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A New Class of Integral Formulas Associated with Generalized Bessel Functions

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Abstract: Integrals involving a variety of special functions have been developed by many authors. Also many interesting integral formulas associated with the Bessel function of the first kind have been established. Very recently, Agarwal et al. presented two interesting integrals involving the Bessel function of the first kind, which are expressed in terms of generalized (Wright) hypergeometric functions. In a similar way, in this paper, we establish two another new interesting integral formulas involving the generalized Bessel functions, which are also expressed in terms of generalized (Wright) hypergeometric functions. Further, some special cases of our main results are also considered.

Keywords: Gamma function, Hypergeometric function ${}_pF_q$, Generalized (Wright) hypergeometric functions ${}_p\Psi_q$, Bessel functions, Generalized Bessel function of the first kind, Lavoie-Trottier integral formula.

1 Introduction

Integrals involving a variety of special functions have been developed by many authors (see [9], [10], [11], [15], see also [8] and [14]). A number of integrals involving the product of Bessel functions play an important role in several diverse field of physics, such as in neutrons physics, palasma physics and radio physics etc. In the present paper, we present two new generalized integral formulas involving the generalized Bessel functions, which are expressed in terms of the generalized (Wright) hypergeometric functions. Further, some interesting special cases of our main results are also considered.

A useful generalization $w^b_{\nu,c}(z)$ of the Bessel function of the first kind is defined for $z \in C\backslash\{0\}$ and $b,c,\nu \in C$ with $\Re(\nu) > -1$ as follows (see [8]):

$$w^b_{\nu,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k\ (z/2)^{\nu+2k}}{k!\ \Gamma(\nu+k+\frac{1+b}{2})}, \tag{1.1}$$

where C denotes the set of complex numbers, $\Gamma(z)$ is the familiar Gamma function and $w^b_{\nu,c}(0) = 0$.

If we consider $c = b = 1$ in (1.1) then $w^b_{\nu,c}(z)$ becomes the Bessel function of the first kind $J_\nu(z)$ and if we consider $c = -1$ and $b = 1$ in (1.1) then $w^b_{\nu,c}(z)$ reduces to the modified Bessel function of purely imaginary argument $I_\nu(z)$. Similarly, if we consider $c = 1$ and $b = 2$ in (1.1) then $w^b_{\nu,c}(z)$ reduces to $\frac{2j_\nu}{\sqrt{\pi}}$, while, if we consider $c = -1$ and $b = 2$ then $w^b_{\nu,c}(z)$ becomes $\frac{2i_\nu}{\sqrt{\pi}}$. Also, $w^b_{\nu,c}(z)$ can be reduces in terms of cosine and sine functions as follows (see [13]):

(i) On setting $\nu = -\frac{b}{2}$ and replacing c by c^2 in (1.1), we get

$$w^b_{-\frac{b}{2},\, c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos cz}{\sqrt{\pi}}. \tag{1.2}$$

(ii) On setting $\nu = -\frac{b}{2}$ and replacing c by $-c^2$ in (1.1), we get

$$w^b_{-\frac{b}{2},\, -c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}}. \tag{1.3}$$

(iii) On setting $\nu = 1-\frac{b}{2}$ and replacing c by c^2 in (1.1), we get

$$w^b_{1-\frac{b}{2},\, c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sin cz}{\sqrt{\pi}}. \tag{1.4}$$

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(iv) On setting $\nu = 1 - \frac{b}{2}$ and replacing c by $-c^2$ in (1.1), we get

$$w^b_{1-\frac{b}{2},\,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sinh cz}{\sqrt{\pi}}. \tag{1.5}$$

The generalization of the generalized hypergeometric series ${}_pF_q$ (1.9) is due to Fox [1] and Wright ([3], [4], [5]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [7, p.21])

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots\ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots\ldots, (\beta_q, B_q);\end{matrix}\; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.6}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{1.7}$$

A special case of (1.6) is

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \ldots\ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots\ldots, (\beta_q, 1);\end{matrix}\; z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1, \ldots\ldots, \alpha_p; \\ \beta_1, \ldots\ldots, \beta_q;\end{matrix}\; z\right], \tag{1.8}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [6, section 1.5])

$${}_pF_q\left[\begin{matrix}\alpha_1, \ldots\ldots, \alpha_p; \\ \beta_1, \ldots\ldots, \beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots \beta_q; z), \tag{1.9}$$

where $(\lambda)_n$, is called the Pochhammer's symbol [2].

