Rayleigh distribution, modified Weibull distribution, Weibull distribution, generalized exponential distribution, exponential distribution, modified generalized Linear failure rate distribution, generalized linear failure rate distribution and linear failure rate distribution is proposed. Various properties of this new distribution are considered and expressions for its moments and moments of the order statistics are obtained. We derive the cumulative distribution function, reliability function, hazard function and stress-strength reliability function. The estimation of the model parameters is performed by the maximum likelihood method. The use of the proposed model is illustrated by application to real data.

Keywords: Modified generalized linear failure rate distribution, reliability function, hazard function, stress-strength reliability function.

1 Introduction

In many of the applied sciences such as finance, engineering and medicine, amongst others, analyzing and modeling lifetime data are crucial. Several lifetime distributions have been proposed in the literature (such as exponential, Rayleigh, Weibull, Modified generalized linear failure rate distribution) and used to model such kinds of data. Still there remain many important problems where the real data does not follow any of the standard or classical probability models. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or a distribution. This is the reason why considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. Adamidis and Loukas [10] introduced the two parameter exponential-geometric distribution with decreasing failure rate. Kus [6] introduced the exponential-Possion distribution with the decreasing failure rate and discussed its various properties. Jain et al. [12] introduced the generalized inverse generalized Weibull distribution and studied its properties. Adamidis et al. [11] proposed the extended exponential- geometric distribution which generalizes the exponential geometric distribution and discussed several of its statistical properties along with its reliability features. The hazard function of the extended exponential- geometric can be monotonic decreasing, increasing or constant.

Reliability has always been a key role for the functionality of the system and safety of people using the products. Lots of research and applications have been carried out in order to understand and explore the applications and methodologies of reliability analysis for the product enhancements and many researchers have investigated statistically and stochastically complex phenomena of real systems to improve their reliability. Survival function (reliability function) and hazard function (failure rate function) are the most frequently used functions in reliability engineering and life time data. The hazard function of the exponential function is constant whereas the hazard functions of linear failure rate, Rayleigh and generalized exponential distribution (Gupta and Kunda, [13]) are monotonic. One of the most frequently used lifetime distribution is Weibull distribution introduced by Fisher and Tippett [14] which is very flexible in modeling lifetime distribution with monotone failure rate. For describing the lifetime of components with variable failure rate Swedish physicist Wallodi Weibull [16] used Weibull distribution to represent the distribution of the breaking strength of materials. Surles and Padgett [8] introduced generalized Rayleigh (two parameter Burr type X) and showed that it could be used in modeling strength data and lifetime data. Khan and Jan [4] discussed the stress-strength problem of the system where the strength follows finite mixture of two parameter Lindley distribution and stress follows exponential, Lindley distribution and mixture of two parameter Lindley distribution and obtained general expressions for the reliabilities of a system. Khan and Jan [3] obtained Bayes estimators of the parameters of the Geeta, Consul and Size-biased Geeta distributions and associated reliability function. Sarhan and Zaindin [2] introduced Modified Weibull Distribution with three parameters. This distribution generalizes generalized exponential distribution, exponential distribution, generalized Rayleigh distribution

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and linear failure rate distribution. Sarhan and Kundu [1] introduced generalized linear failure rate distribution with can have decreasing, increasing and bath tub shaped hazard functions. Ezzatallah [5] introduced Modified generalized linear failure rate distribution. This distribution generalizes linear failure distribution, generalized exponential distribution, exponential distribution, generalized Rayleigh distribution, exponential Weibull distribution, Weibull distribution, generalized linear failure rate distribution and modified Weibull distribution.

2 The New Modified Generalized Linear Failure Rate Distribution

Let X be lifetime random variable whose probability density function with five parameters $(\alpha, \beta, \gamma, \delta, \theta)$ is

$$f(x, \alpha, \beta, \gamma, \delta, \theta) = \theta\delta(\alpha + \beta\gamma x^{\gamma-1})(\alpha x + \beta x^{\gamma})^{\delta-1}\left[1 - e^{-(\alpha x+\beta x^{\gamma})^{\delta}}\right]^{\theta-1} e^{-(\alpha x+\beta x^{\gamma})^{\delta}} \quad (2.1)$$

$$; \; x > 0, \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta > 0, \theta > 0$$

By the introduction of fifth parameter 'δ' the above pdf is the generalization of Modified Generalized Linear Failure Rate Distribution given by Ezzatallah [5] and will be called new modified generalized linear failure rate distribution (NMGLFRD).

Figure 1, 2, 3 and 4 shows the possible shapes of the NMGLFRD for selected values of the parameters involved in the pdf. In fig. 1 for blue colour shape$(\alpha = 1.3, \beta = 1.5, \gamma = 1.5, \delta = 0.9, \theta = 1.6)$, for red colour shape$(\alpha = 1.3, \beta = 0.5, \gamma = 1.5, \delta = 0.8, \theta = 1.6)$ and for green colour shape$(\alpha = 1.3, \beta = 0.5, \gamma = 2.5, \delta = 0.4, \theta = 1.6)$. In fig. 2 for blue colour shape $(\alpha = 1.3, \beta = 0.7, \gamma = 1.5, \delta = 0.6, \theta = 1.6)$, for red colour shape $(\alpha = 1.3, \beta = 0.5, \gamma = 1.5, \delta = 0.5, \theta = 1.6)$ and for green colour shape $(\alpha = 1.3, \beta = 0.5, \gamma = 0.5, \delta = 0.4, \theta = 1.6)$.

