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A Finite Type of Closure Operations on BCK-algebras

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Abstract: In this article we study the (finite-type, cl-closed) closure operations on ideals of a BCK-algebra, with an emphasis on structural properties. Also we give several theorems that make different closure operations, especially on a quotient and Noetherian BCK-algebra. In particular, we show that the intersection and union (by imposing some additional conditions) of closure operations is a closure operation and every closure operation is a finite type on any finitely generated ideal. Moreover by given the notions of residuated quotient ideals and the meet of two ideals, we conclude some related results.

Keywords: BCI-algebra, BCK-algebra, closure operation, finite type closure operation.
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1 Introduction

The study of BCK-algebra was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi and also for the first time, E. H. Moore in 1910 [6] introduced closure operation on a set. After that many researchers have worked on closure operation, see for example [2,4,8,9]. Finally in 2012 , N. Epestein published a paper [4] about closure operation on commutative algebra. The closure operation open a new ideas in commutative algebra. Since ordered algebra have been used in many branches of mathematics and as far as we know, there is no article on closure operation in this area in general, so this paper is the first step of using closure operation in (a special) ordered algebra, and hoping that this manuscript open another new ideas for future researches in algebra structures.

The structure of the article is as follows: first we introduce the notion of closure operations and provide some examples. Then we give some lemmas and theorems that help us to make some more closure operations. Also we consider some closure operations on Noetherian BCK-algebras and we use this notion on a quotient BCK-algebra and obtain some results. Moreover by considering different type of ideals we prove some related results.

2 Preliminaries

In this section, we gather some basic notions relevant to closure operation on ideals which will need in the next sections.

Definition 2.1[10] *An algebra $(X;*,0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions: for any $x, y, z \in X$,*
BCI-1: $((x*y)*(x*z))*(z*y) = 0$
BCI-2: $x*0 = x$
BCI-3: $x*y = 0$ *and* $y*x = 0$ *imply* $x = y$

We call the binary operation $*$ on X the $*$ multiplication on X, and the constant 0 of X the zero element of X. We often write X instead of $(X;*,0)$ for a BCI-algebra in brevity.

Proposition 2.2[10] *Suppose that $(X;*,0)$ is a BCI-algebra. Define a binary relation $\leq$ on X by which $x \leq y$ if and only if $x*y = 0$ for any $x,y \in X$. Then $(X;\leq)$is a partially ordered set with 0 as a minimal element.*

Definition 2.3[10] *Given a BCI-algebra X, if it satisfies the condition*
BCK-1: $0*x = 0$ *for all* $x \in X$*, which means that* $0 \leq x$*. for each* $x \in X$.
We call this algebra a BCK-algebra.

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Definition 2.4[10] *Given an element a in BCI-algebra X, the set*

$$A(a) = \{x \in X | x \leq a\}$$

is called initial section of X determined by a.

Definition 2.5[7] *A BCK-algebra X is called bounded if there exists the greatest element of X, with respect to the ordered relation $\leq$.*

Definition 2.6[10] *A partially ordered set $(X;\leq)$ is called a lower semilattice if any two elements in X have the greatest lower bound. It is called an upper semilattice if each pair of elements in X has its least upper bound.*

Given a BCK-algebra X, if it with respect to its BCI-ordering $\leq$ forms a lower semilattice, then the algebra X is called a lower BCK-semilattice. Similarly we can define an upper BCK-semilattice.
In a lower BCK-semilattice we denote $x \wedge y = glb\{x,y\}$.

Definition 2.7[10] *A subset A of a BCI-algebra X is called an ideal of X if*
(i) $0 \in A$.
(ii) $x \in A$ and $y * x \in A$ imply $y \in A$ for any $x, y \in X$.
Note that X and $\{0\}$ are ideals of X, and they are called the trivial ideals of X.

Theorem 2.8[7] *Let A be an ideal of a BCK-algebra X. Then for any $x, y \in X$, $x \in A$ and $y \leq x$ imply that $y \in A$.*

Definition 2.9[10] *An ideal A of a BCI-algebra X is called closed if A is closed under the $*$ multiplication on X.*

Definition 2.10[10] *Let S be a subset of a BCI-algebra X. We call the least ideal of X, containing S, the generated ideal of X by S, denoted by $< S >$ or $(S]$. An ideal A of a BCK-algebra X is said to be finitely generated if there is a finite subset S of X such that $A =< S >$. The ideal $< a >$ generated by one generator a is also called a* principal ideal *of X.*

Definition 2.11[7] *A BCK-algebra X is called commutative if $x * (x * y) = y * (y * x)$ for any $x, y \in X$.*

Theorem 2.12[7] *A BCK-algebra X is commutative if and only if $(X;\leq)$ is a lower semilattice with $x \wedge y = y * (y * x)$ for any $x, y \in X$.*

Definition 2.13[7] *A BCK-algebra X is said to be Noetherian if each ideal of X is finitly generated.*

Definition 2.14[7] *Given a BCK-algebra X, we say that X satisfies the ascending chain condition, abbreviated by ACC, if there does not exists an infinite properly ascending chain $I_1 \subseteq I_2 \subseteq ...$ in $\mathscr{I}_X$*

Theorem 2.15[7] *In a BCK-algebra X, the following are equivalent:*
(i) X is Noetherian,
(ii) X satisfies ACC,.

Theorem 2.16[7] *Let X be a lower BCK-semilattice. Then X is commutative if and only if $A(a) \bigcap A(b) = A(a \wedge b)$ for any $a, b \in X$ where $a \wedge b = b * (b * a)$ and $A(.)$ is an initial section of X.*

Definition 2.17[10] *Suppose $(X;*,0)$ and $(X';*',0')$ are two BCK-algebra. A mapping $f : X \longrightarrow X'$ is called a homomorphism from X into X' if, for any $x, y \in X$*

$$f(x*y) = f(x)*' f(y)$$

If, in addition,the mapping f is onto, then f is called an epimorphism. The mapping is called an isomorphism if it is both an epimorphism and one-to-one.

