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Some Integral Inequalities in Terms of Supremum Norms of *n*-Time Differentiable Functions

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Abstract: In the paper, the authors establish identities for *n*-time differentiable functions and obtain some integral inequalities in terms of supremum norms of *n*-time differentiable functions. These results generalize Ostrowski's and Simpson's inequalities.

Keywords: integral inequality, differentiable function, identity, supremum norm, Ostrowski's inequality, Simpson's inequality

1 Introduction

Throughout this paper, we use the following notations:

$$\mathbb{R}=(-\infty,\infty),\quad \mathbb{R}_0=[0,\infty),\quad \text{and}\quad \mathbb{R}_+=(0,\infty).$$

In 1938, Ostrowski proved the following integral inequality.

Theorem 1 ([1, p. 468]) *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and let $a,b\in I^\circ$ with $a<b$. If $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le\left[\frac14+\frac{1}{(b-a)^2}\left(x-\frac{a+b}{2}\right)^2\right](b-a)\|f'\|_\infty \tag{1.1}$$

for $x\in(a,b)$ and the constant $\frac14$ is sharp in the sense that it can not be replaced by a smaller one.

In 1976, D. S. Mitrinović and J. E. Pečarić generalized Ostrowski's inequality (1.1) to one for n-time differentiable mappings, the case $n=2$ of which can be formulated as follows.

Theorem 2 ([1, p. 470]) *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that $f'':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$, then*

$$\left|f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le\frac12\left[\frac{1}{12}+\frac{1}{(b-a)^2}\left(x-\frac{a+b}{2}\right)^2\right](b-a)^2\|f''\|_\infty$$

for all $x\in(a,b)$.

The following inequality is well-known in the literature as Simpson's inequality.

Theorem 3 ([1]) *Let $f: I\subset\mathbb{R}_0\to\mathbb{R}$ be a four times continuously differentiable mapping on $[a,b]$ and $\|f^{(4)}\|_\infty=\sup_{t\in(a,b)}|f^{(4)}(t)|<\infty$. Then*

$$\left|\frac16\left[f(a)+f\left(x-\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le\frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty. \tag{1.2}$$

In [2], the authors presented the following inequalities.

Theorem 4 ([2, Theorem 3.1]) *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty([a,b])$. Then for all $x\in[a,b]$ we have*

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$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t-\sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right|\\ \le\frac{\|f^{(n)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\\ \le\frac{\|f^{(n)}\|_\infty(b-a)^{n+1}}{(n+1)!},\end{aligned}\tag{1.3}$$

where $\|f^{(n)}\|_\infty=\sup_{t\in[a,b]}|f^{(n)}(t)|<\infty$.

Theorem 5 ([2, Corollary 3.3]) *Assume that f is as in Theorem 4, then we have*

$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^kf^{(k)}(b)}{2}\right]\right|\\ \le\frac{\|f^{(n)}\|_\infty(b-a)^{n+1}}{(n+1)!}\begin{cases}1, & n=2r,\\ \dfrac{2^{2r+1}-1}{2^{2r+1}}, & n=2r+1.\end{cases}\end{aligned}\tag{1.4}$$

For recent refinements, counterparts, and generalizations on this topic, please refer to [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and closely related references therein.

In this paper, by establishing identities for n-time differentiable functions, we will obtain some integral inequalities in terms of supremum norms of functions.

2 Integral identities

In order to verify our theorems, the following lemma is necessary.

Lemma 1 *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. If $\lambda\in\mathbb{R}$ and $f^{(n)}$ exists for $n\in\mathbb{N}$ and is integrable on $[a,b]$, then*

$$\begin{aligned}&\lambda\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\&+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!(b-a)}\{(x-a)^k[x-a-(k+1)\lambda(b-a)]\\&\quad-(x-b)^k[x-b+(k+1)\lambda(b-a)]\}f^{(k)}(x)\\&=\frac{(-1)^{n-1}}{b-a}\int_a^b K_n(t,x;\lambda)f^{(n)}(t)\,\mathrm{d}t,\end{aligned}\tag{2.1}$$

where

$$K_n(x,t;\lambda)=\begin{cases}\dfrac{(t-a)^{n-1}}{n!}[t-a-n\lambda(b-a)], & t\in[a,x],\\ \dfrac{(t-b)^{n-1}}{n!}[t-b+n\lambda(b-a)], & t\in(x,b].\end{cases}$$

