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On Soft Separation Axioms via Fuzzy α-Open Soft Sets

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Abstract: In the present paper, we continue the study of the properties of fuzzy α-open (closed) soft sets, which were first introduced in [1]. Also, we investigate the concepts of fuzzy α-soft interior (closure), fuzzy α-continuous (open) soft functions and fuzzy α-separation axioms which are important for further research on fuzzy soft topology. In particular we study the relationship between fuzzy α-soft interior fuzzy α-soft closure, which are basic for further research on fuzzy soft topology and will fortify the footing of the theory of fuzzy soft topological space. Further, we study the properties of fuzzy soft α-regular spaces and fuzzy soft α-normal spaces. Moreover, we show that if every fuzzy soft point f_e is fuzzy α-closed soft set in a fuzzy soft topological space $(X,\mathfrak{T},E)$, then $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1- (resp. T_2-) space.

Keywords: Soft set, Fuzzy soft set, Fuzzy soft topological space, Fuzzy α-soft interior, Fuzzy α-soft closure, Fuzzy α-open soft, Fuzzy α-closed soft, Fuzzy α-continuous soft functions, Fuzzy soft α-separation axioms, Fuzzy soft α-T_i-spaces $(i = 1,2,3,4)$, Fuzzy soft α-regular, Fuzzy soft α-normal.

1 Introduction

In real life situation, the problems in economics, engineering, social sciences, medical science etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy set, intuitionistic fuzzy set, rough set, bipolar fuzzy set, i.e. which we can use as mathematical tools for dealings with uncertainties. But, all these theories have their inherent difficulties. The reason for these difficulties Molodtsov [35] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from the above difficulties. In [35,36], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [33], the properties and applications of soft set theory have been studied increasingly [7,29,36]. Xiao et al.[46] and Pei and Miao [39] discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [4,6,10, 18,27,31,32,33,34,36,37,49]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [11].

Recently, in 2011, Shabir and Naz [42] initiated the study of soft topological spaces. They defined soft topology as a collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in [45] investigate some properties of these soft separation axioms. In [19], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Kandil et al. [26] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. The notion of soft ideal was initiated for the first time by Kandil et al.[22]. They also introduced the concept of soft local function. These concepts are discussed with a view to find

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new soft topologies from the original one, called soft topological spaces with soft ideal $(X,\tau,E,\tilde{I})$. Applications to various fields were further investigated by Kandil et al. [20,21,23,24,25,28]. The notion of supra soft topological spaces was initiated for the first time by El-sheikh and Abd El-latif [14]. They also introduced new different types of subsets of supra soft topological spaces and study the relations between them in detail. The notion of b-open soft sets was initiated for the first time by El-sheikh and Abd El-latif [13], which is extended by Abd El-latif et al. in [3]. Maji et al. [31] initiated the study involving both fuzzy sets and soft sets. In [9] the notion of fuzzy soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then, many scientists such as X. Yang et al. [47], improved the concept of fuzziness of soft sets. In [4], Karal and Ahmed defined the notion of a mapping on classes of fuzzy soft sets, which is a fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Chang [12] introduced the concept of fuzzy topology on a set X by axiomatizing a collection $\mathfrak{T}$ of fuzzy subsets of X. [43] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [41] gave the definition of fuzzy soft topology over the initial universe set. Some fuzzy soft topological properties based on fuzzy semi (resp. β-, pre) open soft sets, were introduced in [2,17,18,27].

In the present paper, we investigate some new properties of fuzzy α-open soft sets and fuzzy α-closed soft sets, which were first introduced in [1]. Also, we study the notions of fuzzy α-soft interior, fuzzy α-soft closure and fuzzy α-separation axioms. Further, we study the properties of fuzzy soft α-regular spaces and fuzzy soft α-normal spaces. Moreover, we show that if every fuzzy soft point f_e is fuzzy α-closed soft set in a fuzzy soft topological space $(X,\mathfrak{T},E)$, then $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1- (resp. T_2-) space.

2 Preliminaries

In this section, we present the basic definitions and results of fuzzy soft set theory which will be needed in the paper.

Definition 2.1.[48] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \longrightarrow [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. We denote family of all fuzzy sets by I^X. If $A,B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh [48], as follows:

(1)$A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)\ \forall\, x \in X$.
(2)$A = B \Leftrightarrow \mu_A(x) = \mu_B(x)\ \forall\, x \in X$.
(3)$C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)\ \forall\, x \in X$.
(4)$D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)\ \forall\, x \in X$.
(5)$M = A^c \Leftrightarrow \mu_M(x) = 1 - \mu_A(x)\ \forall\, x \in X$.

Definition 2.2.[35] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F,A) denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F,A) and if $e \notin A$, then $F(e) = \phi$ i.e
$F_A = \{F(e) : e \in A \subseteq E,\ F : A \rightarrow P(X)\}$. The family of all these soft sets over X denoted by $SS(X)_A$.

Definition 2.3.[31] Let $A \subseteq E$. A pair (f,A), denoted by f_A, is called fuzzy soft set over X , where f is a mapping given by $f : A \rightarrow I^X$ defined by $f_A(e) = \mu^e_{f_A}$, where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$ and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0\ \forall\, x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_A$.

Definition 2.4.[40] Let $\mathfrak{T}$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E, then $\mathfrak{T} \subseteq FSS(X)_E$ is called fuzzy soft topology on X if

(1)$\tilde{1}_E, \tilde{0}_E \in \mathfrak{T}$, where $\tilde{0}_E(e) = \overline{0}$ and $\tilde{1}_E(e) = \overline{1}$, $\forall e \in E$,
(2)the union of any members of $\mathfrak{T}$ belongs to $\mathfrak{T}$,
(3)the intersection of any two members of $\mathfrak{T}$ belongs to $\mathfrak{T}$.

The triplet $(X,\mathfrak{T},E)$ is called fuzzy soft topological space over X. Also, each member of $\mathfrak{T}$ is called fuzzy open soft in $(X,\mathfrak{T},E)$. We denote the set of all open soft sets by $FOS(X,\mathfrak{T},E)$, or $FOS(X)$.

Definition 2.5.[40] Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X, if its relative complement f^c_A is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by $FCS(X,\mathfrak{T},E)$, or $FCS(X)$.