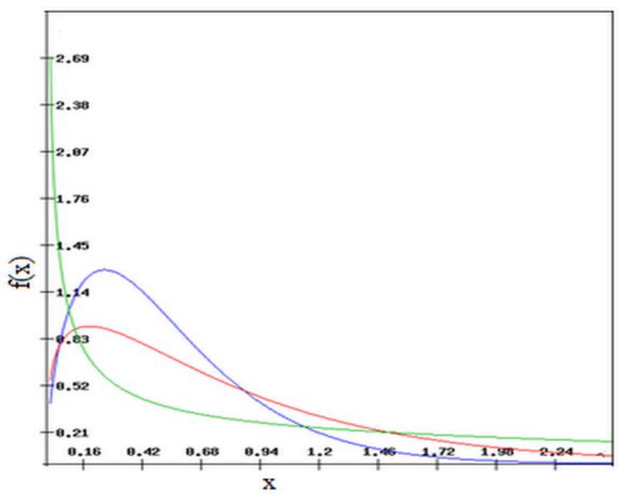

Fig. 1: Possible shapes of NMGLFRD

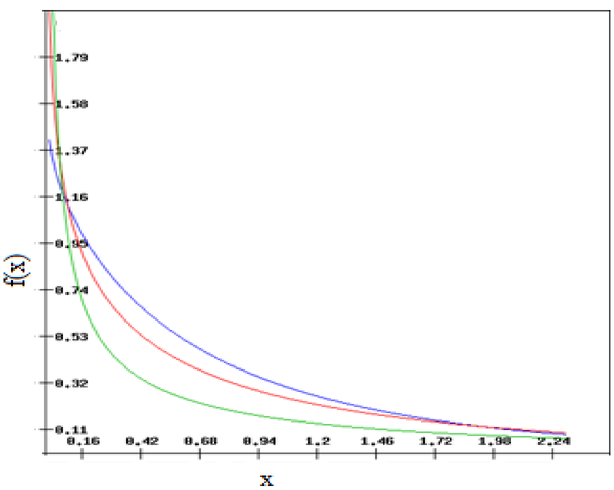

Fig. 2: Possible shapes of NMGLFRD

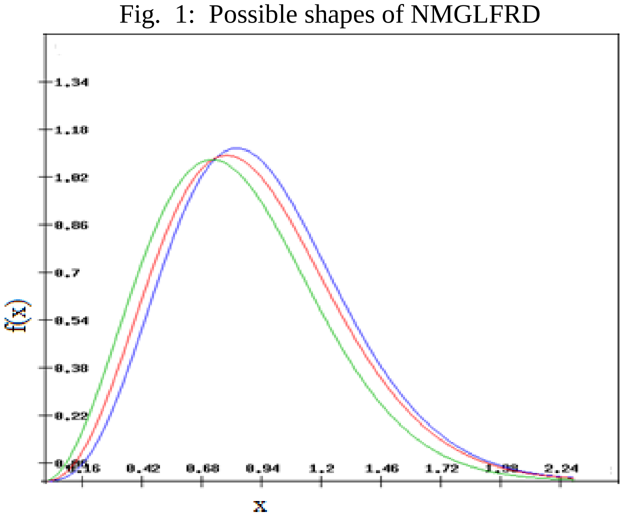

Fig. 3: Possible shapes of NMGLFRD

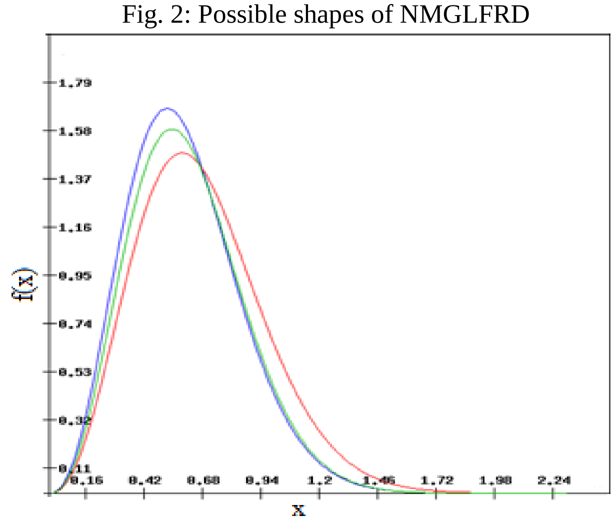

Fig. 4: Possible shapes of NMGLFRD

In fig.3 for blue colour shape$(\alpha = 0.7, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.4)$, for red colour shape$(\alpha = 0.7, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.6)$ and for green colour shape$(\alpha = 0.7, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.8)$ and in fig.4 for blue colour shape$(\alpha = 1.1, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.6)$, for red colour shape$(\alpha = 1.2, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.6)$ and for green colour shape$(\alpha = 1.3, \beta = 0.5, \gamma = 0.9, \delta = 2.1, \theta = 1.6)$.

For different values of parameters involved in NMGLFRD, the distribution has the following distributions as special cases.

1. Exponential Weibull distribution (EWD), $\alpha = 0, \delta = 1$

$$f(x, \beta, \gamma, \theta) = \theta\beta\gamma x^{\gamma-1}\left[1 - e^{-\beta x^{\gamma}}\right]^{\theta-1} e^{-\beta x^{\gamma}} \quad ; x > 0, \beta, \gamma, \theta > 0$$

2. Generalized Rayleigh distribution (GRD), $\alpha = 0,\ \gamma = 2,\ \delta = 1$

$$f(x, \beta, \theta) = 2\theta\beta x\left[1 - e^{-\beta x^2}\right]^{\theta-1} e^{-\beta x^2} \quad ; x > 0, \beta, \theta > 0$$

3. Modified Weibull distribution (MWD), $\theta = 1,\ \delta = 1$

$$f(x, \alpha, \beta, \gamma) = (\alpha + \beta\gamma x^{\gamma-1})e^{-(\alpha x + \beta x^{\gamma})} \ ;\ x > 0, \alpha \geq 0, \beta, \gamma > 0$$

4. Weibull distribution (WD), $\alpha = 0, \delta = \theta = 1$

$$f(x, \beta, \gamma) = \beta\gamma x^{\gamma-1} e^{-\beta x^{\gamma}} \ ;\ x > 0, \beta, \gamma > 0$$

5. Generalized Exponential distribution (GED), $\beta = 0,\ \delta = 1$

$$f(x, \alpha, \theta) = \alpha\theta[1 - e^{-\alpha x}]^{\theta-1} e^{-\alpha x} \ ; x > 0, \alpha, \theta > 0$$

6. Exponential distribution (ED), $\beta = 0,\ \delta = \theta = 1$

$$f(x, \alpha) = \alpha e^{-\alpha x} \ ;\ x > 0, \alpha > 0$$

7. Modified Generalized Linear Failure Rate Distribution (MGLFRD), $\delta = 1$

$$f(x, \alpha, \beta, \gamma, \theta) = \theta(\alpha + \beta\gamma x^{\gamma-1})\left[1 - e^{-(\alpha x + \beta x^{\gamma})}\right]^{\theta-1} e^{-(\alpha x + \beta x^{\gamma})}$$

$$; x > 0, \alpha \geq 0, \beta \geq 0, \gamma \geq 0, , \theta > 0$$

8. Generalized Linear Failure Rate Distribution(GLFRD) , $\gamma = 2, \delta = 1$

$$f(x, \alpha, \beta, \theta) = \theta(\alpha + 2\beta x)\left[1 - e^{-(\alpha x + \beta x^2)}\right]^{\theta-1} e^{-(\alpha x + \beta x^2)}$$

$$; x > 0, \alpha \geq 0, \beta \geq 0, \theta > 0$$

9. Linear Failure Rate Distribution (LFRD), $\gamma = 2, \theta = \delta = 1$

$$f(x, \alpha, \beta) = (\alpha + 2\beta x)e^{-(\alpha x + \beta x^2)} ; x > 0, \alpha \geq 0, \beta \geq 0$$

3 Statistical Properties of the NGMLFRD

In this section we study the statistical properties of the NGMLFRD, specifically distribution function, moments, moment generating function, quartile function, skewness and kurtosis.
Let X follows NMGLFRD with parameters $\alpha, \beta, \gamma, \delta, \theta$. In the sequel, the distribution of X will be referred to $F(x, \alpha, \beta, \gamma, \delta, \theta)$ and given as

$$F(x, \alpha, \beta, \gamma, \delta, \theta) = \left[1 - e^{-(\alpha x + \beta x^{\gamma})^{\delta}}\right]^{\theta} ; x > 0, \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta > 0, \theta > 0 \qquad (3.1)$$

The possible shapes of the cdf for selected values of parameters involved in the distribution function are shown in Fig. 5 and 6. In fig.5 for blue colour shape$(\alpha = 1.4, \beta = 1.9, \gamma = 1.6, \delta = 1.5, \theta = 1.7)$, for red colour shape$(\alpha = 1.3, \beta = 2.4, \gamma = 1.4, \delta = 1.3, \theta = 1.5)$ and for green colour shape$(\alpha = 1.5, \beta = 2.3, \gamma = 1.8, \delta = 1.4, \theta = 1.6)$ and in fig.6 for blue colour shape$(\alpha = 1.4, \beta = 1.9, \gamma = 1.6, \delta = 0.5, \theta = 1.7)$, for red colour shape$(\alpha = 1.3, \beta = 2.4, \gamma = 1.4, \delta = 0.3, \theta = 1.5)$ and for green colour shape$(\alpha = 1.5, \beta = 2.3, \gamma = 1.8, \delta = 0.4, \theta = 1.6)$.

