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Comparison of Some Iterative Algorithms for Approximating Zeroes of Accretive Operators

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Abstract: Construction of iterative algorithms for approximating zeroes of non-linear operators has been studied extensively in the literature. In this paper, we compared two iterative schemes introduced and studied independently by Chidume and Djite [2] and Chidume *et al.*[4] for approximating zeroes of bounded $m-$ accretive operators and found that one of the schemes is more efficient than the other.

Keywords: Accretive operators, Monotone Operators,Proximal point algorithm, Sub differential operator, Banach spaces

1 Introduction

Many problems in applications can be modelled in the form $0 \in Ax$, where for example, $A : H \to 2^H$ is a monotone operator, i.e., A satisfies the following inequality: $\langle u-v, x-y\rangle \geq 0 \ \forall u \in Ax, \ v \in Ay, \ x, y \in H$. Typical example where monotone operators occur and satisfy the inclusion $0 \in Ax$ include the equilibrium state of evolution equations and critical points of some functionals defined on Hilbert spaces H. Let $f : H \to (-\infty, +\infty]$ be a proper, lower-semicontinuous convex function, then, it is known (see, Minty[7] or Rockafellar [12]) that the multi-valued map $T := \partial f$, the subdifferential of f, is maximal monotone, where for $w \in H$,

$$0 \in \partial f(x) \iff f(y) - f(x) \geq \langle y-x, w\rangle \ \forall \, y \in H$$
$$\iff x \in \text{Argmin}(f - \langle ., w\rangle).$$

In this case, the solutions of the inclusion $0 \in \partial f(x)$, if any, corresponds to the critical points of f, which are exactly its minimizers.
In general, consider the following problem:

$$\text{Find } u \in H \text{ such that } \ 0 \in Au \tag{1}$$

where H is a real Hilbert space and A is an m-monotone operator (defined below) on H. One of the classical algorithms for approximating a solution of (1), assuming existence, is the so-called proximal point algorithm introduced by Martinet[10] and studied further by Rockafellar [12] and a host of other authors. More precisely, given $x_k \in H$, an approximation of a solution of (1), the proximal point algorithm generates the next iterate x_{k+1} by solving the following equation

$$x_{k+1} = (I + \frac{1}{\lambda_k}A)^{-1}(x_k) + e_k, \tag{2}$$

where $\lambda_k > 0$ is a regularizing parameter. If the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is bounded above, then the resulting sequence $\{x_k\}_{k=1}^{\infty}$ of proximal point iterates converges weakly to a solution of (1), provided that a solution exists (Rockafellar[12]). Rockafellar then posed the following question:
Q1. Does the proximal point algorithm always converge strongly?

This question was resolved in the negative by Guller [3] who produced a proper, closed convex function G in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges weakly but not strongly, see also [8]. This raised the following question:
Q2. Can the proximal point algorithm be modified to guarantee strong convergence?

It is clear that the proximal point algorithm (2), even if it converges strongly, is not convenient to use. This is because at each step of the iteration process, one has to

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compute $(I+\frac{1}{\lambda_k}A)^{-1}(x_k)$ and this is certainly not convenient. Consequently, Chidume and Djitte [2] posed the following question, which perharps, is more important than **Q2**.

Q3. Can an iteration process be developed which will not involve the computation of $(I+\frac{1}{\lambda_k}A)^{-1}(x_k)$ at each step of the iteration process and which will guarantee strong convergence to solution of (1)?

With respect to **Q2**, many authors have modified the proximal point algorithm to guarantee strong convergence under different settings, see for instance (Solodov and Svaiter[13], Kamimura and Takahashi [5], H. K. Xu[14], Lehdili and Moudafi[6] and the references therein)

Another modification of the proximal point algorithm, perhaps the most significant, which yields strong convergence, is implicitly contained in the following theorem of Reich.

Definition 1. *For a nonlinear operator A with domain $D(A)$, we denote by $N(A)$, the set of zeros of A. That is* $N(A) := \{x \in D(A) : Ax = 0\}$.

Theorem 1.1 (Reich,[11]) Let E be a real uniformly smooth Banach space and $A : D(A) \subseteq E \to E$ be an accretive mapping with cl(D(A)) convex. Suppose A satisfies the range condition $D \subseteq R(I+sA),\ \forall\ s > 0$. Suppose that $0 \in R(A)$, then for each $x \in D$, the strong limit $\lim_{s\to\infty} J_s^A x$ exists and belongs to $N(A)$. If we denote $\lim_{s\to\infty} J_s^A x$ by Qx then $Q : D \to N(A)$ is the unique sunny nonexpansive retraction of D onto $N(A)$.