Definition 2.6.[38] Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A. i.e.,
$Fcl(f_A) = \sqcap\{h_D : h_D\ is\ fuzzy\ closed\ soft\ set\ and\ f_A \sqsubseteq h_D\}$.
The fuzzy soft interior of g_B, denoted by $Fint(f_A)$ is the fuzzy soft union of all fuzzy open soft subsets of f_A.i.e.,
$Fint(g_B) = \sqcup\{h_D : h_D\ is\ fuzzy\ open\ soft\ set\ and\ h_D \sqsubseteq g_B\}$.

Definition 2.7.[30] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha\ (0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = \overline{0}$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.8.[30] The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set (g,A), denoted by $x^e_\alpha \tilde{\in} (g,A)$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x)$.

Theorem 2.1. Mahanta2012f Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and f_e be a fuzzy soft point. Then, the following properties hold:

(1)If $f_e \tilde{\in} g_A$, then $f_e \tilde{\notin} g^c_A$;
(2)$f_e \tilde{\in} g_A \nRightarrow f^c_e \tilde{\in} g^c_A$;

(3)Every non-null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points belonging to f_A.

Definition 2.9.[30] A fuzzy soft set g_B in a fuzzy soft topological space $(X,\mathfrak{T},E)$ is called fuzzy soft neighborhood of the fuzzy soft point x^e_α if there exists a fuzzy open soft set h_C such that $x^e_\alpha \tilde{\in} h_C \sqsubseteq g_B$. A fuzzy soft set g_B in a fuzzy soft topological space $(X,\mathfrak{T},E)$ is called fuzzy soft neighborhood of the soft set f_A if there exists a fuzzy open soft set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$. The fuzzy soft neighborhood system of the fuzzy soft point x^e_α, denoted by $N_{\mathfrak{T}}(x^e_\alpha)$, is the family of all its fuzzy soft neighborhoods.

Definition 2.10.[30] Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $Y \subseteq X$. Let h^Y_E be a fuzzy soft set over (Y,E) such that $h^Y_E : E \to I^Y$ such that $h^Y_E(e) = \mu^e_{h^Y_E}$,

$$\mu^e_{h^Y_E}(x) = \begin{cases} 1\ x \in Y, \\ 0,\ x \notin Y. \end{cases}$$

Let $\mathfrak{T}_Y = \{h^Y_E \sqcap g_B : g_B \in \mathfrak{T}\}$, then the fuzzy soft topology $\mathfrak{T}_Y$ on $(Y,E)$is called fuzzy soft subspace topology for (Y,E) and $(Y,\mathfrak{T}_Y,E)$ is called fuzzy soft subspace of $(X,\mathfrak{T},E)$. If $h^Y_E \in \mathfrak{T}$ (resp. $h^Y_E \in \mathfrak{T}^c$), then $(Y,\mathfrak{T}_Y,E)$ is called fuzzy open (resp. closed) soft subspace of $(X,\mathfrak{T},E)$.

Definition 2.11.[38] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then the map f_{pu} is called fuzzy soft mapping from X to Y and denoted by $f_{pu} : FSS(X)_E \to FSS(Y)_K$ such that,

(1)If $f_A \in FSS(X)_E$. Then the image of f_A under the fuzzy soft mapping f_{pu} is the fuzzy soft set over Y defined by $f_{pu}(f_A)$, where $\forall\, k \in p(E)$, $\forall\, y \in Y$, $f_{pu}(f_A)(k)(y) = \begin{cases} \bigvee_{u(x)=y} [\vee_{p(e)=k}(f_A(e))](x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$

(2)If $g_B \in FSS(Y)_K$, then the pre-image of g_B under the fuzzy soft mapping f_{pu} is the fuzzy soft set over X defined by $f^{-1}_{pu}(g_B)$, where $\forall\, e \in p^{-1}(K)$, $\forall\, x \in X$,

$$f^{-1}_{pu}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & for\ p(e) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also it is said to be constant if p and u are constant.

Definition 2.12.[38] Let $(X,\mathfrak{T}_1,E)$ and $(Y,\mathfrak{T}_2,K)$ be two fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft mapping. Then f_{pu} is called

(1)Fuzzy continuous soft if $f^{-1}_{pu}(g_B) \in \mathfrak{T}_1\ \forall\ (g_B) \in \mathfrak{T}_2$.
(2)Fuzzy open soft if $f_{pu}(g_A) \in \mathfrak{T}_2 \forall\ (g_A) \in \mathfrak{T}_1$.

Theorem 2.2.[5] Let $FSS(X)_E$ and $FSS(Y)_K$ be two families of fuzzy soft sets. For the fuzzy soft function $f_{pu} : FSS(X)_E \to FSS(Y)_K$, the following statements hold,

(a)$f^{-1}_{pu}((g,B)^c) = (f^{-1}_{pu}(g,B))^c \forall\ (g,B) \in FSS(Y)_K$.
(b)$f_{pu}(f^{-1}_{pu}((g,B))) \sqsubseteq (g,B) \forall\ (g,B) \in FSS(Y)_K$. If f_{pu} is surjective, then the equality holds.
(c)$(f,A) \sqsubseteq f^{-1}_{pu}(f_{pu}((f,A))) \forall\ (f,A) \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.
(d)$f_{pu}(\tilde{0}_E) = \tilde{0}_K$, $f_{pu}(\tilde{1}_E) \sqsubseteq \tilde{1}_K$. If f_{pu} is surjective, then the equality holds.
(e)$f^{-1}_{pu}(\tilde{1}_K) = \tilde{1}_E$ and $f^{-1}_{pu}(\tilde{0}_K) = \tilde{0}_E$.
(f)If $(f,A) \sqsubseteq (g,A)$, then $f_{pu}(f,A) \sqsubseteq f_{pu}(g,A)$.
(g)If $(f,B) \sqsubseteq (g,B)$, then $f^{-1}_{pu}(f,B) \sqsubseteq f^{-1}_{pu}(g,B)\ \forall\ (f,B),(g,B) \in FSS(Y)_K$.
(h)$f^{-1}_{pu}(\sqcup_{j\in J}(f,B)_j) = \sqcup_{j\in J} f^{-1}_{pu}(f,B)_j$ and $f^{-1}_{pu}(\sqcap_{j\in J}(f,B)_j) = \sqcap_{j\in J} f^{-1}_{pu}(f,B)_j, \forall\ (f,B)_j \in FSS(Y)_K$.
(I)$f_{pu}(\sqcup_{j\in J}(f,A)_j) = \sqcup_{j\in J} f_{pu}(f,A)_j$ and $f_{pu}(\sqcap_{j\in J}(f,A)_j) \sqsubseteq \sqcap_{j\in J} f_{pu}(f,A)_j\ \forall\ (f,A)_j \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.