For our present investigation, the following interesting and useful result due to Lavoie and Trottier [12] will be required

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.10}$$

$(\Re(\alpha) > 0$ and $\Re(\beta) > 0)$

2 Main results

Two generalized integral formulas which have been established in this section, are expressed in terms of generalized Wright hypergeometric function with suitable arguments in the integrands.

Theorem 2.1. The following integral formula holds true: For $\rho,\ \sigma,\ \nu,\ b,\ c \in C$ with $\Re(\rho+\sigma) > 0,\ \Re(\nu) > -(\frac{1+b}{2}),\ \Re(\rho+\nu) > 0$ and $x > 0$,

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1} w^b_{\nu,c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+\sigma)}\left(\frac{y}{2}\right)^{\nu}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{matrix}(\rho+\nu,2); \\ (\nu+\frac{1+b}{2},1),\quad (2\rho+\sigma+\nu,2);\end{matrix}\; -\frac{cy^2}{4}\right]. \tag{2.1}$$

Proof. In order to derive (2.1), we denote the left-hand side of (2.1) by I, expressing $w^b_{\nu,c}$ as a series with the help of (1.1) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$I = \sum_{k=0}^{\infty} \frac{(-c)^k\,(y/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+\frac{1+b}{2})} \int_0^1 x^{\rho+\sigma-1}(1-x)^{2(\rho+\nu+2k)-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho+\nu+2k-1} dx. \tag{2.2}$$

Now using the result (1.10) in (2.2), we get

$$I = \left(\frac{2}{3}\right)^{2(\rho+\sigma)}\left(\frac{y}{2}\right)^{\nu}\Gamma(\rho+\sigma)\sum_{k=0}^{\infty}\frac{\Gamma(\rho+\nu+2k)}{\Gamma(\nu+\frac{1+b}{2}+k)\,\Gamma(2\rho+\sigma+\nu+2k)}\frac{(-\frac{cy^2}{4})^k}{k!}, \tag{2.3}$$

which upon using (1.6), yields (2.1). This completes the proof of Theorem 2.1.

Theorem 2.2. The following integral formula holds true: For $\rho,\ \sigma,\ \nu,\ b,\ c \in C$ with $\Re(\rho+\sigma) > 0,\ \Re(\nu) > -(\frac{1+b}{2}),\ \Re(\rho+\nu) > 0$ and $x > 0$,

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1} w^b_{\nu,c}\left(xy\left(1-\frac{x}{3}\right)^2\right) dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+\nu)}\left(\frac{y}{2}\right)^{\nu}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{matrix}(\rho+\nu,2); \\ (\nu+\frac{1+b}{2},1),\quad (2\rho+\sigma+\nu,2);\end{matrix}\; -\frac{4cy^2}{81}\right]. \tag{2.4}$$

Proof. It is easy to see that a similar argument as in the proof of Theorem 2.1 will establish the integral formula (2.4).

Next we consider other variations of Theorems 2.1 and 2.2. In fact, we establish some integral formulas for the generalized Bessel function $w^b_{\nu,c}(z)$ expressed in terms of the generalized hypergeometric function ${}_pF_q$.

Corollary 2.1. Let the conditions of Theorem 2.1 be satisfied and $\rho+\sigma,\ \rho+\nu \in C\backslash Z_0^-$. Then the following integral formula holds true:

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1} w^b_{\nu,c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+\sigma)}\left(\frac{y}{2}\right)^{\nu}\frac{\Gamma(\rho+\sigma)\,\Gamma(\rho+\nu)}{\Gamma(\nu+\frac{1+b}{2})\,\Gamma(2\rho+\sigma+\nu)}$$
$$\times\,{}_2F_3\left[\begin{matrix}\left(\frac{\rho+\nu}{2}\right),\quad \left(\frac{\rho+\nu+1}{2}\right); \\ \left(\nu+\frac{1+b}{2}\right),\ \left(\frac{2\rho+\sigma+\nu}{2}\right),\ \left(\frac{\rho+\sigma+\nu+1}{2}\right);\end{matrix}\; -\frac{cy^2}{4}\right]. \tag{2.5}$$

Corollary 2.2. Let the conditions of Theorem 2.2 be satisfied and $\rho+\sigma,\ \rho+\nu \in C\backslash Z_0^-$. Then the following integral formula holds true:

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1} w^b_{\nu,c}\left(xy\left(1-\frac{x}{3}\right)^2\right) dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+\nu)}\left(\frac{y}{2}\right)^{\nu}\frac{\Gamma(\rho+\sigma)\,\Gamma(\rho+\nu)}{\Gamma(\nu+\frac{1+b}{2})\,\Gamma(2\rho+\sigma+\nu)}$$
$$\times\,{}_2F_3\left[\begin{matrix}\left(\frac{\rho+\nu}{2}\right),\quad \left(\frac{\rho+\nu+1}{2}\right); \\ \left(\nu+\frac{1+b}{2}\right),\ \left(\frac{2\rho+\sigma+\nu}{2}\right),\ \left(\frac{\rho+\sigma+\nu+1}{2}\right);\end{matrix}\; -\frac{4cy^2}{81}\right]. \tag{2.6}$$

Proof. In order to proof (2.5), using the results $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ and $(\lambda)_{2n} = 2^{2n}(\frac{\lambda}{2})_n(\frac{\lambda+1}{2})_n \quad (n \in N_0)$, in (2.3) and summing the given series with the help of (1.9), we easily arrive at right-hand side of (2.5). This completes the proof of Corollary 2.1. Similarly, it is easy to see that a similar argument as in proof of Corollary 2.1 will establish the Corollary 2.2.

Remark 1. Setting $b = c = 1$ in (2.1), (2.4), (2.5) and (2.6) and adjusting the parameters, we easily get equations (2.1), (2.3), (2.6) and (2.7) which are known integral formulas involving Bessel functions $J_\nu(z)$ of Agarwal et al. [14]. Further, on setting $b = 1$ and $c = -1$ in (2.1), (2.4), (2.5) and (2.6) and adjusting the parameters, we obtain new integral formulas involving the Modified Bessel functions $I_\nu(z)$.

3 Special cases

In this section, we establish certain new integral formulas for the cosine and sine functions.

Corollary 1. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho) > \frac{b}{2}$, $\Re(\rho+\sigma) > 0$, $\Re(2\rho+\sigma) > \frac{b}{2}$ and $x > 0$,

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-b-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{b}{2}-1}\cos\left(cy\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{2}{3}\right)^{2(\rho+\sigma)}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2},2);\\ (\frac{1}{2},1),\ \ (2\rho+\sigma-\frac{b}{2},2);\end{array}\ -\frac{c^2y^2}{4}\right]. \tag{3.1}$$

Corollary 2. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho) > \frac{b}{2}$, $\Re(\rho+\sigma) > 0$, $\Re(2\rho+\sigma) > \frac{b}{2}$ and $x > 0$,

$$\int_0^1 x^{\rho-\frac{b}{2}-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-b-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\cos\left(cxy\left(1-\frac{x}{3}\right)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{2}{3}\right)^{2\rho-b}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2},2);\\ (\frac{1}{2},1),\ \ (2\rho+\sigma-\frac{b}{2},2);\end{array}\ -\frac{4c^2y^2}{81}\right]. \tag{3.2}$$

The above two corollaries can be established with the help of Theorems 2.1 and 2.2 by setting $\nu = -\frac{b}{2}$, c is replacing by c^2 and then using (1.2).

Corollary 3. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho) > \frac{b}{2}$, $\Re(\rho+\sigma) > 0$, $\Re(2\rho+\sigma) > \frac{b}{2}$ and $x > 0$,

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-b-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{b}{2}-1}\cosh\left(cy\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{2}{3}\right)^{2(\rho+\sigma)}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2},2);\\ (\frac{1}{2},1),\ \ (2\rho+\sigma-\frac{b}{2},2);\end{array}\ \frac{c^2y^2}{4}\right]. \tag{3.3}$$

Corollary 4. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho) > \frac{b}{2}$, $\Re(\rho+\sigma) > 0$, $\Re(2\rho+\sigma) > \frac{b}{2}$ and $x > 0$,

$$\int_0^1 x^{\rho-\frac{b}{2}-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-b-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\cosh\left(cxy\left(1-\frac{x}{3}\right)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{2}{3}\right)^{2\rho-b}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2},2);\\ (\frac{1}{2},1),\ \ (2\rho+\sigma-\frac{b}{2},2);\end{array}\ \frac{4c^2y^2}{81}\right]. \tag{3.4}$$

The above two corollaries can be established with the help of Theorems 2.1 and 2.2 by setting $\nu = -\frac{b}{2}$, c is replacing by $-c^2$ and then using (1.3).