Definition 2.18[10] *An equivalence relation relation θ on a BCI-algebra X is called a congruence on X if it is of the substitution property:*

$$x \sim y(\theta), u \sim v(\theta) \text{ imply } x * u \sim y * v(\theta)$$

for any x, y, u, v $\in X$. Denote $\dfrac{X}{\theta}$ for the quotient set $\{\theta_x | x \in X\}$. If θ is a congruence on X, the operation $*$ on $\dfrac{X}{\theta}$ given by $\theta_x * \theta_y = \theta_{x*y}$ is well-defined. Then $(\dfrac{X}{\theta}, *, \theta_0)$ is an algebra which call the quotient algebra of X induced by θ.

Theorem 2.19[10] *Let $f : X \longrightarrow X'$ be an epimorphism. If A is an ideal of X, then $f(A)$ is an ideal of X'.*

3 Closure Operation

In this section, we define closure operation on ideals and by given some theorems we construct and present some different closure operations.

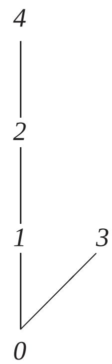
Definition 3.1*By an operation "d" on the set of all ideals of a BCK-algebra X (denoted by $\mathscr{I}_X$), we mean a function $d : \mathscr{I}_X \longrightarrow \mathscr{I}_X$. For simplicity of notation for any $A \in \mathscr{I}_X$ we write $d(A) = A^d$.*

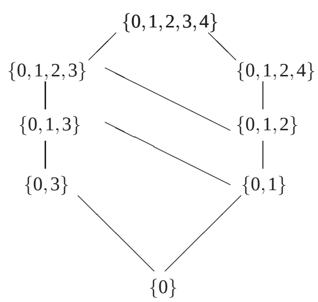
Definition 3.2*Let X be a BCK-algebra. Then a closure operation "cl" on the set $\mathscr{I}_X$, is an operation $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that $A \longmapsto A^{cl}$ satisfies the following conditions:*
***(i)** $A \subseteq A^{cl}$ for all $A \in \mathscr{I}_X$ (Extension).*
***(ii)** $A^{cl} = (A^{cl})^{cl}$ for all $A \in \mathscr{I}_X$ (Idempotence).*
***(iii)** if A and B are in $\mathscr{I}_X$ and $B \subseteq A$. Then $B^{cl} \subseteq A^{cl}$ (Order-preservation).*

Example 3.3*Let X be the set $\{0,1,2,3,4\}$. Define a binary operation $*$ on X by the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then X is a lower BCK-semilattice with 8 ideals. The following figures describe the Hasse diagrams of elements and ideals of X.





Define cl on ideals as follows:

$(\{0\})^{cl} = \{0\}, (\{0,3\})^{cl} = \{0,1,3\}, (\{0,1,3\})^{cl} = \{0,1,3\}, (\{0,1\})^{cl} = \{0,1,2\}, (\{0,1,2\})^{cl} = \{0,1,2\}$
$(\{0,1,2,3\})^{cl} = \{0,1,2,3,4\}, (\{0,1,2,4\})^{cl} = \{0,1,2,3,4\}$ *and* $(\{0,1,2,3,4\})^{cl} = \{0,1,2,3,4\}$.

It is easy to check that "cl" is a closure operation.

Definition 3.4 *We say that an ideal A in $\mathscr{I}_X$ is cl-closed, if $A = A^{cl}$.*
Note that A^{cl} is cl-closed for any ideal A of X, by Definition 3.2.

Example 3.5 *In Example 3.3, the ideals $\{0,1,3\},\{0,1,2\},\{0,1,2,3,4\}$ are $cl-closed$ ideals.*

Theorem 3.6 *A^{cl} is the intersection of all cl-closed ideals containing A.*

Proof. The proof is straightforward.

Definition 3.7 *Let A and B are two arbitrary ideals of a lower BCK-semilattice X. Define:*

$$A \wedge B = <\{x \wedge y | x \in A, y \in B\}> .$$

For an element $x \in X$, we have $x \wedge B = \{x\} \wedge B = <\{x \wedge y | y \in B\}>$.

Definition 3.8 *Let X be a lower BCK-semilattice and $\sum \subseteq \mathscr{I}_X$. Then we say that $\sum$ is $\wedge$-closed if $A \wedge B \in \sum$, for any two ideals $A, B \in \sum$*

Example 3.9 *In Example 3.3, let $\sum = \{\{0\},\{0,3\},\{0,1\},\{0,1,3\},\{0,1,2\}\}$. Then $\sum \subseteq \mathscr{I}_X$ and clearly $\sum$ is a $\wedge$-closed subset. Because for any two elements of $\sum$ like $\{0,1,3\},\{0,1,2\}$, we have $\{0,1,3\} \wedge \{0,1,2\} = \{0,1\}$ and $\{0,1\} \in \sum$.*

Remark 3.10 ***(i)*** *From the above definition we get that: $A^2 = A \wedge A = <\{x \wedge y | x, y \in A\}>$ and $A^3 = A^2 \wedge A, \ldots$*

(ii) *In a lower BCK-semilattice (specially, in a commutative BCK-algebra), we have $\cdots \subseteq A^3 \subseteq A^2 \subseteq A$, because a commutative BCK-algebar is a lower BCK-semilattice and $x \wedge y = x * (x * y) \leq x$, for any $x, y \in X$.*

Definition 3.11 *Let A and B are ideals of a lower BCK-semilattice X. Define the residuated quotient ideal A by B, as follows:*

$$(A :_X B) := <\{x \in X | x \wedge B \subseteq A \wedge B\}> .$$

Also, $(A :_X B^\infty) = \sum_{n \in N} (A :_X B^n) = <\{x \in X |$ there exists $n, x \wedge B^n \subseteq A \wedge B^n\}>$.

Example 3.12 *In Example 3.3 let $A = \{0,1,3\}$ and $B = \{0,1,2\}$. Then by using the Example 3.9 and the Hasse diagram of elements in Example 3.3 we have:*

$$\{x \in X | x \wedge \{0,1,2\} \subseteq \{0,1\}\} = \{0,1,3\}.$$

Thus,
$(A :_X B) = <\{x \in X | x \wedge \{0,1,2\} \subseteq \{0,1\}\}> = <\{0,1,3\}> = \{0,1,3\}$.