In statistical analysis, moments are important and necessary. These can be used to study the most important features and characteristics of a distribution (e.g., dispersion, kurtosis and skewness).

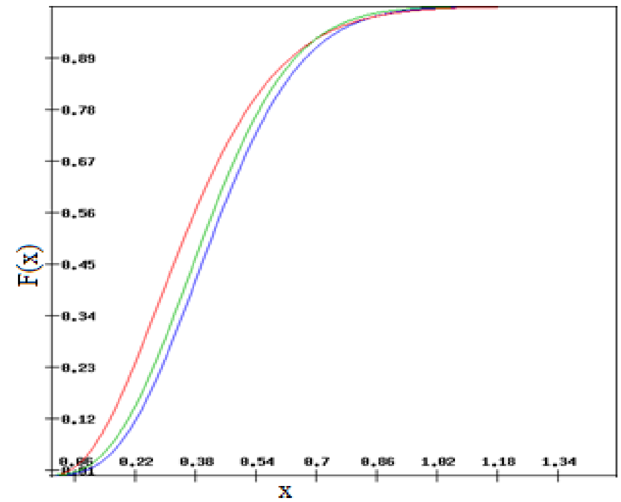

Fig. 5: Possible shapes of cdf

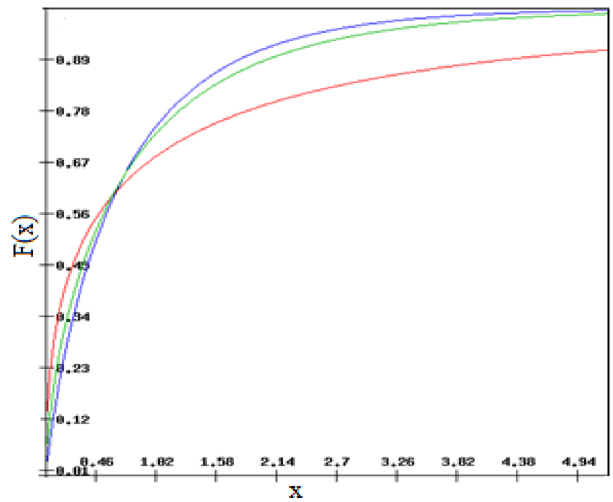

Fig. 6: Possible shapes of cdf

Theorem 3.1: If X has the NGMLFRD, then the k^{th} order moment about zero is given by

$$i)\qquad \mu^{(k)} = \frac{\theta\Gamma\left(\frac{k}{\gamma\delta}+1\right)}{\beta^{k/\gamma}}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{\theta-1}{i}}{(1+i)^{\frac{k}{\gamma}+1}}\ ;\alpha=0,\beta>0$$

$$ii)\quad \mu^{(k)} = \frac{\theta k}{\delta}\sum_{i=0}^{\infty}\frac{(-1)^i}{(1+i)}\binom{\theta-1}{i}\left[\prod_{j=1}^{\infty}\sum_{\lambda=0}^{\infty}\frac{W_{i,j}^{(\lambda)}(0)}{\lambda!}\frac{\Gamma\left(\frac{k+p}{\delta}\right)}{\left(\alpha(1+i)^{\frac{1}{\delta}}\right)^{k+p}}\right]$$

$$;\ \alpha>0,\beta\geq 0\ and\ p=\lambda j(\gamma-1)+\lambda\delta$$

Proof: We know from the definition of the k^{th} moment of the random variable *X* with probability density function $f(x)$ is given by

$$\mu^{(k)} = \int_0^{\infty} x^k f(x)\mathrm{d}x \qquad (3.2)$$

Substituting (2.1) into (3.2), we get

$$\mu^{(k)} = \int_0^{\infty} x^k\theta\delta(\alpha+\beta\gamma x^{\gamma-1})(\alpha x+\beta x^{\gamma})^{\delta-1}\left[1-e^{-(\alpha x+\beta x^{\gamma})^{\delta}}\right]^{\theta-1} e^{-(\alpha x+\beta x^{\gamma})^{\delta}}\,dx$$

Using, $\left[1-e^{-(\alpha x+\beta x^{\gamma})^{\delta}}\right]^{\theta-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\theta-1}{i}e^{-i(\alpha x+\beta x^{\gamma})^{\delta}}$, we obtains

$$\mu^{(k)} = \theta\sum_{i=0}^{\infty}(-1)^i\binom{\theta-1}{i}\left(\frac{k}{1+i}\right)\int_0^{\infty} x^{k-1}e^{-(1+i)(\alpha x+\beta x^{\gamma})^{\delta}}\,dx$$

i) For $\alpha=0,\beta>0$

$$\mu^{(k)} = \frac{\theta\Gamma\left(\frac{k}{\gamma\delta}+1\right)}{\beta^{k/\gamma}}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{\theta-1}{i}}{(1+i)^{\frac{k}{\gamma}+1}}$$

ii) For $\alpha>0,\beta\geq 0$

$$\mu^{(k)} = \theta\sum_{i=0}^{\infty}(-1)^i\binom{\theta-1}{i}\left(\frac{k}{1+i}\right)\int_0^{\infty} x^{k-1}e^{-(1+i)\alpha^{\delta}x^{\delta}\left(1+\frac{\beta}{\alpha}x^{\gamma-1}\right)^{\delta}}\,dx$$

$$\mu^{(k)} = \theta \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} \left(\frac{k}{1+i}\right) \left[\prod_{j=1}^{\infty} \int_0^{\infty} x^{k-1} e^{-(1+i)\alpha^\delta x^\delta} e^{-(1+i)\alpha^\delta \binom{\delta}{j} \left(\frac{\beta}{\alpha}\right)^j x^{j(\gamma-1)+\delta}} dx \right]$$

and by the definition of Taylor expansion

$$\mu^{(k)} = \theta \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} \left(\frac{k}{1+i}\right) \left[\prod_{j=1}^{\infty} \int_0^{\infty} x^{k-1} e^{-(1+i)(\alpha x)^\delta} \sum_{\lambda=0}^{\infty} \frac{W_{i,j}^{(\lambda)}(0)}{\lambda!} x^{\lambda j(\gamma-1)+\lambda\delta} dx \right]$$

$$\mu^{(k)} = \theta \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} \left(\frac{k}{1+i}\right) \left[\prod_{j=1}^{\infty} \sum_{\lambda=0}^{\infty} \frac{W_{i,j}^{(\lambda)}(0)}{\lambda!} \int_0^{\infty} x^{k-1} e^{-(1+i)(\alpha x)^\delta} x^{\lambda j(\gamma-1)+\lambda\delta} dx \right]$$

$$\mu^{(k)} = \theta \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} \left(\frac{k}{1+i}\right) \left[\prod_{j=1}^{\infty} \sum_{\lambda=0}^{\infty} \frac{W_{i,j}^{(\lambda)}(0)}{\lambda!} \int_0^{\infty} x^{k+p-1} e^{-(1+i)(\alpha x)^\delta} dx \right]$$

where, $p = \lambda j(\gamma-1) + \lambda\delta \mu^{(k)} = \frac{\theta k}{\delta} \sum_{i=0}^{\infty} \frac{(-1)^i}{(1+i)} \binom{\theta-1}{i} \left[\prod_{j=1}^{\infty} \sum_{\lambda=0}^{\infty} \frac{W_{i,j}^{(\lambda)}(0)}{\lambda!} \frac{\Gamma\left(\frac{k+p}{\delta}\right)}{\left(\alpha(1+i)^{\frac{1}{\delta}}\right)^{k+p}} \right]$

That completes the proof.

