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On Order Ideal Convergence in a Metric Additive System

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Abstract: In this paper we study the concept of Ideal convergence in a linearly ordered additive system associated with the order convergence endowed with a particular metric and we introduce the idea of order ideal convergence.

Keywords: Additive system, order convergence, natural density, statistical convergence, ideal convergence.

1 Introduction

The idea of I-convergence in real numbers was introduced by Kostyrko, Šalát and Wilezyński [9] in 2000 and it is an interesting generalization of statistical convergence. The notion of statistical convergence was introduced in 1951 by Fast [5] and Schoenberg [14] independently and it was discussed and developed by several authors viz. [6, 10, 15]. Many authors [4, 8, 13, 16, 17, 18, 19, 20] developed the concept of I-convergence based on the notion of ideal I of subsets of the set $\mathbb{N}$ of natural numbers in different spaces.
Recently the concept of statistical convergence has been studied in a linearly ordered additive system associated with the order convergence with respect to a particular metric in [3].
The order convergence is one of the main concept used in this paper and it was described and developed by many authors including [1, 2, 7, 12].
The main purpose of this paper is to examine whether the concept of I-convergence is extendable in a linearly ordered metric additive system mentioned in [3] and we introduce the concept of OI-convergence and study some basic properties of this convergence.

2 Definitions and notations

First we recall the definition of natural density of a subset of natural numbers $\mathbb{N}$ and the idea of statistical convergence.
Definition 2.1. ([11]) If K is a subset of the set of positive integers $\mathbb{N}$ then the natural density of K is defined by,

$$\delta(K) = lim_{n\to\infty} \frac{|K_n|}{n}, \text{ where}$$

$K_n = \{k \leq n : k \in K\}$ and $|K_n|$ is the number of elements of K_n.

Definition 2.2. ([5]) A sequence $\{x_n\}$ of real numbers is said to be statistically convergent to some number ξ, if for any $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0.$$

If $\{x_n\}$ is statistically convergent to ξ, then we write $st - lim_n x_n = \xi$.

We now mention the idea of order convergence and a particular metric γ in a linearly ordered additive system L introduced in the paper [21] and also recall definition of an ideal.

Definition 2.3. Let L be a set of the elements $x, y, z, \ldots$ and $\leq$ is a binary relation defined for all pairs (x, y) for $x, y \in L$.
We say that L is partially ordered set with respect to $\leq$, if for all $x, y, z \in L$
(i) $x \leq x$ for all $x \in L$,
(ii) $x \leq y$ and $y \leq x$ implies $x = y$ and
(iii) $x \leq y$ and $y \leq z$ implies that $x \leq z$.
If $x \leq y$ and $x \neq y$, we write $x < y$. The relation $x \leq y$ is also written as $y \geq x$. Similarly, $x < y$ is also written as $y > x$.
A partially ordered set L is said to be a lattice if every two

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elements $x, y \in L$ possess a least upper bound $x \vee y \in L$ and a greatest lower bound $x \wedge y \in L$.
L is said to be an additive system if for every two elements $x, y \in L$ there exists a least upper bound $x \vee y$ in L and L is said to be a multiplicative system if for every two elements $x, y \in L$ there exists a greatest lower bound $x \wedge y$ in L.

An element θ in L is the null element of L if $x \vee \theta = x$ for all $x \in L$.
If L is a partially ordered set, we say that a sequence $\{x_i\}$ is increasing (decreasing) if $x_i \leq x_j$ $(a_i \geq a_j)$ for $i < j$.

Note 2.4. To denote a monotone increasing (decreasing) sequence $\{x_n\} \in L$ we use the notation $x_n \uparrow$ $(x_n \downarrow)$. The notation $x_n \downarrow x$ means that $x_n \downarrow$ and $\inf x_n = x$. The meaning of the notation $x_n \uparrow x$ is similar.