Proposition 3.13 *Let X be a BCK-algebra and cl be a closure operation on $\mathscr{I}_X$. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a subset of $\mathscr{I}_X$. Then:*

(i) *If every A_α is a "$cl-closed$", then so is $\cap_\alpha A_\alpha$.*
(ii) *$\cap_\alpha A_\alpha^{cl}$ is "$cl-closed$".*
(iii) *$(\sum_\alpha A_\alpha^{cl})^{cl} = (\sum_\alpha A_\alpha)^{cl}$.*

Proof. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be as above. Then:

(i) For any $\beta \in \Lambda$ we have $\cap_\alpha A_\alpha \subseteq A_\beta$, Thus $(\cap_\alpha A_\alpha)^{cl} \subseteq A_\beta^{cl} = A_\beta$. Since this hold for any β ; we have $(\cap_\alpha A_\alpha)^{cl} \subseteq \cap_\alpha A_\alpha$.

(ii) We prove $(\cap_{\alpha \in \Lambda} A_\alpha^{cl})^{cl} = \cap_{\alpha \in \Lambda} A_\alpha^{cl}$. Since for each α in Λ; we have $(A_\alpha^{cl})^{cl} = A_\alpha^{cl}$; so A_α^{cl} is a "$cl-closed$" ideal. Hence (2) follows directly from part (1).

(iii) First we prove $(\sum_\alpha A_\alpha)^{cl} \subseteq (\sum_\alpha A_\alpha^{cl})^{cl}$. By the extension property, $A_\alpha \subseteq A_\alpha^{cl}$ for each $\alpha \in \Lambda$. So $\sum_\alpha A_\alpha \subseteq \sum_\alpha A_\alpha^{cl}$ because $\sum_\alpha A_\alpha = < A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots >$ where $\alpha_1, \alpha_2, \ldots$ are in Λ. Hence the conclusion follows from order-preservation.
Conversly, for any β in Λ we have $A_\beta \subseteq \sum_\alpha A_\alpha \subseteq (\sum_\alpha A_\alpha)^{cl}$ and by using order-preservation and idempotence properties, $A_\beta^{cl} \subseteq (\sum_\alpha A_\alpha)^{cl} \subseteq ((\sum_\alpha A_\alpha)^{cl})^{cl}$ for all $\beta \in \Lambda$. Thus $\sum_\alpha A_\alpha^{cl} \subseteq (\sum_\alpha A_\alpha)^{cl}$.

In the following we mention a theorem which says that: a given closure operation is equivalent with a collection $\sum$ of ideals such that the intersection of any subcollection is in $\sum$.

Theorem 3.14 *Let χ be the set of all closure operations on $\mathscr{I}_X$ and ϕ be the set of all subsets of $\mathscr{I}_X$ which are closed under intersection. Then there is a one-to-one correspondence between χ and ϕ.*

Proof. Suppose "cl" is a closure operation on X. Let $\sum$ be the class of "$cl-closed$" ideals. That is, $A \in \sum$ If and only If $A = A^{cl}$. By Proposition 3.13, the intersection of any subcollection of ideals in $\sum$ is also in $\sum$.
Conversly, suppose that $\sum$ is a collection of ideals for which the intersection of any subcollection is in $\sum$. For any ideal A, let $A^{cl} = \cap\{B|A \subseteq B, B \in \sum\}$. Then we have:
(i) $A \subseteq A^{cl}$.
(ii) By using (i) $A^{cl} \subseteq (A^{cl})^{cl}$. Now suppose that $x \in (A^{cl})^{cl}$, $B \in \sum$ and $A \subseteq B$. By definition A^{cl}, $A^{cl} \subseteq B$. So $x \in B$. Since B is arbitrary, $x \in \cap\{B|A \subseteq B, B \in \sum\}$. Therefore $(A^{cl})^{cl} \subseteq A^{cl}$.
(iii) For ideals A and B of X such that $A \subseteq B$ we have

$$\cap\{K|A \subseteq K, K \in \sum\} \subseteq \cap\{K|B \subseteq K, K \in \sum\}.$$

Thus, $A^{cl} \subseteq B^{cl}$.

Example 3.15 ***(i)*** *The identity operation, sending each ideal to itself, is a closure operation on X.*
(ii) *The indiscrete operation, sending each ideal to X, is also a closure operation on X.*

Theorem 3.16 *Consider a bounded lower BCK-semilattice X with 1 as the greatest element. Define $A^p = \bigcap\{P \in Spec(X); A \subseteq P\}$. Then the operation $cl : I_X \longrightarrow I_X$; $A \longmapsto A^p$ is a closure operation.*

Proof. By Stone-lemma (Theorem 1.4.19 of [10]), $\cap\{P|P \in Spec(X), A \subseteq P\}$ is not empty. Also for each ideal A of X;
(i) Clearly, $A \subseteq A^p$.
(ii) $A^p = (A^p)^p$. The proof is similar to the proof of Theorem 3.14.
(iii) For any two ideals A and B such that $A \subseteq B$; we have $A^p \subseteq B^p$. Hence the operation $cl : I_X \longrightarrow I_X$; $A \longmapsto A^p$ is a closure operation.

Theorem 3.17 *Let X be a bounded lower BCK-semilattice with 1 as the greatest element and define $A^{cl} = \cap\{I|\ I$ is closed ideal and $A \subseteq I\}$. Then the operation $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that $A \longmapsto A^{cl}$, is a closure operation.*

Proof. Since X is a closed ideal containing A, the above set is not empty.
(i) Obviously, $A \subseteq A^{cl}$.
(ii) Because of the intersection of closed ideals, is closed ideal too. Therefore,

$$(A^{cl})^{cl} = A^{cl}.$$

(iii) For any two ideals A and B of X, if $B \subseteq A$ then $B^{cl} \subseteq A^{cl}$ (the proof is similar to the proof of Theorem 3.14).