Definition 2.13.[30] Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space. A fuzzy soft separation of $\tilde{1}_E$ is a pair of non null proper fuzzy open soft sets g_B,h_C such that $g_B \sqcap h_C = \tilde{0}_E$ and $\tilde{1}_E = g_B \sqcup h_C$.

Definition 2.14.[30] A fuzzy soft topological space $(X,\mathfrak{T},E)$ is said to be fuzzy soft connected if and only if there is no fuzzy soft separations of $\tilde{X}$. Otherwise, $(X,\mathfrak{T},E)$ is said to be fuzzy soft disconnected space.

Definition 2.16.[27] Two fuzzy soft sets f_A and g_B are said to be disjoint, denoted by $f_A \sqcap g_B = \tilde{0}_E$, if $A \cap B = \varphi$ and $\mu^e_{f_A} \cap \mu^e_{g_B} = \overline{0}\ \forall e \in E$.

Theorem 2.3.[27] Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then,

(1)$f_A \in FSOS(X)$ if and only if $Fcl(f_A) = Fcl(Fint(f_A))$.
(2)If $g_B \in \mathfrak{T}$, then $g_B \sqcap Fcl(f_A) \sqsubseteq Fcl(g_B \sqcap g_B)$.

Definition 2.15.[19] Let (X,τ,E) be a soft topological space and $F_A \in SS(X)_E$. If $F_A \tilde{\subseteq} int(cl(int(F_A))$, then F_A is called α-open soft set. We denote the set of all α-open soft sets by $\alpha OS(X,\tau,E)$, or $\alpha OS(X)$ and the set of all α-closed soft sets by $\alpha CS(X,\tau,E)$, or $\alpha CS(X)$.

3 Fuzzy α-open (closed) soft sets

In this section, we move one step forward to Investigate new properties of the notions of fuzzy α-open soft sets, fuzzy α-closed soft sets [1] and study various properties and notions related to these structures.

Definition 3.1. [1]. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \sqsubseteq Fint(Fcl(Fint(f_A)))$, then f_A is called fuzzy α-open soft set. We denote the set of all fuzzy α-open soft sets by $F\alpha OS(X,\mathfrak{T},E)$, or $F\alpha OS(X)$ and the set of all fuzzy α-closed soft sets by $F\alpha CS(X,\mathfrak{T},E)$, or $F\alpha CS(X)$.

Theorem 3.1. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in F\alpha OS(X)$.Then

(1)Arbitrary fuzzy soft union of fuzzy α-open soft sets is fuzzy α-open soft.
(2)Arbitrary fuzzy soft intersection of fuzzy α-closed soft sets is fuzzy α-closed soft.

Proof.

(1)Let $\{(f,A)_j : j \in J\} \subseteq F\alpha OS(X)$. Then, $\forall\ j \in J$, $(f,A)_j \sqsubseteq Fint(Fcl(Fint(f,A)_j))$. It follows that, $\sqcup_j(f,A)_j \sqsubseteq \sqcup_j(Fint(Fcl(Fint(f,A)_j))) \sqsubseteq Fint(\sqcup_j Fcl(Fint(f,A)_j)) = Fint(Fcl(\sqcup_j Fint(f,A)_j)) \sqsubseteq Fint(Fcl(Fint(\sqcup_j(f,A)_j)))$. Hence, $\sqcup_j(f,A)_j \in F\alpha OS(X)\ \forall\ j \in J$.
(2)By a similar way.

Theorem 3.2. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\alpha OS(X)$ if and only if $Fcl(f_A) = Fint(Fcl(Fint(f_A)))$.

Proof. Immediate.

Definition 3.2. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \tilde{\in} FSS(X)_E$. Then,

(1)f_e is called fuzzy α-interior soft point of f_A if $\exists\ g_B \in F\alpha OS(X)$ such that $f_e \tilde{\in} g_B \sqsubseteq f_A$. The set of all fuzzy α-interior soft points of f_A is called the fuzzy α-soft interior of f_A and is denoted by $F\alpha int(f_A)$ consequently,
$F\alpha int(f_A) = \sqcup\{g_B : g_B \sqsubseteq f_A,\ g_B \in F\alpha OS(X)\}$.
(2)f_e is called fuzzy α-closure soft point of f_A if $f_A \sqcap h_C \neq \tilde{0}_E\ \forall\ h_D \in F\alpha OS(X)$. The set of all fuzzy α-closure soft points of f_A is called fuzzy α-soft closure of f_A and denoted by $F\alpha cl(f_A)$. Consequently, $F\alpha cl(f_A) = \sqcap\{h_D : h_D \in F\alpha CS(X),\ f_A \sqsubseteq h_D\}$.

Theorem 3.3. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy α-interior operator, denoted by $F\alpha int$.

(1)$F\alpha int(\tilde{1}_E) = \tilde{1}_E$ and $F\alpha int(\tilde{0}_E)) = \tilde{0}_E$.
(2)$F\alpha int(f_A) \sqsubseteq (f_A)$.
(3)$F\alpha int(f_A)$ is the largest fuzzy α-open soft set contained in f_A.
(4)If $f_A \sqsubseteq g_B$, then $F\alpha int(f_A) \sqsubseteq F\alpha int(g_B)$.
(5)$F\alpha int(F\alpha int(f_A)) = F\alpha int(f_A)$.
(6)$F\alpha int(f_A) \sqcup F\alpha int(g_B) \sqsubseteq F\alpha int[(f_A) \sqcup (g_B)]$.
(7)$F\alpha int[(f_A) \sqcap (g_B)] \sqsubseteq F\alpha int(f_A) \sqcap F\alpha int(g_B)$.

Proof. Obvious.

Theorem 3.4. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy α-closure operator, denoted by $F\alpha cl$.