Definition 2.5. ([7]) A sequence $\{x_n\}$ in an additive system L is said to be order convergent (O-convergent) to $\xi \in L$ if there exists a sequence $\{y_n\}$ of elements of L with $y_n \downarrow \theta$ such that

$$|x_n - \xi| < y_n \text{ for each } n \in \mathbb{N},$$

where in L, $|x| = x^+ + x^-$ and $x^+ = x \vee \theta$, $x^- = (-x) \vee \theta$.

Definition 2.6. [21] (i) Let L be an additive system and D be a real valued function defined on L. Then a function γ is defined on L by

$$\gamma(a,b) = 2D(a \vee b) - D(a) - D(b).$$

$D(a)$ is said to be monotone increasing (decreasing) when

$$D(a) \leq D(b) (D(a) \geq D(b)) \text{ for } a < b.$$

(ii) Let L be an additive system and $\gamma(a,b)$ be real valued function defined for every pair $(a,b) \in L$; then define

$$\Delta(a,b,c) = \tfrac{1}{2}\{\gamma(a,b) + \gamma(b,c) - \gamma(a,c)\} \text{ for } a,b,c \in L.$$

The following proposition is immediate.

Proposition 2.7. ([21]) If $D(a)$ is a real valued function defined on an additive system L, then for $a, b \in L$
(i) $D(a) - D(b) = \gamma(a,b)$ if $a \geq b$
(ii) If $D(a)$ is monotone increasing, then $|D(a) - D(b)| \leq \gamma(a,b)$
(iii) $\gamma(a,b) = \gamma(b,a)$, $\gamma(a,a) = 0$
(iv) $\Delta(a, a \vee b, b) = 0$
(v) $D(a)$ is monotone increasing if and only if $\gamma(a,b) \geq 0$
(vi) $D(a)$ is properly monotone increasing if and only if $\gamma(a,b) > 0$ for $a \neq b$.

Note 2.8. If $D(a)$ is monotone increasing and $\Delta(a,b,c) \geq 0$ for every $a,b,c \in L$, then $\gamma(a,b)$ is a metric on L.
In this connection we mention the following result from the paper [21].

Result 2.9. If $D(a)$ is a real valued function defined on an additive system L, then
(A) $\Delta(a,b,c) \geq 0$ for every $a,b,c \in L$ implies the following equivalent statements .
(i) $\gamma(a \vee c, b \vee c) \leq \gamma(a,b)$ for all $a, b \in L$
(ii) $\gamma(a \vee c, b \vee c) \leq \gamma(a,b)$ for all $b \leq a$
(iii) $D(a \vee c) + D(b) \leq D(a) + D(c \vee b)$ for $b \leq a$
(iv) $\gamma(a \vee c, b \vee d) \leq \gamma(a,b) + \gamma(c,d)$
(B) If $D(a)$ is monotone increasing, then $\Delta(a,b,c) \geq 0$ if and only if one of the equivalent statements $(i) - (iv)$ holds.

Here we mention the concept of order statistical convergence in the metric additive system (L, γ).

Definition 2.10. [3] A sequence $\{x_n\}_n$ in a metric additive system (L, γ) is said to be order statistically convergent (i.e *ost*-convergent) to $x \in L$ if, there exists a sequence $\{y_n\}_n$ in L with $y_n \downarrow \theta$ such that

$$\delta(\{k \in \mathbb{N} : \gamma(x_k, x) \geq D(y_k)\}) = 0,$$

where D is a real valued monotone increasing function on L with $D(\theta) = 0$ and $\Delta(a,b,c) \geq 0$ for all $a,b,c \in L$.

We now recall the concept of an ideal and filter of a non-empty set and I-convergence of a sequence.

Definition 2.11. [9] Let $X \neq \emptyset$. A family of sets $I \subseteq 2^X$ is said to be an ideal in X provided I satisfies the following conditions:
(a) $\phi \in I$,
(b) $A \cup B \in I$ if $A, B \in I$,
(c) If $A \in I$ and $B \subseteq A$ then $B \in I$.