Theorem 3.2: The moment generating function of NMGLFRD is given by

$$i)\quad M_X(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^r}{r!} \frac{\theta \Gamma\left(\frac{r}{\gamma\delta}+1\right)}{\beta^{r/\gamma}} \frac{(-1)^i \binom{\theta-1}{i}}{(1+i)^{\frac{r}{\gamma}+1}} \quad ; \alpha = 0, \beta > 0$$

$$ii)\quad M_X(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^r}{r!} \frac{\theta r}{\delta} \frac{(-1)^i}{(1+i)} \binom{\theta-1}{i} \left[\prod_{j=1}^{\infty} \sum_{\lambda=0}^{\infty} \frac{W_{i,j}^{(\lambda)}(0)}{\lambda!} \frac{\Gamma\left(\frac{r+p}{\delta}\right)}{\left(\alpha(1+i)^{\frac{1}{\delta}}\right)^{r+p}} \right]$$

$; \; a > 0, \beta \geq 0 \; and \; p = \lambda j(\gamma-1) + \lambda\delta$

Proof: We know from the definition of the $M_X(t)$ of the random variable *X* with probability density function $f(x)$ given by

$$M_X(t) = \int_0^{\infty} e^{tx} f(x) \mathrm{dx} = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f(x,\alpha,\beta,\gamma,\delta,\theta) \mathrm{dx}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x,\alpha,\beta,\gamma,\delta,\theta) \mathrm{dx} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu^{(k)}$$

and by using theorem 3.1 proof is completed.

3.1 Quartiles, Skewness and Kurtosis

The q^{th} quartile of the NMGLFRD (2.1) is given by

$$q = \int_0^{x_q} \theta\delta(\alpha + \beta\gamma x^{\gamma-1})(\alpha x + \beta x^\gamma)^{\delta-1} \left[1 - e^{-(\alpha x + \beta x^\gamma)^\delta}\right]^{\theta-1} e^{-(\alpha x + \beta x^\gamma)^\delta} dx$$

$$q = \left(1 - e^{-\left(\alpha x_q + \beta x_q^\gamma\right)^\delta}\right)^\theta$$

If $\gamma = 1$ then

$$x_q = \frac{\left[ln\left(1-q^{\frac{1}{\theta}}\right)^{-1}\right]^{\frac{1}{\delta}}}{\alpha+\beta} \quad (3.1.1)$$

If $\gamma = 2$ then

$$x_q = -\frac{\alpha + \sqrt{\alpha^2 + 4\beta\left[ln\left(1-q^{\frac{1}{\theta}}\right)^{-1}\right]^{\frac{1}{\delta}}}}{2\beta} \quad (3.1.2)$$

Substituting $q = \frac{1}{2}$ in (3.1.1) and (3.1.2), we get the distribution median for $\gamma = 1$ and $\gamma = 2$.

The Bowley's skewness [7] is based on quartiles

$$S_K = \frac{q_{0.75} - 2q_{0.5} + q_{0.25}}{q_{0.75} - q_{0.25}}$$

And the Moor's kurtosis [9] is based on octiles

$$K_u = \frac{q_{0.125} - q_{0.375} - q_{0.625} + q_{0.875}}{q_{0.75} - q_{0.25}}$$

3.2 Distribution of order statistics

Let $X_i, i = 1, 2, \dots, n$ be a random sample from the probability density function (2.1). Let $Y_1 \le Y_2 \le \dots \le Y_n$ be the order statistics obtained from the sample, then probability density function of Y_t is given by

$$\begin{aligned} g_t(y) &= \frac{n!}{(t-1)!\,(n-t)!} f(y,\alpha,\beta,\gamma,\delta,\theta)[F(y,\alpha,\beta,\gamma,\delta,\theta)]^{t-1}[1-F(y,\alpha,\beta,\gamma,\delta,\theta)]^{n-t} \\ &= \frac{n!}{(t-1)!\,(n-t)!} f(y,\alpha,\beta,\gamma,\delta,\theta)\sum_{l=0}^{n-t}\binom{n-t}{l}(-1)^l[F(y,\alpha,\beta,\gamma,\delta,\theta)]^{l+t-1} \\ &= \sum_{l=0}^{n-t}\frac{n!}{(t-1)!\,(n-t)!}\binom{n-t}{l}(-1)^l\frac{f\left(y,\alpha,\beta,\gamma,\delta,\theta_{(t+l)}\right)}{t+l} \\ &= \sum_{l=0}^{n-t} k_l(n,t) f\left(y,\alpha,\beta,\gamma,\delta,\theta_{(t+l)}\right) \end{aligned}$$

where, $k_l(n,t) = \frac{n(-1)^l\binom{n-1}{t-1}\binom{n-t}{l}}{t+l}$

Lemma 3.2.1: If $X_i, i = 1, 2, \dots, n$ is a random sample from NMGLFRD $(\alpha,\beta,\gamma,\delta,\theta)$. Then Y_n follows NMGLFRD $(\alpha,\beta,\gamma,\delta,n\theta)$.

Lemma 3.2.2: Let Y_i denote the i^{th} order statistics, then the k^{th} moment of Y_i $\left(\mu^{(k)}\right)$ is given as follows

i) $\alpha = 0, \beta > 0$

$$\mu^{(k)} = \frac{\theta\Gamma\left(\frac{k}{\gamma\delta}+1\right)}{\beta^{k/\gamma}}\sum_{l=o}^{n-t}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{\theta(l+t)-1}{i}}{(1+i)^{\frac{k}{\gamma}+1}}$$

ii) $\alpha > 0, \beta \ge 0$

$$\mu^{(k)} = \frac{k}{\delta}\sum_{l=0}^{n-t}\sum_{i=0}^{\infty}k_l(n,t)\theta_{(t+l)}\frac{(-1)^i}{(1+i)}\binom{\theta-1}{i}\left[\prod_{j=1}^{\infty}\sum_{\lambda=0}^{\infty}\frac{W_{i,j}^{(\lambda)}(0)}{\lambda!}\frac{\Gamma\left(\frac{k+p}{\delta}\right)}{\left(\alpha(1+i)^{\frac{1}{\delta}}\right)^{k+p}}\right]$$

$$; p = \lambda j(\gamma-1) + \lambda\delta$$

4 Reliability function

Let variable T be the lifetime or time to failure of a component having probability density function (2.1) and distribution function (3.1). The probability that the component survives beyond sometime t is called the reliability $R(t,\alpha,\beta,\gamma,\delta,\theta)$ of the component. Thus,

$$R(t,\alpha,\beta,\gamma,\delta,\theta) = \mathrm{P}(\mathrm{T} > t)$$
$$= 1 - F(t,\alpha,\beta,\gamma,\delta,\theta), \qquad t > 0$$
$$= 1 - \left[1 - e^{-(\alpha t+\beta t^{\gamma})^{\delta}}\right]^{\theta}; \ t > 0, \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta > 0, \theta > 0$$

From this expression it is clear for higher values of θ, reliability decreases and also for constant δ, γ and θ, reliability increases when increase in the values of α and β take place at a particular period of time.