(1)$F\alpha cl(\tilde{1}_E) = \tilde{1}_E$ and $F\alpha cl(\tilde{0}_E) = \tilde{0}_E$.
(2)$(f_A) \sqsubseteq F\alpha cl(f_A)$.
(3)$F\alpha cl(f_A)$ is the smallest fuzzy α-closed soft set contains f_A.
(4)If $f_A \sqsubseteq g_B$, then $F\alpha cl(f_A) \sqsubseteq F\alpha cl(g_B)$.
(5)$F\alpha cl(F\alpha cl(f_A)) = F\alpha cl(f_A)$.
(6)$F\alpha cl(f_A) \sqcup F\alpha cl(g_B) \sqsubseteq F\alpha cl[(f_A) \sqcup (g_B)]$.
(7)$F\alpha cl[(f_A) \sqcap (g_B)] \sqsubseteq F\alpha cl(f_A) \sqcap F\alpha cl(g_B)$.

Proof. Immediate.

Lemma 3.1. Every fuzzy open (resp. closed) soft set in a fuzzy soft topological space $(X,\mathfrak{T},E)$ is fuzzy α-open (resp. closed) soft.

Proof. Let $f_A \in FOS(X)$. Then, $Fint(f_A) = f_A$. Since $f_A \sqsubseteq Fcl(f_A)$, then $f_A \sqsubseteq Fint(Fcl(Fint(f_A)))$. Thus, $f_A \in F\alpha OS(X)$.

Remark 3.2. The converse of Lemma 3.1 is not true in general as shown in the following example.

Example 3.1. Let $X = \{a,b,c\}$, $E = \{e_1,e_2,e_3\}$ and $A,B,C,D \subseteq E$ where $A = \{e_1,e_2\}$, $B = \{e_2,e_3\}$, $C = \{e_1,e_3\}$ and $D = \{e_2\}$. Let $\mathfrak{T} = \{\tilde{1}_E, \tilde{0}_E, f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}\}$ where $f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}$ are fuzzy soft sets over X defined as follows:
$\mu^{e_1}_{f_{1A}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{1A}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$,
$\mu^{e_2}_{f_{2B}} = \{a_{0.4}, b_{0.6}, c_{0.3}\}$, $\mu^{e_3}_{f_{2B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{3D}} = \{a_{0.3}, b_{0.6}, c_{0.3}\}$,
$\mu^{e_1}_{f_{4E}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{4E}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$,
$\mu^{e_3}_{f_{4E}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{5B}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{f_{5B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{6D}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$.
Then, $\mathfrak{T}$ defines a fuzzy soft topology on X. Then, the fuzzy soft set k_E where:
$\mu^{e_1}_{k_E} = \{a_{0.4}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{k_E} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$,
$\mu^{e_3}_{k_E} = \{a_{0.3}, b_{0.4}, c_{0.4}\}$.
is fuzzy α-open soft set of $(X,\mathfrak{T},E)$, but it is not fuzzy open soft.

Theorem 3.5. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)$. Then,

(1)$F\alpha int(f^c_A) = \tilde{1} - [F\alpha cl(f_A)]$.
(2)$F\alpha cl(f^c_A) = \tilde{1} - [F\alpha int(f_A)]$.

Proof.

(1)Since $F\alpha cl(f_A) = \sqcap\{h_D : h_D \in F\alpha CS(X),\ f_A \sqsubseteq h_D\}$. Then, $\tilde{1} - F\alpha cl(f_A) = \sqcup\{h^c_D : h^c_D \in F\alpha OS(X),\ h^c_D \sqsubseteq f^c_A\} = F\alpha int(f^c_A)$.
(2)By a similar way.

Theorem 3.6. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space, $f_A \in FOS(X)$ and $g_B \in F\alpha OS(X)$. Then, $f_A \sqcap g_B \in F\alpha OS(X)$.

Proof. Let $f_A \in FOS(X)$ and $g_B \in F\alpha OS(X)$. Then, $f_A \sqcap g_B \sqsubseteq Fint(f_A) \sqcap Fint(Fcl(Fint(g_B))) = Fint[Fcl(Fint(f_A)) \sqcap Fint(g_B)] \sqsubseteq Fint(Fcl[Fint(f_A) \sqcap Fint(g_B)]) = Fint(Fcl(Fint[(f_A) \sqcap (g_B)]))$ from Theorem 2.3 (2). Hence, $f_A \sqcap g_B \in F\alpha OS(X)$.

Theorem 3.7. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\alpha CS(X)$ if and only if $Fcl(Fint(Fcl(f_A))) \sqsubseteq f_A$.

Proof. Obvious.

Corollary 3.1 Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\alpha CS(X)$ if and only if $f_A = f_A \sqcup Fcl(Fint(Fcl(f_A)))$.

4 Fuzzy α-continuous soft functions

In [4], Karal et al. defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Kandil et al. [26] introduced some types of soft function in soft topological spaces. Here, we introduce the notions of fuzzy α-soft function in fuzzy soft topological spaces and study its basic properties.

Definition 4.1. Let $(X,\mathfrak{T}_1,E)$, $(Y,\mathfrak{T}_2,K)$ be fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft function. Then, the function f_{pu} is called;

(1)Fuzzy α-continuous soft if $f_{pu}^{-1}(g_B) \in F\alpha OS(X) \forall g_B \in \mathfrak{T}_2$.
(2)Fuzzy α-open soft if $f_{pu}(g_A) \in F\alpha OS(Y) \forall g_A \in \mathfrak{T}_1$.
(3)Fuzzy α-closed soft if $f_{pu}(f_A) \in F\alpha CS(Y) \forall f_A \in \mathfrak{T}_1^c$.
(4)Fuzzy α-irresolute soft if $f_{pu}^{-1}(g_B) \in F\alpha OS(X) \forall g_B \in F\alpha OS(Y)$.
(5)Fuzzy α-irresolute open soft if $f_{pu}(g_A) \in F\alpha OS(Y) \forall g_A \in F\alpha OS(X)$.
(6)Fuzzy α-irresolute closed soft if $f_{pu}(f_A) \in F\alpha CS(Y) \forall f_A \in F\alpha CS(Y)$.