Definition: 2.12. [9] Let X be a non-empty set. A non-empty family $F \subseteq 2^X$ is said to be a filter on X if the following conditions are satisfied:
(a) $\phi \notin F$,
(b) $A \cap B \in F$ if $A, B \in F$,
(c) If $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

An ideal I is said to be non-trivial if $I \neq \phi$ and $X \notin I$.
A non-trivial ideal I is said to be admissible in X if $\{x\} \in I$ for each $x \in X$.

Lemma 2.13. [9] I is a non-trivial ideal in X if and only if the family of sets $F(I) = \{M \subseteq X : X - M \in I\}$ is a filter in X.
It is called the filter associated with the ideal I.

Definition 2.14. [9] Let I be a non-trivial ideal of subsets of $\mathbb{N}$, the set of natural numbers and (X, ρ) be a metric space. A sequence $x = \{x_n\}$ of elements of X is said to be I-convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \varepsilon\} \in I$.

If $x = \{x_n\}$ is I-convergent to ξ, then ξ is called the I-limit of the sequence x and we denote it by $I - \lim_{n\to\infty} x_n = \xi$.

Definition 2.15. [9] Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (X,ρ) be a metric space. A sequence $x = \{x_n\}$ of elements of X is said to be I^*-convergent to $\xi \in X$ if there exists a set $M \in F(I)$ with $M = \{m_1 < m_2 < m_3 < ....\} \subseteq \mathbb{N}$ such that $\lim_{n\to\infty} \rho(x_{m_n}, \xi) = 0$.

Definition: 2.16. [9] An admissible ideal I of subsets of $\mathbb{N}$ is said to have AP-property if for any sequence $\{A_1, A_2, A_3, ...\}$ of mutually disjoint sets of I, there exists a sequence $\{B_1, B_2, B_3...\}$ such that for each $i \in \mathbb{N}$ the symmetric difference $A_i \Delta B_i$ is finite and $\cup_{i=1}^{\infty} B_i \in I$.

3 Order ideal convergence

Following the idea of *ost*-convergence we introduce the concept of order ideal convergence in the metric additive system (L,γ) where γ is a metric defined in [21] and study some general properties related to this convergence.

Definition 3.1. Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. A sequence $x = \{x_n\}$ of elements of L is said to be order ideal convergent (OI-convergent) to $\xi \in L$ if there exists a sequence $y = \{y_n\} \in L$ with $y_n \downarrow \theta$ such that the set $A = \{n \in \mathbb{N} : \gamma(x_n, \xi) \geq D(y_n)\} \in I$, where D is a real valued monotone increasing function defined on L with $D(\theta) = 0$ and $\Delta(a,b,c) \geq 0$ for all $a,b,c \in L$.

The number ξ is called the order ideal limit (OI-limit) of the sequence $x = \{x_n\}$ and we write $OI - \lim x_n = \xi$.
Throughout the paper we consider D to be a monotone increasing real valued function with $D(\theta) = 0$ and $\Delta(a,b,c) \geq 0$ for all $a,b,c \in L$.

Note 3.2. From the definition of OI-convergence it is clear that an OI-convergent sequence is I-convergent. In particular if D be an identity map and $L = \mathbb{R}$, then γ becomes the usual metric on $\mathbb{R}$. In this case OI-convergence is equivalent to the I-convergence of real numbers.

Example 3.3. If I_f is the family of all finite subsets of $\mathbb{N}$ then I_f is an admissible ideal on $\mathbb{N}$ and the OI-convergence coincides with the ordinary convergence.

Example 3.4. If $I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ then I_δ is an admissible ideal in $\mathbb{N}$ and the OI-convergence coincides with the order statistical convergence.

We give an example of a sequence which is OI-convergent but not convergent in (L,γ) in ordinary sense.