Remark 3.18 *Note that for the above closure operation, we have:*
(i) *A^{cl} is the smallest closed ideal of X containing A.*
(ii) *$A = A^{cl}$ if and only if A is a closed ideal.*

Theorem 3.19 *[10] Suppose that X is a lower BCK-semilattice. Then*

$$< x > \cap < y >=< x \wedge y >$$

Theorem 3.20 *[7] Let X be a lower BCK-semilattice. Then X is commutative if and only if $A(a) \bigcap A(b) = A(a \wedge b)$ for any $a, b \in X$ where $a \wedge b = b * (b * a)$ and $A(.)$ is an initial section of X.*

Theorems 3.19 and 3.20, help us to have a following theorem.

Theorem 3.21 *Let X be a bounded lower BCK-semilattice with 1 as the greatest element. Define $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that for each ideal A of X; $A^{cl} = \cap\{B|A \subseteq B, B \in \sum\}$.*
(i) *If $\sum$ is the set of all principal ideals containing A. Then "cl" is a closure operation.*
(ii) *If $\sum$ is the set of all initial sections containing A and X is a commutative. Then "cl" is a closure operation.*

Proof. **(i)** Since $(1] = X$, $\sum \neq \emptyset$.
(1) Obviously, $A \subseteq A^{cl}$.
(2) By using Theorem 3.19, the intersection of principal ideals is a principal ideal. Therefore, $(A^{cl})^{cl} = A^{cl}$.
(3) The proof of order-preservation is similar to the proof of Theorem 3.14.
(ii) By using the Theorem 3.20, the proof is the same as of the proof of part (i).

Theorem 3.22 *Suppose that B is an ideal of a BCK-algebra X. Define $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that for each ideal A of X; $A^{cl} = A + B =< A \cup B >$. Then "cl" is a closure operation.*

Proof. (i) $A \subseteq A^{cl}$.
(ii) $(A^{cl})^{cl} = (A + B)^{cl} =<< A \cup B > \cup B >=< A \cup B >= A^{cl}$.
(iii) For any two ideals A_1 and A_2 of X. If $A_1 \subseteq A_2$, then $A_1 \cup B \subseteq A_2 \cup B$. So $< A_1 \cup B > \subseteq < A_2 \cup B >$. Thus, $A_1^{cl} \subseteq A_2^{cl}$.

Theorem 3.23 *Let B be an ideal of a lower BCK-semilattice X. Define $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that for each ideal A of X, $A^{cl} = (A :_X B)$. Then "cl" is a closure operation.*

Proof. (i) For every element a in A,

$$a \wedge B \subseteq A \wedge B =< \{\alpha \wedge \beta | \alpha \in A, \beta \in B\} > .$$

Which means that $A \subseteq A^{cl}$.

(ii) Suppose that $a \in (A^{cl})^{cl}$, then $a \wedge B \subseteq A^{cl} \wedge B$. Now if $g \in A^{cl}$ be an arbitrary element, then $g \wedge B \subseteq A \wedge B$. So $A^{cl} \wedge B \subseteq A \wedge B$. Therefore $a \wedge B \subseteq A \wedge B$ and $a \in A^{cl}$.
(iii) For each ideals A_1 , A_2 of X. If $A_1 \subseteq A_2$ and $a \in (A_1 :_X B) = A^{cl}$, then $a \in X$, $a \wedge B \subseteq A_1 \wedge B$. Since $A_1 \wedge B \subseteq A_2 \wedge B$, we have $a \wedge B \subseteq A_2 \wedge B$ and $a \in (A_2 :_X B) = A^{cl}$.

Theorem 3.24*Let A and B be two ideals of a lower BCK-semilattice X. Consider: $(A :_X B^\infty) = \bigcup_{n\in N}(A :_X B^n) =< \{x \in X|$ there exists $n, x \wedge B^n \subseteq A \wedge B^n\} >$. Define $cl : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that for each ideal A of X, $A^{cl} = (A :_X B^\infty)$. Then "cl" is a closure operation.*

*Proof.*The proof is similar to the proof of Theorem 3.23.

Theorem 3.25*Let $\varphi : X_1 \longrightarrow X_2$ be a BCK-epimorphism and "cl" be a closure operation on X_2. For each ideal A of X_1, define $A^{cl'} = \varphi^{-1}((\varphi(A))^{cl})$. Then "cl′" is a closure operation on X_1.*

Proof.(i) Since "*cl*" is a closure operation, $\varphi(A) \subseteq \varphi(A)^{cl}$. So $\varphi^{-1}(\varphi(A)) \subseteq \varphi^{-1}(\varphi(A)^{cl})$. Also $A \subseteq \varphi^{-1}(\varphi(A))$, thus $A \subseteq \varphi^{-1}(\varphi(A)^{cl}) = A^{cl'}$. This prove the extension properti.
(ii) From extension property we know that $A^{cl'} \subseteq (A^{cl'})^{cl'}$. if $\alpha \in (A^{cl'})^{cl'}$, then $\alpha \in \varphi^{-1}(\varphi(A^{cl'})^{cl})$ and $\varphi(\alpha) \in (\varphi(A^{cl'}))^{cl}$. Also $\varphi(A^{cl'}) = \varphi(\varphi^{-1}(\varphi(A)^{cl})) \subseteq \varphi(A)^{cl}$. Hence

$$\varphi(\alpha) \in ((\varphi(A))^{cl})^{cl} = (\varphi(A))^{cl}.$$

So $\alpha \in \varphi^{-1}(\varphi(A))^{cl} = A^{cl'}$.
(iii) Let $A \subseteq B$. Then $\varphi(A) \subseteq \varphi(B)$. Since "*cl*" is a closure operation, $\varphi(A)^{cl} \subseteq \varphi(B)^{cl}$. Hence, $\varphi^{-1}(\varphi(A)^{cl}) \subseteq \varphi^{-1}(\varphi(B)^{cl})$. Therefore $A^{cl'} \subseteq B^{cl'}$.

Suppose that X and X' are two BCK-algebra. We know from [10] that if $f : X \longrightarrow X'$ is an epimorphism and A is an ideal of X, then $f(A)$ is an ideal of X'. This help us for the next theorem.