Proof. When $n=1$, integrating by parts in the right-hand side of (2.1) gives

$$\begin{aligned}&\frac{1}{b-a}\left\{\int_a^x[t-a-\lambda(b-a)]f'(t)\,\mathrm{d}t\right.\\&\qquad\left.+\int_x^b[t-b+\lambda(b-a)]f'(t)\,\mathrm{d}t\right\}\\&=\frac{1}{b-a}\{[x-a-\lambda(b-a)]f(x)+\lambda(b-a)f(a)-[x-b\\&\quad+\lambda(b-a)]f(x)+\lambda(b-a)f(b)\}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\&=\lambda[f(a)+f(b)]+(1-2\lambda)f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t.\end{aligned}$$

When $n=2$, integrating by parts twice in the right-hand side of (2.1) leads to

$$\begin{aligned}&\frac{-1}{2(b-a)}\left\{\int_a^x(t-a)[t-a-2\lambda(b-a)]f''(t)\,\mathrm{d}t\right.\\&\quad\left.+\int_x^b(t-b)[t-b+2\lambda(b-a)]f''(t)\,\mathrm{d}t\right\}\\&=\frac{-1}{2(b-a)}\left\{(x-a)[x-a-2\lambda(b-a)]f'(x)\right.\\&\quad-(x-b)[x-b+2\lambda(b-a)]f'(x)\\&\quad-2\int_a^x[t-a-\lambda(b-a)]f'(t)\,\mathrm{d}t\\&\quad\left.-2\int_x^b[t-b+\lambda(b-a)]f'(t)\,\mathrm{d}t\right\}\\&=\frac{-1}{2(b-a)}\{(x-a)[x-a-2\lambda(b-a)]\\&\quad-(x-b)[x-b+2\lambda(b-a)]\}f'(x)\\&\quad+(1-2\lambda)f(x)+\lambda[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t.\end{aligned}$$

When $n=m-1\ge2$, suppose that the identity (2.1) is valid. When $n=m$, we have

$$\begin{aligned}&\frac{(-1)^{m-1}}{m!(b-a)}\left\{\int_a^x(t-a)^{m-1}[t-a-m\lambda(b-a)]f^{(m)}(t)\,\mathrm{d}t\right.\\&\quad\left.+\int_x^b(t-b)^{m-1}[t-b+m\lambda(b-a)]f^{(m)}(t)\,\mathrm{d}t\right\}\\&=\frac{(-1)^{m-1}}{m!(b-a)}\{(x-a)^{m-1}[x-a-m\lambda(b-a)]f^{(m-1)}(x)\\&\quad-(x-b)^{m-1}[x-b+m\lambda(b-a)]f^{(m-1)}(x)\}\\&\quad-\frac{(-1)^{m-1}}{m!(b-a)}\left\{\int_a^x(t-a)^{m-2}[(m-1)(t-a\right.\\&\quad-m\lambda(b-a))+(t-a)]f^{(m-1)}(t)\,\mathrm{d}t\\&\quad+\int_x^b(t-b)^{m-2}[(m-1)(t-b+m\lambda(b-a))\\&\quad\left.+(t-b)]f^{(m-1)}(t)\,\mathrm{d}t\right\}\\&=\frac{(-1)^{m-1}}{m!(b-a)}\{(x-a)^{m-1}[x-a-m\lambda(b-a)]\end{aligned}$$

$$
\begin{aligned}
&-(x-b)^{m-1}[x-b+m\lambda(b-a)]\big\}f^{(m-1)}(x)\\
&+\frac{(-1)^{m-2}}{(m-1)!(b-a)}\bigg\{\int_a^x (t-a)^{m-2}[t-a\\
&-(m-1)\lambda(b-a)]f^{(m-1)}(t)\,\mathrm{d}t\\
&+\int_x^b (t-b)^{m-2}[t-b+(m-1)\lambda(b-a)]f^{(m-1)}(t)\,\mathrm{d}t\bigg\}\\
=&\,\lambda[f(a)+f(b)]+\sum_{k=0}^{m-1}\frac{(-1)^k}{(k+1)!(b-a)}\big\{(x-a)^k[x-a\\
&-(k+1)\lambda(b-a)]-(x-b)^k[x-b\\
&+(k+1)\lambda(b-a)]\big\}f^{(k)}(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t.
\end{aligned}
$$