Example 3.5. Consider the ideal I_f and let $L = \mathbb{R}$ with D as the identity mapping. Then clearly (L,γ) becomes the usual metric space.
Consider a sequence $\{x_n\}$ in $\mathbb{R}$ as follows:

$$x_n = \begin{cases} 1, \text{ if } n \text{ is a square} \\ 0, \text{ otherwise} \end{cases}$$

Let $\{y_n\}$ be a sequence in $\mathbb{R}$ such that $y_n = \frac{1}{n}$. Then $\{n \in \mathbb{N} : \gamma(x_n, 0) \geq D(y_n)\} \in I$. So $OI - \lim x_n = 0$, but $\{x_n\}$ is not convergent with respect to the metric γ.

Theorem 3.6. If I is a non-trivial ideal, then OI-limit of any sequence if exists, is unique.

Proof: Let $x = \{x_n\}$ be a sequence in L such that x is OI-convergent to ξ as well as ν and suppose $\xi \neq \nu$. Let $\varepsilon = \frac{1}{3}\gamma(\xi,\nu)$. Then $B(\xi,\varepsilon) \cap B(\nu,\varepsilon) = \phi$ where $B(\xi,\varepsilon)$ is the open ball with centre at ξ and ε as the radius.
Since x is OI-convergent to both ξ and ν, then there exists two sequences $\{y_n^{(1)}\}$ and $\{y_n^{(2)}\}$ in L with $y_n^{(1)} \downarrow \theta$ and $y_n^{(2)} \downarrow \theta$ such that $\{k \in \mathbb{N} : \gamma(x_k, \xi) \geq D(y_k^{(1)})\} \in I$ and $\{k \in \mathbb{N} : \gamma(x_k, \nu) \geq D(y_k^{(2)})\} \in I$. Now $y_n^{(1)} \downarrow 0$ and $y_n^{(2)} \downarrow 0$ implies that there exists $n_0 \in \mathbb{N}$ such that $D(y_n^{(1)}) < \varepsilon/2$ and $D(y_n^{(2)}) < \varepsilon/2$ for all $n \geq n_0$. Then for $k \geq n_0$, $\{k \in \mathbb{N} : \gamma(x_k, \xi) < D(y_k^{(1)})\} \subseteq \{k \in \mathbb{N} : \gamma(x_k, \xi) < \varepsilon/2\}$. So for $k \geq n_0$, $A = \{k \in \mathbb{N} : \gamma(x_k, \xi) < \varepsilon/2\} \in F(I)$ since $\{k \in \mathbb{N} : \gamma(x_k, \xi) < D(y_k^{(1)})\} \in F(I)$. Similarly for $k \geq n_0$, $B = \{k \in \mathbb{N} : \gamma(x_k, \nu) < \varepsilon/2\} \in F(I)$. Thus for $k \geq n_0$, $A \cap B \in F(I)$ and $A \cap B \neq \phi$ which is a contradiction and hence the proof.

Lemma 3.7. If $x = \{x_n\} \in L$ is such that $\lim_{n\to\infty} x_n = \xi$ with respect to the metric γ, then there exists a sequence $\{\alpha_n\} \in L$ with $\alpha_n \downarrow \theta$ such that $\gamma(x_n, \xi) < D(\alpha_n)$, for all $n \in \mathbb{N}$.

Proof: Since $\lim_{n\to\infty} x_n = \xi$, then for $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\gamma(x_n, \xi) < \varepsilon$ for all $n \geq m$.
Let $\{y_n\}$ be a sequence in L such that $y_n \downarrow \theta$. Then for each y_i there exists a smallest positive integer m_i such that $\gamma(x_n, \xi) < D(y_i)$ for all $n \geq m_i$, $i = 1,2,3,...$
Choose $z_1 \in L$ such that, $D(z_1) \geq max\{D(y_1), \gamma(x_1, \xi), \gamma(x_2, \xi), ..., \gamma(x_{m_1-1}, \xi)\}$,

Choose $z_2 \in L$ such that,
$\gamma(x_{m_1}, \xi) \geq D(z_2) > max\{D(y_2), \gamma(x_{m_1+1}, \xi), \gamma(x_{m_1+2}, \xi), ..., \gamma(x_{m_2-1}, \xi)\}$,