Corollary 5. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho+\sigma) > 0$, $\Re(\rho) > \frac{b}{2}-1$, $\Re(2\rho+\sigma) > \frac{b}{2}-1$ and $x > 0$,

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-b-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{b}{2}-1}\sin\left(cy\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{y}{2}\right)\left(\frac{2}{3}\right)^{2(\rho+\sigma)}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2}+1,2);\\ (\frac{3}{2},1),\ \ (2\rho+\sigma-\frac{b}{2}+1,2);\end{array}\ -\frac{c^2y^2}{4}\right]. \tag{3.5}$$

Corollary 6. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho+\sigma) > 0$, $\Re(\rho) > \frac{b}{2}-1$, $\Re(2\rho+\sigma) > \frac{b}{2}-1$ and $x > 0$,

$$\int_0^1 x^{\rho-\frac{b}{2}-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-b-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\sin\left(cxy\left(1-\frac{x}{3}\right)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{y}{2}\right)\left(\frac{2}{3}\right)^{2\rho-b+2}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2}+1,2);\\ (\frac{3}{2},1),\ \ (2\rho+\sigma-\frac{b}{2}+1,2);\end{array}\ -\frac{4c^2y^2}{81}\right]. \tag{3.6}$$

The above two corollaries can be established with the help of Theorems 2.1 and 2.2 by setting $\nu = 1-\frac{b}{2}$, c is replacing by c^2 and then using (1.4).

Corollary 7. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho+\sigma) > 0$, $\Re(\rho) > \frac{b}{2}-1$, $\Re(2\rho+\sigma) > \frac{b}{2}-1$ and $x > 0$,

$$\int_0^1 x^{\rho+\sigma-1}(1-x)^{2\rho-b-1}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{b}{2}-1}\sinh\left(cy\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{y}{2}\right)\left(\frac{2}{3}\right)^{2(\rho+\sigma)}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2}+1,2);\\ (\frac{3}{2},1),\ \ (2\rho+\sigma-\frac{b}{2}+1,2);\end{array}\ \frac{c^2y^2}{4}\right]. \tag{3.7}$$

Corollary 8. The following integral formula holds true: For $\rho,\ \sigma,\ b \in C$ with $\Re(\rho+\sigma) > 0$, $\Re(\rho) > \frac{b}{2}-1$, $\Re(2\rho+\sigma) > \frac{b}{2}-1$ and $x > 0$,

$$\int_0^1 x^{\rho-\frac{b}{2}-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-b-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\sinh\left(cxy\left(1-\frac{x}{3}\right)^2\right)dx$$
$$=\sqrt{\pi}\left(\frac{y}{2}\right)\left(\frac{2}{3}\right)^{2\rho-b+2}\Gamma(\rho+\sigma)\,{}_1\Psi_2\left[\begin{array}{c}(\rho-\frac{b}{2}+1,2);\\ (\frac{3}{2},1),\ \ (2\rho+\sigma-\frac{b}{2}+1,2);\end{array}\ \frac{4c^2y^2}{81}\right]. \tag{3.8}$$

The above two corollaries can be established with the help of Theorems 2.1 and 2.2 by setting $\nu = 1 - \frac{b}{2}$, c is replacing by $-c^2$ and then using (1.5).

Remark 2. On setting $b = c = 1$ in (3.1), (3.2), (3.5) and (3.6), we see that these results reduces to the known results (3.2), (3.3), (3.7) and (3.8) of Agarwal et al. [14]. Also, we notice that for $\nu = -\frac{b}{2}$, c replaced by c^2 and $-c^2$ in (2.5) and (2.6) and then using (1.2) and (1.3) respectively, we will obtain some interesting integral formulas. Further, for $\nu = 1 - \frac{b}{2}$, c replaced by c^2 and $-c^2$ in (2.5) and (2.6) and then using (1.4) and (1.5) respectively, we may obtain some more interesting integral formulas.

4 Concluding Remarks

In the present paper, we have investigated some new integral formulas involving the generalized Bessel function $w^b_{\nu,c}(z)$, which are expressed in terms of generalized (Wright) hypergeometric functions. Also, it can be easily seen that $J_\nu(z)$, $I_\nu(z)$, $\frac{2j_\nu}{\sqrt{\pi}}$ and $\frac{2i_\nu}{\sqrt{\pi}}$ are special cases of the generalized Bessel function $w^b_{\nu,c}(z)$. Therefore, the results presented in this paper are easily converted in terms of various Bessel functions after some suitable parametric replacements. Bessel functions are associated with a wide range of problems in diverse areas of mathematical physics, for example, neutrons physics, plasma physics and radio physics etc. So the results presented in this paper may be applicable in the theory of mathematical physics.

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