Example 4.1. Let $X = Y = \{a,b,c\}$, $E = \{e_1,e_2,e_3\}$ and $A \subseteq E$ where $A = \{e_1,e_2\}$. Let $f_{pu} : (X,\mathfrak{T}_1,E) \to (Y,\mathfrak{T}_2,K)$ be the constant soft mapping where $\mathfrak{T}_1$ is the indiscrete fuzzy soft topology and $\mathfrak{T}_2$ is the discrete fuzzy soft topology such that $u(x) = a \forall x \in X$ and $p(e) = e_1 \forall e \in E$. Let f_A be fuzzy soft set over Y defined as follows:
$\mu_{f_A}^{e_1} = \{a_{0.1}, b_{0.5}, c_{0.6}\}$, $\mu_{f_A}^{e_2} = \{a_{0.6}, b_{0.2}, c_{0.5}\}$.
Then $f_A \in \mathfrak{T}_2$. Now, we find $f_{pu}^{-1}(f_A)$ as follows:
$f_{pu}^{-1}(f_A)(e_1)(a) = f_A(p(e_1))(u(a)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_1)(b) = f_A(p(e_1))(u(b)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_1)(c) = f_A(p(e_1))(u(c)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_2)(a) = f_A(p(e_2))(u(a)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_2)(b) = f_A(p(e_2))(u(b)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_2)(c) = f_A(p(e_2))(u(c)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_3)(a) = f_A(p(e_3))(u(a)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_3)(b) = f_A(p(e_3))(u(b)) = f_A(e_1)(a) = 0.6$,
$f_{pu}^{-1}(f_A)(e_3)(c) = f_A(p(e_3))(u(c)) = f_A(e_1)(a) = 0.6$.
Hence, $f_{pu}^{-1}(f_A) \notin F\alpha OS(X)$. Therefore, f_{pu} is not fuzzy α-continuous soft function. on the other hand, if we consider $\mathfrak{T}_1$ is the discrete fuzzy soft topology.In this case, f_{pu} will be fuzzy α-continuous soft function.

Theorem 4.1. Every fuzzy continuous soft function is fuzzy α-continuous soft.

Proof. Immediate from Lemma 3.1.

Theorem 4.2. Let $(X,\mathfrak{T}_1,E)$, $(Y,\mathfrak{T}_2,K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then, the following are equivalent:

(1)f_{pu} is a fuzzy α-continuous soft function.
(2)$f_{pu}^{-1}(h_B) \in F\alpha CS(X) \forall h_B \in FCS(Y)$.
(3)$f_{pu}(F\alpha cl(g_A)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)) \forall g_A \in FSS(X)_E$.
(4)$F\alpha cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B)) \forall h_B \in FSS(Y)_K$.
(5)$f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\alpha int(f_{pu}^{-1}(h_B)) \forall h_B \in FSS(Y)_K$.

Proof.

(1) $\Rightarrow$ (2)Let h_B be a fuzzy closed soft set over Y. Then $h_B^c \in FOS(Y)$ and $f_{pu}^{-1}(h_B^c) \in F\alpha OS(X)$ from Definition 4.1. Since $f_{pu}^{-1}(h_B^c) = (f_{pu}^{-1}(h_B))^c$ from Theorem 2.2. Thus, $f_{pu}^{-1}(h_B) \in F\alpha CS(X)$.

(2) $\Rightarrow$ (3)Let $g_A \in FSS(X)_E$. Since $g_A \sqsubseteq f_{pu}^{-1}(f_{pu}(g_A)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))) \in F\alpha CS(X)$ from (2) and Theorem 2.2. Then, $g_A \sqsubseteq F\alpha cl(g_A) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))$. Hence, $f_{pu}(F\alpha cl(g_A)) \sqsubseteq f_{pu}(f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))$ from Theorem 2.2. Thus, $f_{pu}(F\alpha cl(g_A)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))$.

(3) $\Rightarrow$ (4)Let $h_B \in FSS(Y)_K$ and $g_A = f_{pu}^{-1}(h_B)$. Then, $f_{pu}(F\alpha cl f_{pu}^{-1}(h_B)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))$ From (3). Hence, $F\alpha cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(f_{pu}(F\alpha cl(f_{pu}^{-1}(h_B)))) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B))))$ $\sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))$ from Theorem 2.2. Thus, $F\alpha cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))$.

(4) $\Rightarrow$ (2)Let h_B be a fuzzy closed soft set over Y. Then, $F\alpha cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B)) \forall h_B \in FSS(Y)_K$ from (4). But clearly $f_{pu}^{-1}(h_B) \sqsubseteq F\alpha cl(f_{pu}^{-1}(h_B))$. This means that, $f_{pu}^{-1}(h_B) = F\alpha cl(f_{pu}^{-1}(h_B))$, and consequently $f_{pu}^{-1}(h_B) \in F\alpha CS(X)$.

(1) $\Rightarrow$ (5)Let $h_B \in FSS(Y)_K$. Then, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \in F\alpha OS(X)$ from (1). Hence, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = F\alpha int(f_{pu}^{-1} Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\alpha int(f_{pu}^{-1}(h_B))$. Thus, $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\alpha int(f_{pu}^{-1}(h_B))$.

(5) $\Rightarrow$ (1)Let h_B be a fuzzy open soft set over Y. Then, $Fint_{\mathfrak{T}_2}(h_B) = h_B$ and $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = f_{pu}^{-1}((h_B)) \sqsubseteq F\alpha int(f_{pu}^{-1}(h_B))$ from (5). But, we have $F\alpha int(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(h_B)$. This means that, $F\alpha int(f_{pu}^{-1}(h_B)) = f_{pu}^{-1}(h_B) \in F\alpha OS(X)$. Thus, f_{pu} is a fuzzy α-continuous soft function.

Theorem 4.3. Let $(X,\mathfrak{T}_1,E)$ and $(Y,\mathfrak{T}_2,K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then, the following are equivalent,

(1)f_{pu} is a fuzzy α-open soft function.

(2) $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\alpha int(f_{pu}(g_A))\ \forall\ g_A \in FSS(X)_E$.

Proof.