Choose $z_3 \in L$ such that,
$\gamma(x_{m_2}, \xi) \geq D(z_3) > max\{D(y_3), \gamma(x_{m_2+1}, \xi), \gamma(x_{m_2+2}, \xi), ..., \gamma(x_{m_3-1}, \xi)\}$,

and so on.
Now set,

$$\begin{aligned}\alpha_i &= z_1;\ i = 1,2,.....,m_1-1\\ &= y_1;\ i = m_1\\ &= z_2;\ i = m_1+1, m_1+2,.....,m_2-1\\ &= y_2;\ i = m_2\\ &..\ ....\end{aligned}$$

Then

$$\gamma(x_n,\xi) < D(\alpha_n), \text{ for all } n \in \mathbb{N} \text{ and } \alpha_n \downarrow \theta.$$

Theorem 3.8. If I is a non-trivial ideal and $x = \{x_n\} \in L$ be such that $\lim_{n\to\infty} x_n = \xi$ with respect to the metric γ, then $OI - \lim x_n = \xi$.

Proof: Let $x = \{x_n\} \in L$ be a sequence such that $\lim_{n\to\infty} x_n = \xi$ with respect to the metric γ. Then by Lemma 3.7 there exists a sequence $\{\alpha_n\} \in L$ with $\alpha_n \downarrow \theta$ such that $\gamma(x_n,\xi) < D(\alpha_n)$, for all $n \in \mathbb{N}$.
Then $\{n \in \mathbb{N} : \gamma(x_n,\xi) \geq D(\alpha_n)\} = \phi \in I$. So, $OI - \lim x_n = \xi$.

Theorem 3.9. If I is a non-trivial ideal and if $\{x_n\}$ and $\{y_n\}$ are two sequences in L such that $OI - \lim x_n = \xi$ and $OI - \lim y_n = \nu$, then $OI - \lim(x_n \vee y_n) = \xi \vee \nu$.

Proof: Since $OI - \lim x_n = \xi$ and $OI - \lim y_n = \nu$, then there exists sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in L with $\alpha_n \downarrow \theta$ and $\beta_n \downarrow \theta$ such that
$A = \{n \in \mathbb{N} : \gamma(x_n,\xi) \geq D(\alpha_n)\} \in I$ and $B = \{n \in \mathbb{N} : \gamma(y_n,\nu) \geq D(\beta_n)\} \in I$.
Let $p \in A^c \cap B^c$. Clearly $\gamma(x_p,\xi) < D(\alpha_p)$ and $\gamma(y_p,\nu) < D(\beta_p)$. Since D is an increasing function, then by using Result 2.9(B) we have

$$\gamma(x_p \vee y_p, \xi \vee \nu) \leq \gamma(x_p,\xi) + \gamma(y_p,\nu) < D(\alpha_p) + D(\beta_p).$$

Since $\alpha_n \downarrow \theta$ and $\beta_n \downarrow \theta$ we can consider a sequence $\{\delta_n\} \in L$ with $\delta_n \downarrow \theta$ and $D(\delta_n) \geq D(\alpha_n) + D(\beta_n)$ for all $n \in \mathbb{N}$. Then $\gamma(x_p \vee y_p, \xi \vee \nu) < D(\delta_p)$.
Let $C = \{n \in \mathbb{N} : \gamma(x_n \vee y_n, \xi \vee \nu) \geq D(\delta_n)\}$. Then $p \in C^c$ and hence $A^c \cap B^c \subseteq C^c$. This implies that $C \subseteq A \cup B \in I$ since $A, B \in I$ and consequently $OI - \lim(x_n \vee y_n) = \xi \vee \nu$.

Definition 3.10 Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. A sequence $x = \{x_n\}$ of elements in L is said to be order ideally bounded(i.e. OI-bounded) in L if there exists $B \in \mathbb{R}$ such that the set $\{n \in \mathbb{N} : D(x_n) \geq B\} \in I$.