This means that the identity (2.1) holds also for $n=m$. By induction, the identity (2.1) holds for all $n\in\mathbb{N}$. The proof of Lemma 1 is complete.

Corollary 1 *Under the conditions of Lemma 1, we have*

$$
\begin{aligned}
&\lambda[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\quad+\sum_{k=0}^{n-1}\frac{(b-a)^k[1-(k+1)\lambda]}{(k+1)!}f^{(k)}(a)\\
&=\frac{(-1)^{n-1}}{n!(b-a)}\int_a^b (t-b)^{n-1}[t-b+n\lambda(b-a)]f^{(n)}(t)\,\mathrm{d}t,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&+\sum_{k=1}^{n-1}\frac{(b-a)^k[1+(-1)^k][1-2(k+1)\lambda]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\\
&\quad=\frac{(-1)^{n-1}}{n!(b-a)}\bigg\{\int_a^{(a+b)/2}(t-a)^{n-1}\\
&\quad\times[t-a-n\lambda(b-a)]f^{(n)}(t)\,\mathrm{d}t+\\
&\int_{(a+b)/2}^b (t-b)^{n-1}[t-b+n\lambda(b-a)]f^{(n)}(t)\,\mathrm{d}t\bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\quad+\sum_{k=0}^{n-1}\frac{(-1)^k(b-a)^k[1-(k+1)\lambda]}{(k+1)!}f^{(k)}(b)\\
&=\frac{(-1)^{n-1}}{n!(b-a)}\int_a^b (t-a)^{n-1}[t-a-n\lambda(b-a)]f^{(n)}(t)\,\mathrm{d}t,
\end{aligned}
\tag{2.3}
$$

where the sum above takes 0 *when* $n=1$.

Proof. These are special cases of Lemma 1 for $x=a,\frac{a+b}{2},b$ respectively.

Adding the identities (2.2) and (2.3) and then dividing by 2 result in the following corollary.

Corollary 2 *With the assumptions of Lemma 1, we have*

$$
\begin{aligned}
&\lambda[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&+\sum_{k=0}^{n-1}\frac{(b-a)^k[1-(k+1)\lambda]}{2(k+1)!}\big[f^{(k)}(a)+(-1)^kf^{(k)}(b)\big]\\
&\quad=\frac{(-1)^{n-1}}{2n!(b-a)}\int_a^b\big\{(t-b)^{n-1}[t-b+n\lambda(b-a)]\\
&\qquad+(t-a)^{n-1}[t-a-n\lambda(b-a)]\big\}f^{(n)}(t)\,\mathrm{d}t\\
&\quad=\frac{1}{2n!(b-a)}\int_a^b\big\{(b-t)^{n-1}[t-b+n\lambda(b-a)]\\
&+(-1)^{n-1}(t-a)^{n-1}[t-a-n\lambda(b-a)]\big\}f^{(n)}(t)\,\mathrm{d}t.
\end{aligned}
$$