Lemma 4.1: If in a parallel system the k components have NMGLFRD with reliability function $R(t,\alpha,\beta,\gamma,\delta,\theta)$, then the reliability of the whole system is $R(t,\alpha,\beta,\gamma,\delta,k\theta)$.

The hazard rate function or failure rate of NMGLFRD is given by

$$h(t,\alpha,\beta,\gamma,\delta,\theta) = \frac{f(t,\alpha,\beta,\gamma,\delta,\theta)}{R(t,\alpha,\beta,\gamma,\delta,\theta)}$$
$$= \frac{\theta\delta(\alpha+\beta\gamma t^{\gamma-1})(\alpha t+\beta t^{\gamma})^{\delta-1}\left[1-e^{-(\alpha t+\beta t^{\gamma})^{\delta}}\right]^{\theta-1} e^{-(\alpha t+\beta t^{\gamma})^{\delta}}}{1-\left[1-e^{-(\alpha t+\beta t^{\gamma})^{\delta}}\right]^{\theta}}$$

In fig. 7 the hazard function of NMGLFRD can be non-decreasing, non-increasing or bathtub shaped for particular values of the parameters involved in the hazard rate function, for example, for $\alpha = 1.5, \beta = 0.6, \gamma = 0.6, \delta = 0.5, \theta = 1.5$, the hazard rate function is non-increasing (red curve), for $\alpha = 0.5, \beta = 0.6, \gamma = 1.2, \delta = 1.1, \theta = 1.2$, the hazard rate function is non-decreasing (green curve) and for $\alpha = 0.8, \beta = 0.1, \gamma = 3.2, \delta = 0.7, \theta = 1.3$, the hazard rate function is bathtub shaped (blue curve).

If $\theta = 1$ and $\delta = 1$

$$h(t,\alpha,\beta,\gamma) = \alpha + \beta\gamma t^{\gamma-1}$$

In fig. 8, the graph of hazard rate function is a straight line (blue) parallel to time axis for $\gamma = 1$ i.e. constant. For $\gamma = 2$, the graph of hazard rate function is a straight line (red) with constant slope 2β i.e. increasing. For $\gamma > 2$, the graph of hazard rate function is a increasing curve (green) with positive slope.

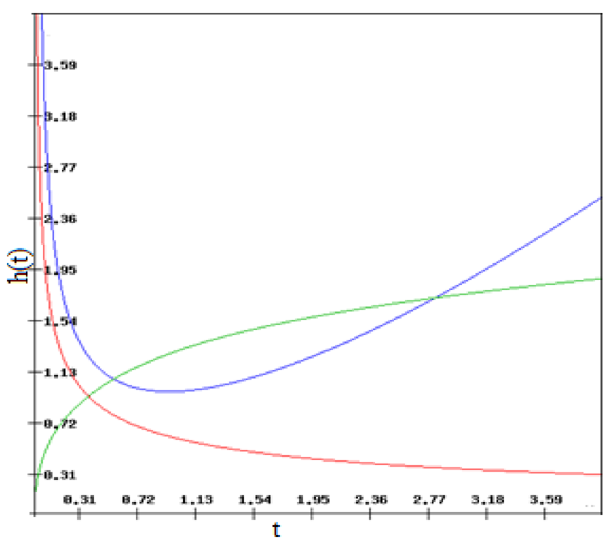

Fig. 7: Shapes of hazard function of NMGLFRD

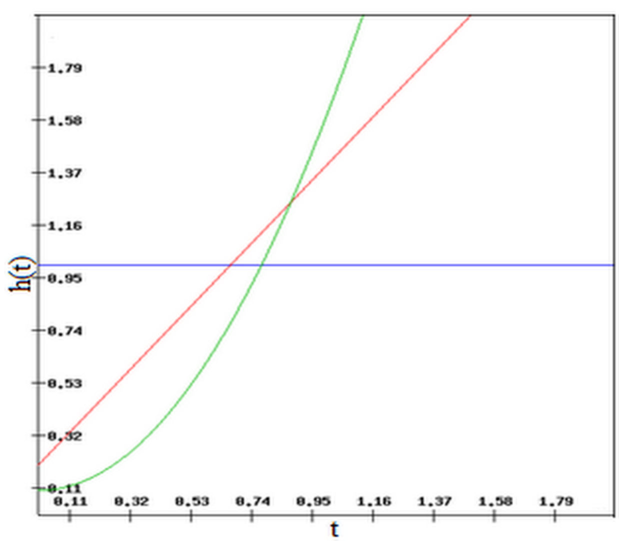

Fig. 8: Shapes of hazard function for $\theta = 1$ and $\delta = 1$

The reversed hazard rate function of NMGLFRD is given by

$$r(t,\alpha,\beta,\gamma,\delta,\theta) = \frac{f(t,\alpha,\beta,\gamma,\delta,\theta)}{F(t,\alpha,\beta,\gamma,\delta,\theta)}$$
$$= \frac{\theta\delta(\alpha+\beta\gamma t^{\gamma-1})(\alpha t+\beta t^{\gamma})^{\delta-1}e^{-(\alpha t+\beta t^{\gamma})^{\delta}}}{1-e^{-(\alpha t+\beta t^{\gamma})^{\delta}}}$$
$$= \theta\frac{f(t,\alpha,\beta,\gamma,\delta,1)}{F(t,\alpha,\beta,\gamma,\delta,1)} = \theta r(t,\alpha,\beta,\gamma,\delta,1)$$

4.1 Stress –Strength Reliability

The term “stress- strength reliability” refers to the quantity P(X > Y), where a system with random strength X is subjected to a random stress Y such that a system fails, if the stress exceeds the strength. Suppose X and Y are two independent random variables both having the pdf (2.1) with parameters $(\alpha,\beta,\gamma,\delta,\theta_1)$ and $(\alpha,\beta,\gamma,\delta,\theta_2)$ respectively and let Y represents the 'stress' which is applied to a certain appliance and X represents the 'strength' to sustain the stress, then the stress-strength reliability is denoted by

$$R = P(Y < X) = \int_0^\infty P(Y < X/Y = y) f_Y(y) dy$$
$$= \int_0^\infty \left(1 - F_X(y,\alpha,\beta,\gamma,\delta,\theta_1)\right) f_Y(y,\alpha,\beta,\gamma,\delta,\theta_2) dy$$
$$= 1 - \int_0^\infty \theta_2 \delta(\alpha + \beta\gamma y^{\gamma-1})(\alpha y + \beta y^\gamma)^{\delta-1} \left[1 - e^{-(\alpha y+\beta y^\gamma)^\delta}\right]^{\theta_1+\theta_2-1} e^{-(\alpha y+\beta y^\gamma)^\delta} dy$$
$$= \frac{\theta_1}{\theta_1 + \theta_2}$$