Theorem 3.11. Let I be a non-trivial ideal of subsets of $\mathbb{N}$. An OI-convergent sequence in the metric additive system (L,γ) is OI-bounded.

Proof: Let $x = \{x_n\}$ be a sequence in L such that $OI - \lim x_n = \xi$. Then there exists a sequence $\{y_n\}$ in L with $y_n \downarrow \theta$ such that $\{n \in \mathbb{N} : \gamma(x_n,\xi) \geq D(y_n)\} \in I$. i.e. $A = \{n \in \mathbb{N} : \gamma(x_n,\xi) < D(y_n)\} \in F(I)$.
Let $p \in A$. Then $\gamma(x_p,\xi) < D(y_p)$
i.e., $2D(x_p \vee \xi) - D(x_p) - D(\xi) < D(y_p)$. Then $D(x_p) \leq 2D(x_p \vee \xi) - D(x_p) < D(y_p) + D(\xi)$.
Since $y_n \downarrow \theta$, then $D(y_n) \downarrow 0$ and consequently, $\{D(y_n)\}$ is bounded and we can choose a real number M such that $M = \sup\{D(y_p) : p \in A\}$. Clearly $D(x_p) < D(\xi) + M$ and so $A \subseteq \{k \in \mathbb{N} : D(x_k) < D(\xi) + M\} \in F(I)$. Hence the proof.

Theorem 3.12 Let I be an admissible ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. If I contains an infinite set, then there exists an OI-convergent sequence $\{x_n\}$ in L, which has subsequence, which does not converge to the same limit.

Proof: Let A be an infinite set in I and $A = \{n_1, n_2, n_3, ...\}$ with $n_1 < n_2 < n_3 < ....$
Again let $B = \mathbb{N} - A = \{m_1, m_2, m_3, ...\}$ with $m_1 < m_2 < m_3 < ....$
Since I is admissible then B is also an infinite set.
Let us choose $\eta, \xi \in L$ such that $\eta \neq \xi$ and consider a sequence $\{x_n\} \in L$ such that

$$\begin{aligned}x_k &= \eta;\ if\ k \in A,\\ &= \xi;\ if\ k \in B.\end{aligned}$$

We choose a sequence $\{y_n\}$ of non-null elements in L such that $y_n \downarrow \theta$. This implies that $\{n \in \mathbb{N} : \gamma(x_n,\xi) \geq D(y_n)\} \subseteq A \in I$. Clearly, $OI - \lim x_k = \xi$. But $\{n_k \in \mathbb{N} : \gamma(x_{n_k},\eta) \geq D(y_n)\} = \phi \in I$ and consequently the subsequence $\{x_{n_k}\}$ is OI-convergent to η.

Theorem 3.13. Let I be an admissible ideal of subsets of natural numbers and each sequence $x = \{x_n\}$ in L has a subsequence which is OI-convergent to ξ, then x is OI-convergent to ξ.

Proof: Let $x = \{x_n\}$ be a sequence in L such that each subsequence of x has a subsequence that is OI-convergent to ξ but $OI - \lim x_n \neq \xi$.
Then for each $\{y_n\} \in L$ with $y_n \downarrow \theta$, $A = \{n \in \mathbb{N} : \gamma(x_n,\xi) \geq D(y_n)\} \notin I$. i.e. $A \in F(I)$ and A is an infinite set since I is admissible.
Let $A = \{n_1 < n_2 < n_3 < ...\}$ and $\{x_{n_k}\}$ be a subsequence of x. Then if we choose any subsequence $\{x_{p_k}\}$ of $\{x_{n_k}\}$, then clearly
$\{p_k \in \mathbb{N} : \gamma(x_{p_k},\xi) \geq D(y_{p_k})\} \notin I$ which is a contradiction. Therefore, $OI - \lim x_n = \xi$.