Theorem 3.26*Let $\varphi : X_1 \longrightarrow X_2$ be a BCK-epimorphism and "cl" be a closure operation on X_1. For each ideal A of X_2, define $A^c = \varphi((\varphi^{-1}(A))^{cl})$. Then "c" is a closure operation on X_2.*

*Proof.*The proof is similar to the proof of Theorem 3.25.

Lemma 3.27*Let $\{cl_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of closure operations on ideals of a BCK-algebra X. Then $A^{cl} = \bigcap_{\lambda\in\Lambda} A^{cl_\lambda}$ defines a closure operation.*

*Proof.*Suppose that A is an ideal of X.
For each $\lambda \in \Lambda$ since $A \subseteq A^{cl_\lambda}$, $A \subseteq \bigcap_{\lambda\in\Lambda} A^{cl_\lambda}$. Thus $A \subseteq A^{cl}$ and the extension property holds.
For idempotence, suppose that $\alpha \in (A^{cl})^{cl}$. Then for every $\lambda \in \Lambda$, we have $\alpha \in (A^{cl})^{cl_\lambda}$. Since $A^{cl} \subseteq A^{cl_\lambda}$ and cl_λ preserve order, we get that $\alpha \in (A^{cl})^{cl_\lambda} \subseteq (A^{cl_\lambda})^{cl_\lambda} = A^{cl_\lambda}$. Hence $\alpha \in \bigcap_{\lambda\in\Lambda} A^{cl_\lambda} = A^{cl}$ and $A^{cl} = (A^{cl})^{cl}$.
Let A and B are two arbitrary ideals of X such that $A \subseteq B$. Since for each $\lambda \in \Lambda$, cl_λ is a closure operation, $A^{cl_\lambda} \subseteq B^{cl_\lambda}$. Hence $\bigcap_{\lambda\in\Lambda} A^{cl_\lambda} \subseteq \bigcap_{\lambda\in\Lambda} B^{cl_\lambda}$. Thus $A^{cl} \subseteq B^{cl}$ and we have order-preservation property.

Definition 3.28*Let cl_1 and cl_2 be two closure opertions on a BCK-algebra X. Then we write $cl_1 \leq cl_2$ if for every ideal A, $A^{cl_1} \subseteq A^{cl_2}$.*

Lemma 3.29*Suppose that $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals such that for every λ_1 and λ_2 in Λ, there exists some $\beta \in \Lambda$ such that $A_{\lambda_i} \subseteq A_\beta$ for $i = 1, 2$. Then $A = \bigcup_{\lambda\in\Lambda} A_\lambda$ is an ideal.*

*Proof.*The proof is straightforward.

Lemma 3.30*Let $\{cl_\lambda\}_{\lambda\in\Lambda}$ be a direct set of closure operations, that is for any $\lambda_1, \lambda_2 \in \Lambda$, there exists some $\mu \in \Lambda$ such that $c_{\lambda_i} \leq c_\mu$ for i=1,2. Moreover, assume that every ideal of X is finitly generated. Then $A^{cl} = \bigcup_{\lambda\in\Lambda} A^{cl_\lambda}$ defines a closure operation.*

*Proof.*By Lemma 3.29, A^{cl} is an ideal and every ideal of X is finitely generated, so $A^{cl} =< a_1, a_2, \ldots, a_n >$. Since $a_i \in A^{cl}$ (i=1,2,...,n), for each i=1,2,...,n there exists $\lambda_i \in \Lambda$ such that $a_i \in A^{cl_{\lambda_i}}$. Now by assumption there exists cl_β such that $cl_{\lambda_i} \leq cl_\beta$ (i=1,2,...,n). So $A^{cl_{\lambda_i}} \subseteq A^{cl_\beta}$. Hence $a_i \in A^{cl_\beta}$. Therefore $A^{cl} \subseteq A^{cl_\beta}$. On the other hand, by definition of A^{cl}, $A^{cl_\beta} \subseteq A^{cl}$. Hence $A^{cl} = A^{cl_\beta}$. Extension and order-preservation properties are clear.

Idempotence: for any ideal A, by above argument there exists λ_1 and λ_2 in Λ such that $A^{cl} = A^{cl_{\lambda_2}}$ and $(A^{cl})^{cl} = (A^{cl})^{cl_{\lambda_1}}$. Hence

$$(A^{cl})^{cl} = (A^{cl})^{cl_{\lambda_1}} = (A^{cl_{\lambda_2}})^{cl_{\lambda_1}}.$$

By assumption, there exist β in Λ such that $cl_{\lambda_1} \leq cl_\beta$. So

$$(A^{cl_{\lambda_2}})^{cl_{\lambda_1}} \subseteq (A^{cl_\beta})^{cl_\beta} = A^{cl_\beta}.$$

Now from $A^{cl} = \bigcup_{\lambda\in\Lambda} A^{cl_\lambda}$, we have $A^{cl_\beta} \subseteq A^{cl}$. Thus $(A^{cl})^{cl} \subseteq A^{cl}$.

Also by extension property, $A^{cl} \subseteq (A^{cl})^{cl}$. Therefore $(A^{cl})^{cl} = A^{cl}$.

Remark 3.31*Since in Noetherian BCK-algebra every ideal is finitely generated, the above result is true for every Noetherian BCK-algebra.*

Theorem 3.32*Let ξ be a $\wedge$-closed set of ideals of a Noetherian lower BCK-semilattice X. Define $\xi : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ such that for each ideal A of X; $A^\xi = \sum_{K\in\xi}(A :_X K)$. Then, "$\xi$" is a closure operation.*

*Proof.*First for each ideal K in ξ, define $A^K = (A :_X K)$. By Theorem 3.23, $(-)^K$ is a closure operation on $\mathscr{I}_X$. Now we show that $\{(-)^K; K \in \xi\}$ is a direct set of closure operations. To this end let K_1 and K_2 be in ξ. Then $K_1 \wedge K_2$ is in ξ and $A^{K_1} \subseteq A^{K_1\wedge K_2}$. Also, $A^{K_2} \subseteq A^{K_1\wedge K_2}$. Now since $A^\xi = \bigcup_{K\in\xi}(A :_X K)$, by Lemma 3.30, $(-)^\xi$ is a closure operation on $\mathscr{I}_X$.