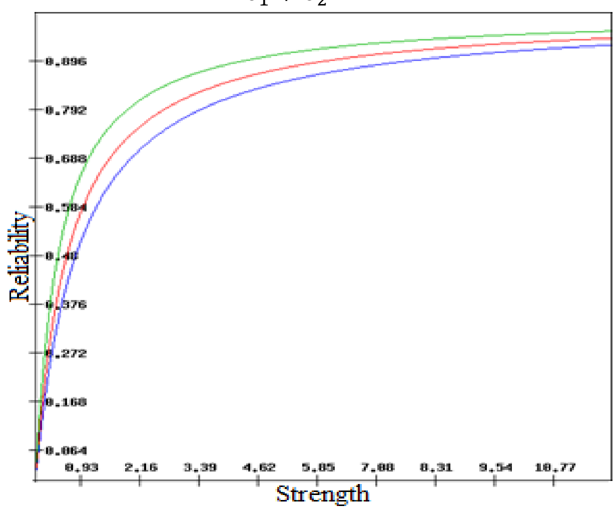

Fig. 9: Variation in R for constant Stress

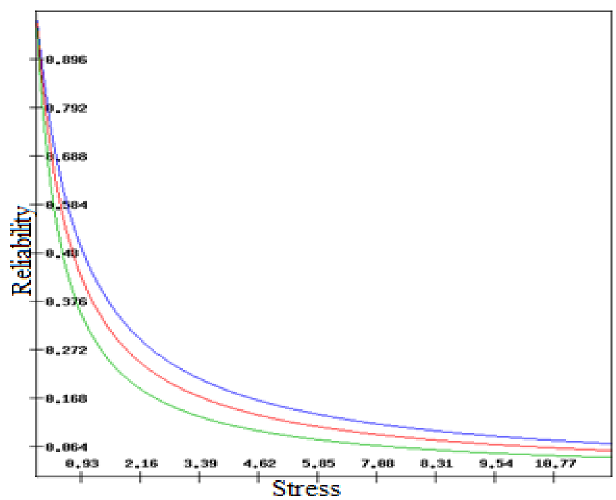

Fig. 10: Variation in R for constant Strength

Fig. 9 shows with increase in strength parameter (θ_2) and keeping stress constant the reliability of system increases, for example, for $\theta_1 = 0.5, 0.7$ and 0.9 the increase in reliability is shown by green, red and blue curves respectively. Also, fig. 10 shows with increase in stress parameter (θ_1) the reliability of the system decreases, for example, for $\theta_2 = 0.5, 0.7$ and 0.9 the decrease in reliability is shown by green, red and blue curves respectively.

5 Estimation

Let $x = x_1, x_2, \dots, x_n$ be a random sample of the NMGLFRD with unknown parameter vector $\varphi = (\alpha,\beta,\gamma,\delta,\theta)^T$. The log likelihood for $l = l(\varphi; x)$ for φ is

$$l = nlog\theta + nlog\delta + \sum_{i=1}^n log\left(\alpha + \beta\gamma x_i^{\gamma-1}\right) + (\delta-1)\sum_{i=1}^n log\left(\alpha x_i + \beta x_i^\gamma\right) + (\theta-1)\sum_{i=1}^n log\left(1 - e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right) - \sum_{i=1}^n \left(\alpha x_i + \beta x_i^\gamma\right)^\delta$$

The score function $U(\varphi) = \left(\frac{\partial l}{\partial\alpha}, \frac{\partial l}{\partial\beta}, \frac{\partial l}{\partial\gamma}, \frac{\partial l}{\partial\delta}, \frac{\partial l}{\partial\theta}\right)^T$ has components

$$\frac{\partial l}{\partial\alpha} = \sum_{i=1}^n \left(\alpha + \beta\gamma x_i^{\gamma-1}\right)^{-1} + (\delta-1)\sum_{i=1}^n \frac{x_i}{\left(\alpha x_i + \beta x_i^\gamma\right)} + \delta(\theta-1)\sum_{i=1}^n \frac{x_i e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\left(\alpha x_i + \beta x_i^\gamma\right)^{\delta-1}}{\left(1 - e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)} - \delta\sum_{i=1}^n x_i\left(\alpha x_i + \beta x_i^\gamma\right)^{\delta-1}$$

$$\frac{\partial l}{\partial \beta}=\sum_{i=1}^{n}\frac{\gamma x_i^{\gamma}}{\left(\alpha x_i+\beta\gamma x_i^{\gamma}\right)}+(\delta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}}{\left(\alpha x_i+\beta x_i^{\gamma}\right)}+\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\right)}-\delta\sum_{i=1}^{n}x_i^{\gamma}\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta-1}$$

$$\frac{\partial l}{\partial \gamma}=\beta\sum_{i=1}^{n}\frac{x_i^{\gamma}(1+\gamma log x_i)}{\left(\alpha x_i+\beta x_i^{\gamma}\right)}+\beta(\delta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}\, log x_i}{\left(\alpha x_i+\beta x_i^{\gamma}\right)}+\beta\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}\, log x_i e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\right)}-\beta\delta\sum_{i=1}^{n}x_i^{\gamma}\, log x_i\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta-1}$$

$$\frac{\partial l}{\partial \delta}=\frac{n}{\delta}+\sum_{i=1}^{n}log\left(\alpha x_i+\beta x_i^{\gamma}\right)+(\theta-1)\sum_{i=1}^{n}\frac{e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}log\left(\alpha x_i+\beta x_i^{\gamma}\right)}{\left(1-e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\right)}-\sum_{i=1}^{n}\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}log\left(\alpha x_i+\beta x_i^{\gamma}\right)$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}log\left(1-e^{-\left(\alpha x_i+\beta x_i^{\gamma}\right)^{\delta}}\right)$$

The maximum likelihood estimate (MLE) $\hat{\varphi}$ of φ can be obtained by solving non-linear equations $U(\hat{\varphi})=0$. These equations cannot be solved analytically but statistical software can be used to solve them numerically, for example, through the R-language or any iterative methods such as the BFGS (Broyden-Fletcher-Goldfarb-Shanno), NR (Newton-Raphson), NM (Nelder-Mead), BHHH (Berndt-Hall-Hall-Hausman), L-BFGS-B (Limited-Memory Quasi-Newton code for Bound-Constrained Optimization) and SANN (Simulated-Annealing).

The observed 5×5 information matrix for the interval estimation and hypothesis testing for parameters in φ is given by

$$K=K(\varphi)=-\begin{pmatrix} K_{\alpha,\alpha} & K_{\alpha,\beta} & K_{\alpha,\gamma} & K_{\alpha,\delta} & K_{\alpha,\theta} \\ & K_{\beta,\beta} & K_{\beta,\gamma} & K_{\beta,\delta} & K_{\beta,\theta} \\ & & K_{\gamma,\gamma} & K_{\gamma,\delta} & K_{\gamma,\theta} \\ & & & K_{\delta,\delta} & K_{\delta,\theta} \\ & & & & K_{\theta,\theta} \end{pmatrix}$$

$K(\varphi)$ is observed and not the expected information matrix because the expressions turn out to be very complicated for writing the elements of the expected information matrix. The expressions for the elements of K are given in appendix. Under certain regularity conditions (fulfilled for parameters in the interior of the parameter space but not on the boundary),

$$\sqrt{n}(\hat{\varphi}-\varphi)\sim N_5(0, I(\varphi)^{-1})$$

$I(\varphi)$ is the expected information matrix used for construction of tests of hypotheses and appropriate confidence regions for the parameters and can be replaced by the observed information matrix $K(\varphi)$. The asymptotic normality is useful for testing goodness of fit of GIGW distribution versus some of its sub models.