(1) $\Rightarrow$ (2)Let $g_A \in FSS(X)_E$. Since $Fint_{\mathfrak{T}_1}(g_A) \in \mathfrak{T}_1$. Then, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \in F\alpha OS(Y)\ \forall\ g_A \in \mathfrak{T}_1$ by (1). It follow that, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = F\alpha int(f_{pu}Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\alpha int(f_{pu}(g_A))$. Therefore, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\alpha int(f_{pu}(g_A))\ \forall\ g_A \in FSS(X)_E$.

(2) $\Rightarrow$ (1)Let $g_A \in \mathfrak{T}_1$. By hypothesis, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = f_{pu}(g_A) \sqsubseteq F\alpha int(f_{pu}(g_A)) \in F\alpha OS(Y)$, but $F\alpha int(f_{pu}(g_A)) \sqsubseteq f_{pu}(g_A)$. So, $F\alpha int(f_{pu}(g_A)) = f_{pu}(g_A) \in F\alpha OS(Y)\ \forall\ g_A \in \mathfrak{T}_1$. Hence, f_{pu} is a fuzzy α-open soft function.

Theorem 4.5. Let $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$ be a fuzzy α-open soft function. If $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1^c$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$, then there exists $h_B \in F\alpha CS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) \sqsubseteq l_C$.

Proof. Let $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1^c$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$. Then, $f_{pu}(l_C^c) \sqsubseteq k_D^c$ from Theorem 2.2 where $l_C^c \in \mathfrak{T}_1$. Since f_{pu} is fuzzy α-open soft function. Then, $f_{pu}(l_C^c) \in F\alpha OS(Y)$. Take $h_B = [f_{pu}(l_C^c)]^c$. Hence, $h_B \in F\alpha CS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) = f_{pu}^{-1}([f_{pu}(l_C^c)]^c) \sqsubseteq f_{pu}^{-1}(k_D^c)^c = f_{pu}^{-1}(k_D) \sqsubseteq l_C$. This completes the proof.

Theorem 4.6. Let $(X,\mathfrak{T}_1,E)$ and $(Y,\mathfrak{T}_2,K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K$. Then, the following are equivalent:

(1) f_{pu} is a fuzzy α-closed soft function.
(2) $F\alpha cl(f_{pu}(h_A)) \sqsubseteq f_{pu}(Fcl_{\mathfrak{T}_1}(h_A))\ \forall\ h_A \in FSS(X)_E$.

Proof. It follows immediately from Theorem 4.3.

5 Fuzzy soft α-separation axioms

Soft separation axioms for soft topological spaces were studied by Shabir and Naz [42]. Kandil et al. [26] introduced and studied the notions of soft α-separation axioms in soft topological spaces. Here, we introduce the notions of fuzzy soft α-separation axioms in fuzzy soft topological spaces and study some of its basic properties.

Definition 5.1. A fuzzy soft topological space $(X,\mathfrak{T},E)$ is said to be a fuzzy soft α-T_o-space if for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy α-open soft set containing one of the points but not the other.

Examples 5.1.

(1)Let $X = \{a,b,c,d\}$, $E = \{e_1,e_2,e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_o-space.

(2)Let $X = \{a,b,c\}$, $E = \{e_1,e_2\}$ and $\mathfrak{T}$ be the indiscrete fuzzy soft topology on X. Then, $\mathfrak{T}$ is not fuzzy soft α-T_o-space.

Theorem 5.1. A soft subspace $(Y,\mathfrak{T}_Y,E)$ of a fuzzy soft α-T_o-space $(X,\mathfrak{T},E)$ is fuzzy soft α-T_o.

Proof. Let h_e, g_e be two distinct fuzzy soft points in (Y,E). Then, these fuzzy soft points are also in (X,E). Hence, there exists a fuzzy α-open soft set f_A in $\mathfrak{T}$ containing one of the fuzzy soft points but not the other. Thus, $h_E^Y \sqcap f_A$ is a fuzzy α-open soft set in $(Y,\mathfrak{T}_Y,E)$ containing one of the fuzzy soft points but not the other from Definition 2.10. Therefore, $(Y,\mathfrak{T}_Y,E)$ is fuzzy soft α-T_o.

Definition 5.2. A fuzzy soft topological space $(X,\mathfrak{T},E)$ is said to be a fuzzy soft α-T_1-space if for every pair of distinct fuzzy soft points f_e, g_e there exist fuzzy α-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$.

Example 5.1. Let $X = \{a,b,c\}$, $E = \{e_1,e_2,e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1-space.

Theorem 5.2. A fuzzy soft subspace $(Y,\mathfrak{T}_Y,E)$ of a fuzzy soft α-T_1-space $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1.

Proof. It is similar to the proof of Theorem 5.1.

Theorem 5.3. If every fuzzy soft point of a fuzzy soft topological space $(X,\mathfrak{T},E)$ is fuzzy α-closed soft, then $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1.

Proof. Suppose that f_e and g_e be two distinct fuzzy soft points of (X,E). By hypothesis, f_e and g_e are fuzzy α-closed soft sets. Hence, f_e^c and g_e^c are distinct fuzzy α-open soft sets where $f_e \tilde{\in} g_e^c$, $g_e \tilde{\notin} g_e^c$; and $f_e \tilde{\notin} f_e^c$, $g_e \tilde{\in} f_e^c$. Therefore, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1.

Definition 5.3. A fuzzy soft topological space $(X,\mathfrak{T},E)$ is said to be a fuzzy soft α-T_2-space if for every pair of distinct fuzzy soft points f_e, g_e there exist disjoint fuzzy α-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$.

Example 5.1. Let $X = \{a,b,c,d\}$, $E = \{e_1,e_2,e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_2-space.

Proposition 5.1. For a fuzzy soft topological space $(X,\mathfrak{T},E)$ we have:
fuzzy soft α-T_2-space $\Rightarrow$ fuzzy soft α-T_1-space $\Rightarrow$ fuzzy soft α-T_o-space.

Proof.

(1)Let $(X,\mathfrak{T},E)$ be a fuzzy soft α-T_2-space and f_e, g_e be two distinct fuzzy soft points. Then, there exist disjoint fuzzy α-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$. Since $f_A \sqcap g_B = \tilde{0}_E$. Then, $f_e \tilde{\notin} g_B$ and $g_e \tilde{\notin} f_A$. Therefore, there exist fuzzy α-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Thus, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_1-space.