Definition 3.14. Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. A sequence $x = \{x_n\}$ of elements of L is said to be OI^*-convergent to $\xi \in L$ if there exists a set $M \in F(I)$ with $M = \{m_1 < m_2 < m_3 < .....\}$ and $\lim_{k\to\infty} x_{m_k} = \xi$ with respect to the metric γ.

Theorem 3.15. Let I be a non-trivial ideal of subsets of $\mathbb{N}$. If $\{x_n\}$ is a sequence in L such that $OI^* - \lim x_n = \xi$, then $OI - \lim x_n = \xi$.

Proof: Let $OI^* - \lim x_n = \xi$. Then there exists $M \in F(I)$ with $M = \{m_1 < m_2 < m_3 < .....\}$ such that $\lim_{k\to\infty} x_{m_k} = \xi$. Then we can choose $\{\beta_n\}$ in L with $\beta_n \downarrow \theta$ by using Lemma 3.7 such that $\gamma(x_{m_k}, \xi) < D(\beta_{m_k})$, for all $k \in \mathbb{N}$.
Therefore, $\{k \in \mathbb{N} : \gamma(x_k, \xi) \geq D(\beta_k)\} \subseteq \mathbb{N} - M \in I$. Consequently $OI - \lim x_n = \xi$.

The following example ensures that for an ideal I a sequence $\{x_n\}$ in L, $OI - \lim x_n$ and $OI^* - \lim x_n$ may not be equal.

Example 3.16. Let $N_p = \{p, p^2, p^3, ...\}$, where $p \in P$, the set of all prime numbers and $N_1 = \mathbb{N} - \cup_{p\in P} N_p$ Then $\mathbb{N} = \cup_{j=1, j\in P} N_j$ where each N_j is infinite and $N_i \cap N_j = \phi$ for $i \neq j$. Consider $I = \{\ A \subset \mathbb{N} : A\ intersects\ only\ a\ finite\ number\ of\ N_j\text{'s}\ \}$.
Let L has an accumulation point ξ in L. Then there exists a sequence $\{x_n\}$ in L so that $\lim_{n\to\infty} x_n = \xi$.
Using Lemma 3.7 we can choose a sequence $\{\alpha_n\}$ in L with $\alpha_n \downarrow \theta$ so that $\gamma(x_n, \xi) < D(\alpha_n)$ for all $n \in \mathbb{N}$.
Define a sequence $\{y_n\}$ in L with $y_n = x_j$ if $n \in N_j$, where j is either 1 or a prime number. We assert that $OI - \lim y_n = \xi$. If not, then for each $\beta = \{\beta_n\}$ in L with $\beta_n \downarrow \theta$, $A(\beta) = \{n \in \mathbb{N} : \gamma(y_n, \xi) \geq D(\beta_n)\} \notin I$. i.e. $A(\beta)$ intersects infinite number of N_j's. Then there exists a subsequence $\{p_n\}$ of prime numbers such that $\gamma(y_n, \xi) \geq D(\beta_n)$ when $n \in N_1 \cup N_{p_1} \cup N_{p_2} \cup N_{p_3} \cup ....$
Since N_{p_1} is infinite, there exists a natural number $n = q_1 \in N_{p_1}$ such that $\gamma(x_{p_1}, \xi) \geq D(\beta_{q_1})$.
Further N_{p_2} is infinite, so there exists a natural number $n = q_2 \in N_{p_2}$ with $q_2 > q_1$ such that $\gamma(x_{p_2}, \xi) \geq D(\beta_{q_2})$.
Continuing this process we can construct a subsequence $\{q_n\}$ of natural numbers such that $\gamma(x_{p_n}, \xi) \geq D(\beta_{q_n})$ for all $n \in \mathbb{N}$ but in particular if $\beta_{q_n} = \alpha_{p_n}$, this contradicts the choice of $\{\alpha_n\}$.