Remark. We mention here that in response to **Q2**, all modifications of the classical proximal point algorithm to obtain strong convergence involved the computation of inverse of some operators at each step of the process.

In the case that A is maximal monotone and bounded, Chidume and Djitte [2] gave an affirmative answer to **Q3** by proving the following important theorem. The reader can also see [9].

Theorem CD (Chidume and Djitte [2]. Let E be a 2-uniformly smooth real Banach space and let $A : E \to E$ be a bounded m-accretive map. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n \theta_n (x_n - x_1),\ n \geq 1, \qquad (3)$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim \theta_n = 0$; and $\{\theta_n\}$ is decreasing;

(2) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty,\ \ \lambda_n = o(\theta_n)$;

(3) $\lim_{n\to\infty} \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n \theta_n} = 0,\ \sum_{n=1}^{\infty} \lambda_n^2 < \infty$

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0 \theta_n\ \forall\ n \geq 1$, $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Remark. We note that 2-uniformly smooth Banach spaces include L_p spaces, $2 \leq p < \infty$ but do not include L_p spaces, $1 < p < 2$.

Inspired by Theorem CD [2], Chidume *et al.*[4] introduced a recursive sequence and proved that it converges strongly to a zero of $m-$accretive operator in uniformly smooth Banach spaces which include $2-$uniformly smooth Banach spaces and claimed that their recursive sequence is **simpler** than that of Theorem CD. To prove their results, they employed the two celebrated theorems of Simeon Reich ([1], [11]).
Before stating the result of Chidume *et al.*[4], we first present the result of Reich which was used in Chidume *et al.*[4].

Theorem Reich (S. Reich, [1]) Let E be a real uniformly smooth Banach space. Then, there exists a nondecreasing continuous function $\beta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) $\beta(ct) \leq c\beta(t)\ \forall c \geq 1$,
(ii) $\lim_{t\to 0^+} \beta(t) = 0$ and

$$||x+y||^2 \leq ||x||^2 + 2Re\langle y, j(x)\rangle + \max\{||x||, 1\}||y||\beta(||y||)\ \forall x, y \in E.$$

Theorem C (Chidume *et al.*[4]) Let E be a uniformly smooth real Banach space and let $A : E \to E$ be a bounded accretive map which satisfies the range condition. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be iteratively defined by

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n (x_n - x_1),\ n \geq 1, \qquad (4)$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(1) $\lim \lambda_n = 0$;

(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$, .

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta(\lambda_n) \leq \gamma_0\ \forall\ n \geq 1$, (where β is the one appearing in Theorem Reich), $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Remark. Let $A : H \to H$ be monotone map. A is called m-monotone if $R(I+\lambda A) = H$ for some $\lambda > 0$. It is well known that if A is $m-$monotone, it satisfies the range condition, that is, $R(I+\lambda A) = H$ for all $\lambda > 0$ (see, Chidume and Djitte [2]).

Motivated by the claim of Chidume *et al.*[4] and the ongoing research in this direction, it is our purpose in this note to compare the two algorithms used in Theorem CD and Theorem C.

2 Main Result

Let $E = \mathbb{R}$ the set of real numbers and $A : E \to E$ be defined by $Ax = \tanh x$. Observe that $E = \mathbb{R}$ is uniformly smooth and A is monotone, continuous and bounded. Hence it is $m-$accretive (see [7]). Furthermore, $0 \in N(A)$, the set of zeroes of A. Consequently, E and A satisfy the hypotheses of Theorem CD and Theorem C respectively.
We also note that $\lambda_n = \frac{1}{(n+1)^{\frac{3}{5}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$ satisfy the conditions required on λ_n and θ_n in Theorem CD and Theorem C.
We now re- state without proof, Theorem CD and Theorem C using the above parameters. Next, we implement the algorithms using MATLAB. The table below and the corresponding graphs were obtained.