Corollary 3 *Under the conditions of Lemma 1, we have*

$$
\begin{aligned}
&\lambda[f(a)+f(b)]+(1-2\lambda)f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\qquad=\frac{1}{b-a}\int_a^b K_1(t,x;\lambda)f'(t)\,\mathrm{d}t,\\
&\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\quad=\frac{1}{b-a}\bigg\{\int_a^{(a+b)/2}[t-a-\lambda(b-a)]f'(t)\,\mathrm{d}t\\
&\qquad+\int_{(a+b)/2}^b[t-b+\lambda(b-a)]f'(t)\,\mathrm{d}t\bigg\},\\
&\lambda[f(a)+f(b)]+(1-2\lambda)f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\qquad-\frac{(1-2\lambda)(2x-a-b)}{2}f'(x)\\
&\qquad\qquad=\frac{-1}{b-a}\int_a^b K_2(t,x;\lambda)f''(t)\,\mathrm{d}x,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&=\frac{-1}{2(b-a)}\bigg\{\int_a^{(a+b)/2}\big[(t-a)^2-2\lambda(b-a)(t-a)\big]f''(t)\,\mathrm{d}t\\
&+\int_{(a+b)/2}^b\big[(t-b)^2+2\lambda(b-a)(t-b)\big]f''(t)\,\mathrm{d}t\bigg\}.
\end{aligned}
\tag{2.4}
$$

Proof. These follow from taking $n=1$, $n=1$ and $x=\frac{a+b}{2}$, $n=2$, and $n=2$ and $x=\frac{a+b}{2}$ in Lemma 1 respectively.

The following Taylor-like formula with an integral remainder also holds.

Corollary 4 *Let $f:[a,y]\to\mathbb{R}$ be a mapping such that $f^{(n)}$ is absolutely continuous on $[a,y]$, then for all $x\in[a,y]$, we have*

$$\begin{aligned} f(y) = f(a) + (y-a)\lambda[f'(a)+f'(y)] + \\ \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\{(x-a)^k[x-a-(k+1)\lambda(y-a)] \\ -(x-y)^k[x-y+(k+1)\lambda(y-a)]\}f^{(k+1)}(x) \\ +(-1)^n\int_a^y K_n(t,x;\lambda)f^{(n+1)}(t)\,\mathrm{d}t. \end{aligned} \tag{2.5}$$

Proof. This can be deduced from replacing f by f' and letting $b=y$ in Lemma 1.

3 Some integral inequalities in terms of supremum norms

We are now in a position to establish some integral inequalities in terms of supremum norms of differentiable functions.

Theorem 6 *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable function such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty([a,b])$. Then*

$$\begin{aligned} &\Bigg|\lambda[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \\ &+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!(b-a)}\{(x-a)^k[x-a-(k+1)\lambda(b-a)] \\ &-(x-b)^k[x-b+(k+1)\lambda(b-a)]\}f^{(k)}(x)\Bigg| \\ &\le \frac{\|f^{(n)}\|_\infty}{(n+1)!(b-a)} \\ &\times\begin{cases} 4n^n[\lambda(b-a)]^{n+1}+(b-x)^{n+1}+(x-a)^{n+1} \\ \quad -(n+1)\lambda(b-a)[(b-x)^n+(x-a)^n], \\ \quad 0\le\lambda\le\lambda_m(x;n); \\ 2n^n[\lambda(b-a)]^{n+1} \\ \quad +n^{n+1}(b-a)^{n+1}\left[\lambda_M^{n+1}(x;n)-\lambda_m^{n+1}(x;n)\right] \\ \quad -(n+1)\lambda n^n(b-a)^{n+1}[\lambda_M^n(x;n)-\lambda_m^n(x;n)], \\ \quad \lambda_m(x;n)\le\lambda\le\lambda_M(x;n); \\ (n+1)\lambda(b-a)[(b-x)^n+(x-a)^n] \\ \quad -[(b-x)^{n+1}+(x-a)^{n+1}], \quad \lambda\ge\lambda_M(x;n) \end{cases} \\ &\le \frac{\|f^{(n)}\|_\infty(b-a)^n}{(n+1)!} \\ &\times\begin{cases} 4n^n\lambda^{n+1}+1-(n+1)\lambda, \quad 0\le\lambda\le\lambda_m(x;n); \\ 2n^n\lambda^{n+1}+n^{n+1}[\lambda_M^{n+1}(x;n)-\lambda_m^{n+1}(x;n)] \\ \quad -(n+1)\lambda n^n[\lambda_M^n(x;n)-\lambda_m^n(x;n)], \\ \quad \lambda_m(x;n)\le\lambda\le\lambda_M(x;n); \\ (n+1)\lambda-1, \quad \lambda\ge\lambda_M(x;n) \end{cases} \end{aligned} \tag{3.1}$$

holds for all $t\in[a,b]$, where $n\in\mathbb{N}$, $x\in[a,b]$, and

$$\lambda_m(x;n)=\min\left\{\frac{x-a}{n(b-a)},\frac{b-x}{n(b-a)}\right\}, \tag{3.2}$$

$$\lambda_M(x;n)=\max\left\{\frac{x-a}{n(b-a)},\frac{b-x}{n(b-a)}\right\}. \tag{3.3}$$