Remark 3.33*We call the above closure operation as the ξ-closure of an ideal A.*

Example 3.34 *Suppose that X be a BCK-algebra as in Example 3.3, and Put $\xi = \{\{0\},\{0,1\},\{0,3\}\}$. clearly ξ is a $\wedge$-closed subset of ideals of a Noetherian lower BCK-semilattice X. For ideal $A_0 = \{0\}$ we have*

$$A_0^{\xi} = < (A_0 :_X \{0\}) \cup (A_0 :_X \{0,1\}) \cup (A_0 :_X \{0,3\}) > .$$

But $(A_0 :_X \{0\}) = \{x \in X \mid x \wedge \{0\} \subseteq A_0 \wedge \{0\}\} = X$. By routine verification, for any ideal A of X we have $A^{\xi} = \sum_{K \in \xi} (A :_X K) = X$. Thus ξ is an indiscrete closure operation.

Proposition 3.35 *Let X be a BCK-algebra and $d : \mathscr{I}_X \longrightarrow \mathscr{I}_X$ be an operation which it satisfies the extension and order-preservation conditions. Let S be the set of all closure operations on X defined by the property that $c \in S$ if and only if $A^d \subseteq A^c$ for all ideal A of X (It means that $d \leq c$). Then by Lemma 3.27 the assignment $A \longmapsto A^{d^S} = \bigcap_{c \in S} A^c$ is itself a closure operation.*

Proof. By Lemma 3.27 it is straightforward.

Lemma 3.36 *d^S is the smallest closure operation lying above d.*

Proof. Let e be a closure operation and $d \leq e$. Then for each ideal A of X, $A^d \subseteq A^e$. Since $e \in S$, we have $A^{d^S} = \bigcap_{c \in S} A^c \subseteq A^e$.

We inductively define a new closure operation on X and show that if X is a Noetherian, then it is equivalent to d^S.

Definition 3.37 *Suppose that X be a BCK-algebra and d be an operation on the set of ideals of X, which satisfies the extension and order-preservation conditions. We define inductively d^n for a natural number n as follows:*

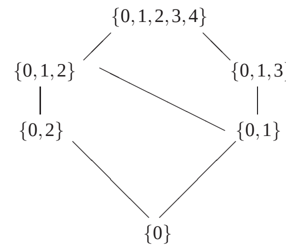
Suppose $d = d^1$ and for a natural number n ($n \geq 2$), $A^{d^n} = (A^{d^{n-1}})^d$.

Moreover, for an ideal A of X, we let $A^{d^{\infty}} = \bigcup_{n \geq 1} A^{d^n}$

Example 3.38 *Let X be the set $\{0,1,2,3,4\}$. Define a binary operation $*$ on X by the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

X is a BCK-algebra. The following figure describe the Hasse diagram of ideals of X.



Define d on ideals as follows:

$(\{0\})^d = \{0,2\}, (\{0,2\})^d = \{0,1,2\}, (\{0,1,2\})^d = \{0,1,2,3,4\}, (\{0,1\})^d = \{0,1,2\}$
$(\{0,1,3\})^d = \{0,1,2,3,4\}$ *and* $(\{0,1,2,3,4\})^d = \{0,1,2,3,4\}$

d is an operation on ideals of X, which satisfies the extension and order-preservation conditions. For the ideal $\{0,1\}$ of X we have.

$(\{0,1\})^d = \{0,1,2\}, (\{0,1\})^{d^2} = ((\{0,1\})^d)^d = (\{0,1,2\})^d = \{0,1,2,3,4\}$

$$(\{0,1\})^{d^3} = ((\{0,1\})^{d^2})^d = \{0,1,2,3,4\}^d = \{0,1,2,3,4\}$$

Similarly, we have $(\{0,1\})^{d^n} = \{0,1,2,3,4\}$ for all $n \geq 3$ and,

$$(\{0,1\})^{d^{\infty}} = \bigcup_{n \geq 1} (\{0,1\})^{d^n} = \{0,1,2,3,4\}$$

Theorem 3.39 *Let X be a Noetherian BCK-algebra. Then for each ideal A of X;*

(i) d^{∞} is a closure operation on X.

(ii) $A^{d^{\infty}} = A^{d^S}$

Proof. (i) Since d has extension property, for each ideal A, we have $A \subseteq A^d$. Now by induction for each natural number n, $A \subseteq A^{d^n}$. Hence $A \subseteq \bigcup_n A^{d^n} = A^{d^{\infty}}$. Therefore d^{∞} has extension property. For any two ideals A and B of X such that $A \subseteq B$. By induction on n, we can prove, operation d^n has order-preservation property. So for any natural number n, $A^{d^n} \subseteq B^{d^n}$. Hence $A^{d^{\infty}} \subseteq B^{d^{\infty}}$. For idempotence, since X is Noetherian BCK-algebra and the sequence $\{A^{d^n}\}$ is ascending sequence, there exists a natural number n such that $A^{d^{\infty}} = A^{d^n}$. Hence $(A^{d^{\infty}})^{d^{\infty}} = (A^{d^n})^{d^{\infty}} = \bigcup_{l \in N} (A^{d^n})^{d^l} = \bigcup_{l \in N} A^{d^{n+l}} = A^{d'}$.

(ii) We show that $A^{d^{\infty}} \subseteq A^{d^S}$. It is enough to prove that for each operation c in S and each ideal A of X, $A^{d^n} \subseteq A^c$. Using induction on natural number n, we have: For $n = 1$ it is obviouse. Let $n \geq 2$ and for each operation $c \in S$ and each ideal A of X, $A^{d^{n-1}} \subseteq A^c$. Then $A^{d^n} = (A^{d^{n-1}})^d \subseteq (A^c)^d$ and since $d \leq c$, $(A^c)^d \subseteq (A^c)^c = A^c$. Hence $A^{d^n} \subseteq A^c$. By using Lemma 3.36, d^S is the smallest closure operation above d and $d \leq d^1 \leq d^S$. Therefore $d^1 = d^S$.

After some fundamental results concerning closure operation, we present some result regarding quotient algebra. From [10], we know that if A be an ideal of a

BCK-algebra X, Then quotient algebra $\frac{X}{A}$ is still be BCK-algebra. Also, if X is bounded, so is $\frac{X}{A}$.