(2)Let $(X,\mathfrak{T},E)$ be a fuzzy soft α-T_1-space and f_e, g_e be two distinct fuzzy soft points. Then, there exist fuzzy α-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Then, we have a fuzzy α-open soft set containing one of the fuzzy soft point but not the other. Thus, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_o-space.

Theorem 5.4. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space . If $(X,\mathfrak{T},E)$ is fuzzy soft α-T_2-space, then for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy α-closed soft set k_A such that containing one of the fuzzy soft points $g_e \tilde{\in} k_A$, but not the other $f_e \tilde{\notin} k_A$ and $g_e \tilde{\notin} F\alpha cl(k_A)$.

Proof. Let f_e, g_e be two distinct fuzzy soft points. By assumption, there exists disjoint fuzzy α-open soft sets b_A and h_B such that $f_e \tilde{\in} b_A$, $g_e \tilde{\in} h_B$. Hence, $g_e \tilde{\in} b_A^c$ and $f_e \tilde{\notin} b_A^c$ from Theorem 2.1. Thus, b_A^c is a fuzzy α-closed soft set containing g_e but not f_e and $f_e \tilde{\notin} F\alpha cl(b_A^c) = b_A^c$.

Theorem 5.5. A fuzzy soft subspace $(Y,\mathfrak{T}_Y,E)$ of fuzzy soft α-T_2-space $(X,\mathfrak{T},E)$ is fuzzy soft α-T_2.

Proof. Let j_e, k_e be two distinct fuzzy soft points in (Y,E). Then, these fuzzy soft points are also in (X,E). Hence, there exist disjoint fuzzy α-open soft sets f_A and g_B in $\mathfrak{T}$ such that $j_e \in f_A$ and $k_e \in g_B$. Thus, $h_E^Y \sqcap f_A$ and $h_E^Y \sqcap g_B$ are disjoint fuzzy α-open soft sets in $\mathfrak{T}_Y$ such that $j_e \in h_E^Y \sqcap f_A$ and $k_e \in h_E^Y \sqcap g_B$. So, $(Y,\mathfrak{T}_Y,E)$ is fuzzy soft α-T_2.

Theorem 5.6. If every fuzzy soft point of a fuzzy soft topological space $(X,\mathfrak{T},E)$ is fuzzy α-closed soft, then $(X,\mathfrak{T},E)$ is fuzzy soft α-T_2.

Proof. It similar to the proof of Theorem 5.3.

Definition 5.4. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space, h_C be a fuzzy α-closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If there exist disjoint fuzzy α-open soft sets f_S and f_W such that $g_e \tilde{\in} f_S$ and $g_B \sqsubseteq f_W$. Then, $(X,\mathfrak{T},E)$ is called fuzzy soft α-regular space. A fuzzy soft α-regular T_1-space is called a fuzzy soft α-T_3-space.

Proposition 5.2. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space, h_C be a fuzzy α-closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If $(X,\mathfrak{T},E)$ is fuzzy soft α-regular space. Then, there exists a fuzzy α-open soft set f_A such that $g_e \tilde{\in} f_A$ and $f_A \sqcap h_C = \tilde{0}_E$.

Proof. Obvious from Definition 5.4.

Theorem 5.7. Let $(X,\mathfrak{T},E)$ be a fuzzy soft α-regular space and be a fuzzy α-open soft set g_B such that $f_e \tilde{\in} g_B$. Then, there exists a fuzzy α-open soft set f_S such that $f_e \tilde{\in} f_S$ and $F\alpha cl(f_S) \sqsubseteq g_B$.

Proof. Let g_B be a fuzzy α-open soft set containing a fuzzy soft point f_e in a fuzzy soft α-regular space $(X,\mathfrak{T},E)$. Then, g_B^c is a fuzzy α-closed soft such that $f_e \tilde{\notin} g_B^c$ from Theorem 2.1. By hypothesis, there exist disjoint fuzzy α-open soft sets f_S and f_W such that $f_e \tilde{\in} f_S$ and $g_B^c \sqsubseteq f_W$. It follows that, $f_W^c \sqsubseteq g_B$ and $f_S \sqsubseteq f_W^c$. Thus, $F\alpha cl(f_S) \sqsubseteq f_W^c \sqsubseteq g_B$. So, we have a fuzzy α-open soft set f_S containing f_e such that $F\alpha cl(f_S) \sqsubseteq g_B$.

Theorem 5.8. Every fuzzy soft α-T_3-space, in which every fuzzy soft point is fuzzy α-closed soft, is fuzzy soft α-T_2-space.

Proof. Let f_e, g_e be two distinct fuzzy soft points of a fuzzy soft α-T_3-space $(X,\mathfrak{T},E)$. By hypothesis, g_e is fuzzy α-closed soft set and $f_e \tilde{\notin} g_e$. From the fuzzy soft α-regularity, there exist disjoint fuzzy α-open soft sets k_A and h_B such that $f_e \tilde{\in} k_A$ and $g_e \sqsubseteq h_B$. Thus, $f_e \tilde{\in} k_A$ and $g_e \tilde{\in} h_B$. Therefore, $(X,\mathfrak{T},E)$ is fuzzy soft α-T_2-space.

Theorem 5.9. A fuzzy soft subspace $(Y,\mathfrak{T}_Y,E)$ of a fuzzy soft α-T_3-space $(X,\mathfrak{T},E)$ is fuzzy soft α-T_3.

Proof. By Theorem 5.2, $(Y,\mathfrak{T}_Y,E)$ is fuzzy soft α-T_1-space. Now, we want to prove that $(Y,\mathfrak{T}_Y,E)$ is fuzzy soft α-regular space. Let k_E be a fuzzy α-closed soft set in (Y,E) and g_e be a fuzzy soft point in (Y,E) such that $g_e \tilde{\notin} k_E$. Then, $k_E = h_E^Y \sqcap g_B$ for some fuzzy α-closed soft set g_B in (X,E). Hence, $g_e \tilde{\notin} h_E^Y \sqcap g_B$. But $g_e \tilde{\in} h_E^Y$, so $g_e \tilde{\notin} g_B$. Since $(X,\mathfrak{T},E)$ is fuzzy soft α-T_3. Then, there exist disjoint fuzzy α-open soft sets f_S and f_W in $\mathfrak{T}$ such that $g_e \tilde{\in} f_S$ and $g_B \sqsubseteq f_W$. It follows that, $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy α-open soft sets in $\mathfrak{T}_Y$ such that $g_e \tilde{\in} h_E^Y \sqcap f_S$ and $k_E \sqsubseteq h_E^Y \sqcap f_W$. Therefore, $(Y,\mathfrak{T}_Y,E)$ is fuzzy soft α-T_3.