6 Application

In this section we compare the results of fitting the New Modified Generalized Linear Failure Rate Distribution (NMGLFRD), Exponential Weibull distribution (EWD), Generalized Rayleigh distribution (GRD), Modified Weibull distribution (MWD), Weibull distribution (WD), Generalized Exponential distribution (GED), Exponential distribution (ED), Modified Generalized Linear failure rate distribution (MGLFRD), Generalized Linear failure rate distribution (GLFRD), and Linear failure rate distribution (LFRD) to the data set studied by Meeker and Escobar [13], which gives the times of failure and running times for a sample of devices from a eld-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by normal product wear and failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms. The times are:

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.

In order to compare the distribution models, we consider criteria like $-2\log(L)$, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Corrected Akaike Information Criterion) for the data set. The better distribution corresponds to smaller $-2l$, AIC and CAIC values:

$$\mathrm{AIC} = 2k - 2\log(L), \mathrm{BIC} = k(\log n) - 2\log(L) \text{ and } \mathrm{CAIC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1}$$

Where, n the sample size, k is the number of parameters in the statistical model, and l is the maximized value of the log-likelihood function under the considered model.

Table 1: The ML estimates, standard error, AIC, BIC and CAIC of the models based on data set

Model	$-2log(L)$	**Estimates**	**St. Error**	**AIC**	**BIC**	**CAIC**
NMGLFRD		$\hat{\alpha} = 0.101539$	0.0111931			
		$\hat{\beta} = 0.019350$	0.0714287			
	69.34327	$\hat{\gamma} = 3.138587$	3.3239828	79.34327	86.34926	81.84327
		$\hat{\delta} = 5.431130$	0.0098645			
		$\hat{\theta} = 0.116232$	0.0281048			
EWD		$\hat{\beta} = 0.003129$	0.0022435			
	80.07011	$\hat{\gamma} = 4.992084$	0.6304445	86.07011	90.2737	86.99319
		$\hat{\theta} = 0.184281$	0.0406443			
GRD	87.56398	$\hat{\beta} = 0.139866$	0.0413616	91.56398	94.36637	92.00842
		$\hat{\theta} = 0.485812$	0.1027996			
MWD		$\hat{\alpha} = 0.246418$	0.0735728			
	77.09773	$\hat{\beta} = 0.007050$	0.0039493	83.91758	87.30132	84.84066
		$\hat{\gamma} = 4.474707$	0.5997677			
WD	92.31747	$\hat{\beta} = 0.449800$	0.1156533	96.31747	99.11986	96.76191
		$\hat{\gamma} = 1.265047$	0.2044284			
GED	93.91389	$\hat{\alpha} = 0.616110$	0.1369297	97.91389	100.7163	98.35833
		$\hat{\theta} = 1.154287$	0.2733672			
ED	94.27007	$\hat{\alpha} = 0.564864$	0.1031293	96.27007	97.67127	96.41293
MGLFRD		$\hat{\alpha} = 0.726295$	0.1714571			
	91.12194	$\hat{\beta} = 2.581178$	1.5422796	99.12194	104.7267	100.7219
		$\hat{\gamma} = 0.064292$	0.0522157			
		$\hat{\theta} = 27.92919$	43.760853			
GLFRD		$\hat{\alpha} = 0.130047$	0.1542428			
	86.86245	$\hat{\beta} = 0.131477$	0.0452863	92.86245	97.06604	93.78553
		$\hat{\theta} = 0.721135$	0.2398295			
LFRD	87.94711	$\hat{\alpha} = 0.274949$	0.1205749	91.94711	94.7495	92.39155
		$\hat{\beta} = 0.116322$	0.0488434			

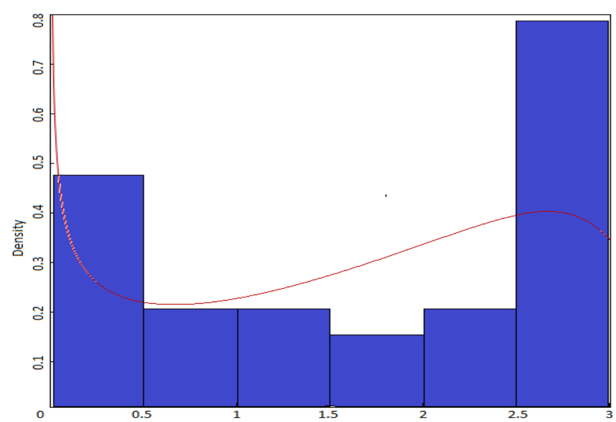

Fig. 11: The estimated NMGLFRD density superimposed on the histogram for the data set.

Table 1 shows parameter MLEs to each one of the two fitted distributions for data set, values of $-2log(L)$, AIC, BIC and AICC. The values in Table 1 indicate that the New Modified Generalized Linear Failure Rate Distribution model performs signicantly better than its sub-models used here for fitting data set. Also, it can be easily seen in figure 9 that fitted density for the New Modified Generalized Linear Failure Rate model is closer to the empirical histogram.

7 Conclusion

A new model so called the New Modified Generalized Linear Failure Rate Distribution (NMGLFRD) has been introduced. It is shown that various existing distribution can be obtained from this new distribution. We have derived some mathematical properties and plots of pdf, cdf and hazard functions are presented to show the versatility of new distribution. It is observed that NMGLFRD can have non-increasing, non-decreasing and bathtub shaped hazard function which are quite desirable for data analysis purposes. The model parameters are estimated by maximum likelihood. We prove that the proposed model can be superior to some models generated from other know families in terms of model fitting by means of an application to a real data set.

Appendix

$$K_{\alpha,\alpha}=\frac{\partial^2 l}{\partial\alpha^2}=\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^2 e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[(\delta-1)\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)-\delta(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$
$$-\sum_{i=1}^{n}\left(\alpha+\beta\gamma x_i^{\gamma-1}\right)^{-2}-(\delta-1)\sum_{i=1}^{n}\frac{x_i^2}{(\alpha x_i+\beta x_i^\gamma)^2}-\delta(\delta-1)\sum_{i=1}^{n}x_i^2(\alpha x_i+\beta x_i^\gamma)^{\delta-2}$$

$$K_{\alpha,\beta}=\frac{\partial^2 l}{\partial\alpha\partial\beta}=\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma+1} e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[(\delta-1)\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)-\delta(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$
$$-\sum_{i=1}^{n}\frac{\gamma x_i^{\gamma+1}}{(\alpha x_i+\beta\gamma x_i^\gamma)^2}-(\delta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma+1}}{(\alpha x_i+\beta x_i^\gamma)^2}-\delta(\delta-1)\sum_{i=1}^{n}x_i^{\gamma+1}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}$$

$$K_{\alpha,\gamma}=\frac{\partial^2 l}{\partial\alpha\partial\gamma}=\beta\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma+1}\log x_i e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[(\delta-1)\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)-\delta(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$
$$-\beta\sum_{i=1}^{n}\frac{x_i^{\gamma+1}(1+\gamma\log x_i)}{(\alpha x_i+\beta\gamma x_i^\gamma)^2}-\beta(\delta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma+1}\log x_i}{(\alpha x_i+\beta x_i^\gamma)^2}-\beta\delta(\delta-1)\sum_{i=1}^{n}x_i^{\gamma+1}\log x_i(\alpha x_i+\beta x_i^\gamma)^{\delta-2}$$