Now if possible let $OI^* - \lim y_n = \xi$. Then there exists a set $M = \{m_1 < m_2 < m_3 < ...\} \in F(I)$ such that $\lim_{k\to\infty} x_{m_k} = \xi$ with respect to the metric γ. Using Lemma 3.7 we can have a sequence $\{\beta_n\}$ in L with $\beta_n \downarrow \theta$ such that

$$\gamma(x_{m_k}, \xi) < D(\beta_{m_k}) \text{ for all } k \in \mathbb{N} \qquad .....(1)$$

Let $H = \mathbb{N} - M$, then $H \in I$ and there exists prime numbers p_1, p_2, ..., p_r such that $H \subseteq N_1 \cup N_{p_1} \cup N_{p_2} \cup ... N_{p_r}$. Thus, $N_{r+1} \subseteq \mathbb{N} - H = M$ and $n \in N_{r+1}$ implies that $y_n = x_{r+1}$ for infinitely many n. i.e. for infinitely many m_k, $\gamma(y_{m_k}, \xi) = \gamma(x_{r+1}, \xi) > 0$ which contradicts the relation (1) since $\beta_n \downarrow \theta$ and D is monotone increasing. Hence $OI^* - \lim y_n \neq \xi$.

Theorem 3.17 Let I be an admissible ideal of subsets of $\mathbb{N}$ and I has the AP-property. Then for a sequence $\{x_n\}$ in (L, γ), $OI - \lim x_n = \xi$ if and only if $OI^* - \lim x_n = \xi$, $\xi \in L$.

Proof: Since $OI - \lim x_n = \xi$, then we can choose a sequence $\{y_n\}$ of distinct elements in L with $y_n \downarrow \theta$ such that $\{n \in \mathbb{N} : \gamma(x_n, \xi) \geq D(y_n)\} \in I$.
Consider $A_1 = \{n \in \mathbb{N} : \gamma(x_n, \xi) \geq D(y_1)\}$ and
$A_n = \{k \in \mathbb{N} : D(y_n) \leq \gamma(x_k, \xi) < D(y_{n-1})\}$, for $n \geq 2$.
Clearly A_i's are pairwise disjoint. By AP-property there exists a sequence of subsets $\{B_n\}$ such that $A_i \Delta B_i$ is finite for all $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in I$
Let $M = \mathbb{N} - B = \{m_1, m_2, m_3, ...\}$.
For $\varepsilon > 0$ we choose the smallest positive integer $k \in \mathbb{N}$ such that $D(y_{k+1}) < \varepsilon$. Then $\{n \in \mathbb{N} : \gamma(x_n, \xi) \geq \varepsilon\} \subseteq \cup_{i=1}^{k+1} A_i$.
$A_i \Delta B_i$, $i = 1, 2, 3, ..., k+1$ are all finite sets and so there is some $m \in \mathbb{N}$ such that $\cup_{i=1}^{k+1} B_i \cap \{n \in \mathbb{N} : n > m\} = \cup_{i=1}^{k+1} A_i \cap \{n \in \mathbb{N} : n > m\}$.
If $n > m$ and $n \notin B$ then $n \notin \cup_{i=1}^{k+1} B_i$ and this implies that $n \notin \cup_{i=1}^{k+1} A_i$. Then $\gamma(x_n, \xi) < D(y_{k+1}) < \varepsilon$.
Therefore, $\gamma(x_n, \xi) < \varepsilon$ for $n > m$ and $n \in M$ and hence $\lim_{n\to\infty} x_{m_n} = \xi$ and consequently $OI^* - \lim x_n = \xi$.
The converse follows from Theorem 3.15.

4 Conclusion

In this paper, two new concepts, namely the concepts of OI-convergence and OI^*-convergence in a linearly ordered additive system have been introduced and investigated. In this investigation we have also shown by an example that OI-convergence and OI^*-convergence need not be equivalent. Further we have introduced the idea of OI-bounded sequences and investigated some basic properties. The present paper also contains a generalization of the results of the papers [3] and [9]. In this perspective we think that these results could provide a more general frame work for the investigation on convergence of sequences with respect to order.

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