Theorem CD Let $E = \mathbb{R}$ and let $A : E \to E$ be as defined above. For arbitrary $z_1 \in E$, let the sequence $\{z_n\}$ be iteratively defined by

$$z_{n+1} = z_n - \frac{1}{(n+1)^{\frac{3}{5}}}\tanh(z_n) - \frac{1}{(n+1)^{\frac{3}{5}}}\frac{1}{(n+1)^{\frac{1}{4}}}(z_n - z_1), \quad n \geq 1. \tag{5}$$

Then, there exists a constant $\gamma_0 > 0$ such that if $\lambda_n) \leq \gamma_0\theta_n \ \forall \ n \geq 1$, $\{z_n\}$ converges strongly to 0, a solution of the equation $Az = 0$. **Theorem C** Let $E = \mathbb{R}$ and let $A : E \to E$ be as defined above. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be iteratively defined by

$$x_{n+1} = x_n - \frac{1}{(n+1)^{\frac{3}{5}}}\tanh(x_n) - \frac{1}{(n+1)^{\frac{3}{5}}}(x_n - x_1), \quad n \geq 1. \tag{6}$$

Then, there exists a constant $\gamma_0 > 0$ such that if $\beta(\lambda_n) < \gamma_0$, $\{x_n\}$ converges strongly to 0, a solution of the equation $Ax = 0$.

3 Conclusion

(i). From table (1) and the graphs in Figure (1), we deduce the following facts;
(a) the iterative scheme of Theorem C converged faster than that of Theorem CD to 0, a zero of the operator, A.
(b) the iterative scheme of Theorem CD gave a better approximation of 0, a zero of the operator A than the iterative scheme of Theorem C. This suggests that the algorithm of Theorem CD is preferred to that of Theorem C in approximating zero of A..
(c). Another fact discovered which is common to the two algorithms, though not reflected in the graphs is that the closer the initial point is to 0, the better approximation of 0, one obtains. (ii). The result obtained above is for a particular operator. It is of interest to analytically verify that the facts established here are true in general.
(iii). For further studies, we shall check if the result of this paper is true in spaces of higher dimension and with different operators.

Table 1: Convergence of x(n+1) and z(n+1) with their corresponding errors

Number of iteration (n)	x(n+1)	z(n+1)	x(n+1) error	z(n+1) error
1	0.0342	0.0342	0.0342	0.0342
2	0.0506	0.0420	0.0506	0.0420
3	0.0501	0.0417	0.0501	0.0417
4	0.0500	0.0407	0.0500	0.0407
5	0.0500	0.0397	0.0500	0.0397
6	0.0500	0.0389	0.0500	0.0389
7	0.0500	0.0382	0.0500	0.0382
8	0.0500	0.0375	0.0500	0.037
9	0.0500	0.0369	0.0500	0.0369
10	0.0500	0.0364	0.0500	0.0364
⋮	⋮	⋮	⋮	⋮
18	0.0500	0.0333	0.0500	0.0333
19	0.0500	0.0333	0.0500	0.0333
20	0.0500	0.0327	0.0500	0.0327
⋮	⋮	⋮	⋮	⋮
27	0.0500	0.0311	0.0500	0.0311
28	0.0500	0.0309	0.0500	0.0309
⋮	⋮	⋮	⋮	⋮
49	0.0500	0.0280	0.0500	0.0280
50	0.0500	0.0279	0.0500	0.0247
⋮	⋮	⋮	⋮	⋮
98	0.0500	0.0246	0.0500	0.0246
99	0.0500	0.0246	0.0500	0.0246
100	0.0500	0.0246	0.0500	0.0246

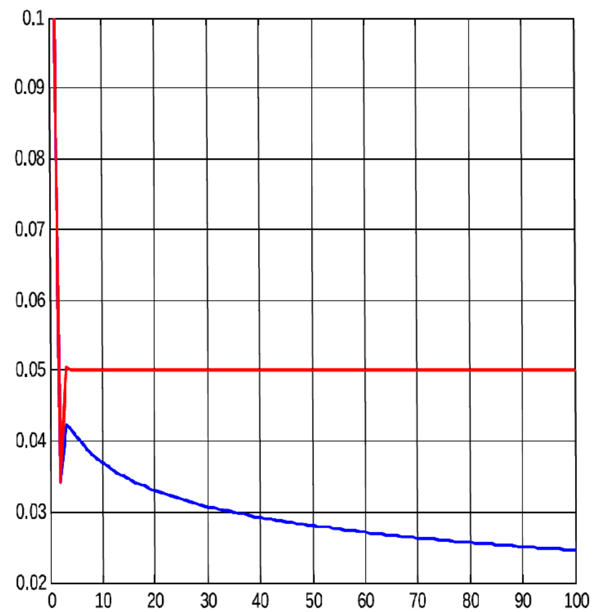

Fig. 1: Graphs of x(n+1), z(n+1) & x(n+1) with z(n+1)

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