Definition 5.5. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and h_C, g_B be disjoint fuzzy α-closed soft sets. If there exist disjoint fuzzy α-open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B \sqsubseteq f_W$. Then, $(X,\mathfrak{T},E)$ is called fuzzy soft α-normal space. A fuzzy soft α-normal T_1-space is called a fuzzy soft α-T_4-space.

Theorem 5.10. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space. Then, the following are equivalent:

(1)$(X,\mathfrak{T},E)$ is a fuzzy soft α-normal space.
(2)For every fuzzy α-closed soft set h_C and fuzzy α-open soft set g_B such that $h_C \sqsubseteq g_B$, there exists a fuzzy α-open soft set f_S such that $h_C \sqsubseteq f_S$, $F\alpha cl(f_S) \sqsubseteq g_B$.

Proof.

(1) $\Rightarrow$ (2) Let h_C be a α-closed soft set and g_B be a fuzzy α-open soft set such that $h_C \sqsubseteq g_B$. Then, h_C, g_B^c are disjoint fuzzy α-closed soft sets. It follows by (1), there exist disjoint fuzzy α-open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B^c \sqsubseteq f_W$. Now, $f_S \sqsubseteq f_W^c$, so $F\alpha cl(f_S) \sqsubseteq F\alpha cl\, f_W^c = f_W^c$, where g_B is fuzzy α-open soft set. Also, $f_W^c \sqsubseteq g_B$. Hence, $F\alpha cl(f_S^c) \sqsubseteq f_W^c \sqsubseteq g_B$. Thus, $h_C \sqsubseteq f_S$, $F\alpha cl(f_S) \sqsubseteq g_B$.

(2) $\Rightarrow$ (1) Let g_A and g_B be disjoint fuzzy α-closed soft sets. Then, $g_A \sqsubseteq g_B^c$. By hypothesis, there exists a fuzzy α-open soft set f_S such that $g_A \sqsubseteq f_S$, $F\alpha cl(f_S) \sqsubseteq g_B^c$. So $g_B \sqsubseteq [F\alpha cl(f_S)]^c$, $g_A \sqsubseteq f_S$ and $[F\alpha cl(f_S)]^c \sqcap f_S = \tilde{0}_E$, where f_S and $[F\alpha cl(f_S)]^c$ are fuzzy α-open soft sets. Thus, $(X,\mathfrak{T},E)$ is fuzzy soft α-normal space.

Theorem 5.11. A fuzzy α-closed fuzzy soft subspace $(Y,\mathfrak{T}_Y,E)$ of a fuzzy soft α-normal space $(X,\mathfrak{T},E)$ is fuzzy soft α-normal.

Proof. Let g_A and g_B be disjoint fuzzy α-closed soft sets in $\mathfrak{T}_Y$. Then, $g_A = h_E^Y \sqcap f_C$ and $g_B = h_E^Y \sqcap f_D$ for some fuzzy α-closed soft sets f_C, f_D in (X,E). Hence, f_C, f_D are disjoint fuzzy α-closed soft sets in $\mathfrak{T}$. Since $(X,\mathfrak{T},E)$ is fuzzy soft α-normal. Then, there exist disjoint fuzzy α-open soft sets f_S and f_W in $\mathfrak{T}$ such that $f_C \sqsubseteq f_S$, $f_D \sqsubseteq f_W$. It follows that, $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy α-open soft sets in $\mathfrak{T}_Y$ such that

$g_A = h_E^Y \sqcap f_C \sqsubseteq h_E^Y \sqcap f_S$ and $g_B = h_E^Y \sqcap f_D \sqsubseteq h_E^Y \sqcap f_W$. Therefore, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft α-normal.

Theorem 5.12. Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a fuzzy soft function which is bijective, fuzzy α-irresolute soft and fuzzy α-irresolute open soft. If $(X, \mathfrak{T}_1, E)$ is a fuzzy soft α-normal space, then $(Y, \mathfrak{T}_2, K)$ is also a fuzzy soft α-normal space.

Proof. Let f_A, g_B be disjoint fuzzy α-closed soft sets in Y. Since f_{pu} is fuzzy α-irresolute soft, then $f_{pu}^{-1}(f_A)$ and $f_{pu}^{-1}(g_B)$ are fuzzy α-closed soft set in X such that $f_{pu}^{-1}(f_A) \sqcap f_{pu}^{-1}(g_B) = f_{pu}^{-1}[f_A \sqcap g_B] = f_{pu}^{-1}[\tilde{0}_K] = \tilde{0}_E$ from Theorem 2.2. By hypothesis, there exist disjoint fuzzy α-open soft sets k_C and h_D in X such that $f_{pu}^{-1}(f_A) \sqsubseteq k_C$ and $f_{pu}^{-1}(g_B) \sqsubseteq h_D$. It follows that, $f_A = f_{pu}[f_{pu}^{-1}(f_A)] \sqsubseteq f_{pu}(k_C)$,$g_B = f_{pu}[f_{pu}^{-1}(g_B)] \sqsubseteq f_{pu}(h_D)$ from Theorem 2.2 and $f_{pu}(k_C) \sqcap f_{pu}(h_D) = f_{pu}[k_C \sqcap h_D] = f_{pu}[\tilde{0}_E] = \tilde{0}_K$ from Theorem 2.2. Since f_{pu} is fuzzy α-irresolute open soft function. Then, $f_{pu}(k_C), f_{pu}(h_D)$ are fuzzy α-open soft sets in Y. Thus, $(Y, \mathfrak{T}_2, K)$ is a fuzzy soft α-normal space.

6 Conclusion

Since the authors introduced topological structures on fuzzy soft sets [9,16,43], so the α-topological properties, which introduced by Kandil et al.[26], is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems [46,39], we can use the results deducted from the studies on fuzzy soft topological space to improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life

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