Theorem 3.40*Let $(X,*,0)$ be a BCK-algebra and A be an ideal of X. Define $\varphi : \frac{X}{A} \longrightarrow \frac{X}{A^{cl}}$ by $\varphi(A_x) = (A^{cl})_x$. Then φ is a BCK-epimorphism with $Ker(\varphi) = \frac{A^{cl}}{A}$*

*Proof.*Since X is a BCK-algebra, $\frac{X}{A}$ and $\frac{X}{A^{cl}}$ are BCK-algebras too. Also,

$$\varphi(A_x * A_y) = \varphi(A_{x*y}) = (A^{cl})_{x*y} = A^{cl}_x * A^{cl}_y = \varphi(A_x) * \varphi(A_y)$$

Therefore φ is a BCK-homomorphism. It is clear that φ is onto. Now we have

$$Ker(\varphi) = \{A_x \in \frac{X}{A}; \varphi(A_x) = A^{cl}_0\} = \{A_x \in \frac{X}{A}; A^{cl}_x = A^{cl}_0\}$$

Note that $A^{cl}_x = A^{cl}_0$ means that $x \in A^{cl}_0$. Hence $x * 0 = x \in A^{cl}$. Therefore $Ker(\varphi) = \frac{A^{cl}}{A}$.

Theorem 3.41*Let $(X,*,0)$ be a BCK-algebra, A be an ideal of X and "cl" be a closure operation on $\mathscr{I}_X$. Define $\eta : \mathscr{I}(\frac{X}{A^{cl}}) \longrightarrow \mathscr{I}(\frac{X}{A^{cl}})$ such that for each ideal $\frac{I}{A^{cl}}$ of $\frac{X}{A^{cl}}$, $(\frac{I}{A^{cl}})^{\eta} = \frac{I^{cl}}{A^{cl}}$. Then, "$\eta$" is a closure operation.*

*Proof.*Extension property: Since "cl" is a closure operation, $I \subseteq I^{cl}$. Hence

$$\frac{I}{A^{cl}} \subseteq \frac{I^{cl}}{A^{cl}} = (\frac{I}{A^{cl}})^{\eta}.$$

Order-preservation: Suppose that I and J are two ideals of X such that $\frac{I}{A^{cl}} \subseteq \frac{J}{A^{cl}}$. So $I \subseteq J$ and $I^{cl} \subseteq J^{cl}$. Therefore $(\frac{I}{A^{cl}})^{\eta} = \frac{I^{cl}}{A^{cl}} \subseteq \frac{J^{cl}}{A^{cl}} = (\frac{J}{A^{cl}})^{\eta}$.
Idempotency:

$$((\frac{I}{A^{cl}})^{\eta})^{\eta} = (\frac{I^{cl}}{A^{cl}})^{\eta} = \frac{(I^{cl})^{cl}}{A^{cl}} = \frac{I^{cl}}{A^{cl}} = (\frac{I}{A^{cl}})^{\eta}$$

4 Closure Operations of Finite Type

In this section, we study finite type closure operations on ideals of a BCK-algebra.

Lemma 4.1*Let "c" be a closure operation. Consider "c_f" by setting $A^{c_f} = \bigcup\{B^c | B$ is a finitely generated ideal such that $B \subseteq A\}$. Then "c_f" is a closure operation.*

*Proof.*Extension property: let $x \in A$. If $B = (x]$, then $B \subseteq A$. Now by extension property of c, $B \subseteq B^c$. Hence $x \in B^c$. Therefore $A \subseteq A^{c_f}$. Order-preservation: Suppose A and B are two ideals of X such that $A \subseteq B$. If $x \in A^{c_f}$, then there exists a finitely generated ideal B' such that $B' \subseteq A$ and $x \in (B')^c$. Now by assumption $B' \subseteq A \subseteq B$. Hence by order-preservation of c, $x \in B^{c_f}$. Therefore $A^{c_f} \subseteq B^{c_f}$. Idempotency: Suppose $z \in (A^{c_f})^{c_f}$; then there is a finitely generated ideal $B \subseteq A^{c_f}$ such that $z \in B^c$. Let $\{z_1, z_2, \ldots, z_n\}$ be a finite generating set for B. Since each $z_i \in A^{c_f}$, then there exists a finitely generated ideal $K_i \subseteq A$ such that $z_i \in K_i^c$. Now let $K = \sum_{i=1}^{n} K_i$. Then for each i, $K_i^c \subseteq K^c$. So $B \subseteq K^c$. Hence $z \in B^c \subseteq (K^c)^c = K^c$ and since K is a finitely generated sub-ideal of A; it follows that $z \in A^{c_f}$. Therefore $A^{c_f} = (A^{c_f})^{c_f}$.

Definition 4.2*If $c = c_f$ (Lemma 4.1), we say that "c" is a closure operation of finite type.*

Example 4.3*The identity and indiscrete closure operations are of finite type. To show this, it is enough to prove that $c \leq c_f$. Because by definition of finite type closure operation, we have $c_f \leq c$.*
(i) Let c be an identity closure operation on X and $x \in A^c = A$ for an arbitrary ideal A. Then $x \in (x]^c = (x]$ and $(x] \subseteq A$. Thus by using Definition 4.2, $x \in A^{c_f}$. Therefore $A \subseteq A^{c_f}$ and c is a finite type closure operation.
(ii) Similarly, if c be an indiscrete closure operation on X and $x \in A^c = X$ for an arbitrary ideal A, then $x \in (x]^c = X$ and $(x] \subseteq X$. Therefore $x \in A^{c_f}$ and c is a finite type closure operation.

Theorem 4.4*Let "c" be a closure operation on a BCK-algebra X. Then:*
(i) For every finitely generated ideal A of X; $A^c = A^{c_f}$
(ii) The closure operation "c_f"; is a finite type closure operation.
(iii) The closure operation "c_f" is the greatest finite type closure operation in which $c_f \leq c$.