$$K_{\alpha,\delta}=\frac{\partial^2 l}{\partial\alpha\partial\delta}=\sum_{i=1}^{n}\frac{x_i}{(\alpha x_i+\beta x_i^\gamma)}+(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-1}}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)}-\delta\sum_{i=1}^{n}x_i(\alpha x_i+\beta x_i^\gamma)^{\delta-1}\log(\alpha x_i+\beta x_i^\gamma)$$
$$+\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i\log(\alpha x_i+\beta x_i^\gamma)e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-1}\left[1-(\alpha x_i+\beta x_i^\gamma)^\delta-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$
$$-\sum_{i=1}^{n}x_i(\alpha x_i+\beta x_i^\gamma)^{\delta-1}$$

$$K_{\alpha,\theta}=\frac{\partial^2 l}{\partial\alpha\partial\theta}=\delta\sum_{i=1}^{n}\frac{x_i e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-1}}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)}$$

$$K_{\beta,\beta}=\frac{\partial^2 l}{\partial\beta^2}=\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{2\gamma} e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[(\delta-1)\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)-\delta(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$
$$-\sum_{i=1}^{n}\frac{\gamma^2 x_i^{2\gamma}}{(\alpha x_i+\beta\gamma x_i^\gamma)^2}-(\delta-1)\sum_{i=1}^{n}\frac{x_i^{2\gamma}}{(\alpha x_i+\beta x_i^\gamma)^2}-\delta(\delta-1)\sum_{i=1}^{n}x_i^{2\gamma}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}$$

$$K_{\beta,\gamma}=\frac{\partial^2 l}{\partial\beta\partial\gamma}=\alpha\sum_{i=1}^{n}\frac{x_i^{\gamma+1}(1+\gamma\log x_i)}{(\alpha x_i+\beta\gamma x_i^\gamma)^2}+\alpha(\delta-1)\sum_{i=1}^{n}\frac{x_i^{\gamma+1}\log x_i}{(\alpha x_i+\beta x_i^\gamma)^2}-\delta\sum_{i=1}^{n}x_i^\gamma\log x_i(\alpha x_i+\beta x_i^\gamma)^{\delta-2}(\alpha x_i+\beta\delta x_i^\gamma)$$
$$+\beta\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^{2\gamma}\log x_i\, e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[(\delta-1)\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)-\delta(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$

$$K_{\beta,\delta} = \frac{\partial^2 l}{\partial\beta\partial\delta} = \sum_{i=1}^{n}\frac{x_i^\gamma}{\left(\alpha x_i + \beta x_i^\gamma\right)} + \left(\theta-1\right)\sum_{i=1}^{n}\frac{x_i^\gamma e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\left(\alpha x_i + \beta x_i^\gamma\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)} - \delta\sum_{i=1}^{n}x_i^\gamma\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-1}log\left(\alpha x_i+\beta x_i^\gamma\right)$$

$$+\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^\gamma log(\alpha x_i+\beta x_i^\gamma)e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-1}\left[1-(\alpha x_i+\beta x_i^\gamma)^\delta - e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$

$$-\sum_{i=1}^{n}x_i^\gamma(\alpha x_i+\beta x_i^\gamma)^{\delta-1}$$

$$K_{\beta,\theta} = \frac{\partial^2 l}{\partial\beta\partial\theta} = \delta\sum_{i=1}^{n}\frac{x_i^\gamma e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)}$$

$$K_{\gamma,\gamma} = \frac{\partial^2 l}{\partial\gamma^2} = \beta\sum_{i=1}^{n}\frac{x_i^\gamma\, logx_i\left[\alpha x_i\left(2+\gamma logx_i\right)+\beta x_i^\gamma\right]}{\left(\alpha x_i+\beta x_i^\gamma\right)^2} + \alpha\beta\left(\delta-1\right)\sum_{i=1}^{n}\frac{x_i^{\gamma+1}\left(logx_i\right)^2}{\left(\alpha x_i+\beta x_i^\gamma\right)^2}$$

$$-\beta\delta\sum_{i=1}^{n}x_i^{\gamma+1}\left(logx_i\right)^2\left(\alpha+\beta\delta x_i^{\gamma-1}\right)\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-2}$$

$$+\beta\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^\gamma(logx_i)^2 e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-2}\left[\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)(\alpha x_i+\beta\delta x_i^\gamma)-\beta\delta x_i^\gamma(\alpha x_i+\beta x_i^\gamma)^\delta\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2}$$

$$K_{\gamma,\delta} = \frac{\partial^2 l}{\partial\gamma\partial\delta} = \beta\sum_{i=1}^{n}\frac{x_i^\gamma\, logx_i}{\left(\alpha x_i+\beta x_i^\gamma\right)} + \beta\left(\theta-1\right)\sum_{i=1}^{n}\frac{x_i^\gamma\, logx_i e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)}$$

$$-\beta\delta\sum_{i=1}^{n}x_i^\gamma logx_i\, log\left(\alpha x_i+\beta x_i^\gamma\right)\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-1}$$

$$+\beta\delta(\theta-1)\sum_{i=1}^{n}\frac{x_i^\gamma logx_i\, log(\alpha x_i+\beta x_i^\gamma)e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}(\alpha x_i+\beta x_i^\gamma)^{\delta-1}\left[1-(\alpha x_i+\beta x_i^\gamma)^\delta-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right]}{\left(1-e^{-(\alpha x_i+\beta x_i^\gamma)^\delta}\right)^2} - \beta\sum_{i=1}^{n}x_i^\gamma\, logx_i\,(\alpha x_i+\beta x_i^\gamma)^{\delta-1}$$

$$K_{\gamma\theta} = \frac{\partial^2 l}{\partial\gamma\partial\theta} = \beta\delta\sum_{i=1}^{n}\frac{x_i^\gamma\, logx_i e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\left(\alpha x_i+\beta x_i^\gamma\right)^{\delta-1}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)}$$

$$K_{\delta,\delta} = \frac{\partial^2 l}{\partial\delta^2} = -\frac{n}{\delta^2} - \sum_{i=1}^{n}\left(\alpha x_i+\beta x_i^\gamma\right)^\delta\left(log\left(\alpha x_i+\beta x_i^\gamma\right)\right)^2$$

$$+\left(\theta-1\right)\sum_{i=1}^{n}\frac{e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\left(\alpha x_i+\beta x_i^\gamma\right)^\delta\left(log\left(\alpha x_i+\beta x_i^\gamma\right)\right)^2\left[1-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right]}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)^2}$$

$$K_{\delta,\theta} = \frac{\partial^2 l}{\partial\delta\partial\theta} = \sum_{i=1}^{n}\frac{\left(\alpha x_i+\beta x_i^\gamma\right)^\delta log\left(\alpha x_i+\beta x_i^\gamma\right)e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}}{\left(1-e^{-\left(\alpha x_i+\beta x_i^\gamma\right)^\delta}\right)}$$

$$K_{\theta,\theta} = \frac{\partial^2 l}{\partial\theta^2} = -\frac{n}{\theta^2}$$

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