Proof.(i) Let K be a finitely generated ideal of X. Then by using Lemma 4.1, $K^{c_f} \subseteq K^c$. On the other hand since K is finitely generated, $K^c \subseteq K^{c_f} = \cup\{B^c | B$ is a finitely generated ideal, $B \subseteq K\}$. Thus $K^c = K^{c_f}$.
(ii) By using part (i) for each finitely generated ideal K of X, we have $K^c = K^{c_f}$. So the following two sets are equal. $A^{(c_f)_f} = \cup\{K^{c_f} | K \subseteq A$ and K is finitely generated ideal$\}$ and $A^{c_f} = \cup\{K^c | K \subseteq A$ and K is finitely generated ideal $\}$.
(iii) Let d be a finite type closure operation and $d \leq c$. Then by Lemma 4.1, we have $d_f \leq c_f$.

Lemma 4.5*Let "c" be a finite type closure operation on X and $\{A_\lambda\}_{\lambda \in \Lambda}$ be a chain of $c-closed$ ideals of X. Then the union of this chain is a $c-closed$ ideal too.*

*Proof.*By extension property of "c" we know $(\cup_{\lambda \in \Lambda} A_\lambda) \subseteq (\cup_{\lambda \in \Lambda} A_\lambda)^c$. Now we show that $(\cup_{\lambda \in \Lambda} A_\lambda)^{c_f} \subseteq (\cup_{\lambda \in \Lambda} A_\lambda)$. Let B be a finitely generated ideal of X and $B \subseteq (\cup_{\lambda \in \Lambda} A_\lambda)$. If $B = (a_1, a_2, \ldots, a_n]$, then for each i there exists $\lambda_i \in \Lambda$

such that $a_i \in A_{\lambda_i}$. Since $\{A_\Lambda\}_{\lambda\in\Lambda}$ is a chain of $c-closed$ ideals, then there exists $\beta \in \Lambda$ such that $a_i \in A_\beta$ for each i, $1 \le i \le n$. Thus $B \subseteq A_\beta$ and $B^c \subseteq A^c_\beta$. So $B^c \subseteq A_\beta$. Now by definition of c_f;

$$(\cup_{\lambda\in\Lambda}A_\lambda)^{c_f} \subseteq (\cup_{\lambda\in\Lambda}A_\lambda).$$

Since "c" is a finite type closure operation, we have:

$$(\cup_{\lambda\in\Lambda}A_\lambda)^{c_f} = (\cup_{\lambda\in\Lambda}A_\lambda).$$

Theorem 4.6*Let "c" be a closure operation of finite type on a BCK-algebra X. Then every $c-closed$ ideal contained in a maximal $c-closed$ ideal.*

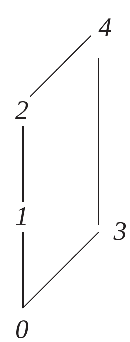
*Proof.*We prove the theorem by Zorn's lemma. Suppose that A be a "$c-closed$" ideal of X. Let $\sum = \{B|B$ is "$c-closed$" ideal of X and $A \subseteq B\}$. $\sum$ is not empty set because $A \in \sum$ and $\sum$ is a partial ordered set by inclusion. Now let T be a chain in $\sum$. If $K = \cup_{B\in T}B$, then K is an ideal of X and $A \subseteq K$. Also by Lemma 4.5, K is a "$c-closed$" ideal. Hence $K \in \sum$. For each ideal H of $\sum$, $H \subseteq K$. So K is an upper bound for T in $\sum$. By Zorn lemma $\sum$ has a maximal element.

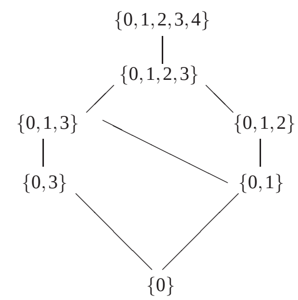
In the following example, we show that the converse of Theorem 4.6 is not true.

Example 4.7*Suppose that X is the set $\{0,1,2,3,4\}$. Define a binary operation $*$ on X by the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

X is a BCK-algebra. The following figures describe the Hasse diagrams of elements and ideals of X.





Define cl on ideals as follows:
$(\{0\})^{cl} = \{0\}, (\{0,3\})^{cl} = \{0,1,3\}, (\{0,1,3\})^{cl} = \{0,1,3\}, (\{0,1\})^{cl} = \{0,1,2\}, (\{0,1,2\})^{cl} = \{0,1,2\}$
$(\{0,1,2,3\})^{cl} = \{0,1,2,3,4\}$ *and* $(\{0,1,2,3,4\})^{cl} = \{0,1,2,3,4\}$.

cl is a closure operation and "$cl-closed$" ideals of X are $\{0\}, \{0,1,3\}, \{0,1,2\}, \{0,1,2,3,4\}$. Clearly every "$cl-closed$" ideal contained in a maximal "$cl-closed$" ideal. But cl is not a finite type closure operation. Because $(\{0,1,3\})^{cl} = \{0,1,3\}$ *and* $(\{0,1,3\})^{cl_f} = \cup\{B^{cl} \mid B \subseteq \{0,1,3\}, B \text{ is a finitely generated ideal of } X\} =$

$$(\{0\})^{cl} \cup (\{0,3\})^{cl} \cup (\{0,1\})^{cl} \cup (\{0,1,3\})^{cl} = \{0,1,2,3\}.$$

Therefore $(\{0,1,3\})^{cl_f} = \{0,1,2,3\} \neq (\{0,1,3\})^{cl} = \{0,1,3\}$ *and cl is not a finite type closure operation.*

Remark 4.8*Since every commutative BCK-algebra is a lower BCK-semilattice, all of the above results hold for a commutative BCK-algebra too.*

5 Conclusions and future works

As we mentioned in the abstract, in this article we give the notions of closure operation, cl-closed, finite type and then obtain some different closure oprations together with somemore related results. Now there are some ideas and questions:
(i) How we can define some other types of closure operation, e.g. semi-prime, meet and prime closure operation.
(ii) Can we obtain some relationship between different types of closure operations.
(iii) Can we generalized these ideas to hyper BCK (K)-algebra.
We will try to work on these ideas and give the results in the forthcoming papers.

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