Proof. Making use of the identity (2.1) yields

$$\begin{aligned} &\Bigg|\lambda[f(a)+f(b)]+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!(b-a)}\{(x-a)^k[x-a \\ &-(k+1)\lambda(b-a)]-(x-b)^k[x-b+(k+1)\lambda(b-a)]\} \\ &\times f^{(k)}(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\Bigg| \\ &\le\frac{\|f^{(n)}\|_\infty}{b-a}\int_a^b|K_n(t,x;\lambda)|\,\mathrm{d}t \\ &=\frac{\|f^{(n)}\|_\infty}{n!(b-a)}\left[\int_a^x(t-a)^{n-1}|t-a-n\lambda(b-a)|\,\mathrm{d}t\right. \\ &\left.+\int_x^b(b-t)^{n-1}|t-b+n\lambda(b-a)|\,\mathrm{d}t\right]. \end{aligned} \tag{3.4}$$

A straightforward computation gives

$$\int_a^x(t-a)^{n-1}|t-a-n\lambda(b-a)|\,\mathrm{d}t= \begin{cases} \dfrac{2n^n[\lambda(b-a)]^{n+1}+(x-a)^n[(x-a)-(n+1)\lambda(b-a)]}{n+1}, \\ \quad 0\le\lambda\le\dfrac{x-a}{n(b-a)}; \\ \dfrac{(x-a)^n}{n+1}[(n+1)\lambda(b-a)-(x-a)], \quad \lambda>\dfrac{x-a}{n(b-a)} \end{cases}$$

and

$$\int_x^b(b-t)^{n-1}|t-b+n\lambda(b-a)|\,\mathrm{d}t= \begin{cases} \dfrac{2n^n[\lambda(b-a)]^{n+1}+(b-x)^n[(b-x)-(n+1)\lambda(b-a)]}{n+1}, \\ \quad 0\le\lambda\le\dfrac{b-x}{n(b-a)}; \\ \dfrac{(b-x)^n}{n+1}[(n+1)\lambda(b-a)-(b-x)], \quad \lambda>\dfrac{b-x}{n(b-a)}. \end{cases}$$

Substituting the above equations into (3.4) leads to the first part of the inequality (3.1).

We observe that $(x-a)^{n+1}+(b-x)^{n+1}\le(b-a)^{n+1}$ for all $t\in[a,b]$. Consequently, the second part of the inequality (3.1) follows. The proof of Theorem 6 is complete.

Remark 1 *If letting $\lambda=0$, then Theorem 6 becomes Theorem 4.*

Corollary 5 *Under the conditions of Theorem 6, we have*

$$\Bigg|\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \sum_{k=1}^{n-1}\frac{(b-a)^k[1+(-1)^k][1-2(k+1)\lambda]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\Bigg| \le \frac{\|f^{(n)}\|_\infty (b-a)^n}{(n+1)!2^n} \times \begin{cases} 2n^n(2\lambda)^{n+1}+1-2(n+1)\lambda, & 0\le\lambda\le\frac{1}{2n}, \\ 2(n+1)\lambda-1, & \lambda\ge\frac{1}{2n}. \end{cases} \tag{3.5}$$

Proof. The mapping $h_n(x)=(x-a)^n+(b-x)^n$ on $[a,b]$ has the property

$$\inf_{x\in[a,b]}h_n(x)=h_n\left(\frac{a+b}{2}\right)=\frac{(b-a)^n}{2^{n-1}},$$

so we obtain (3.5) from (3.1) for $x=\frac{a+b}{2}$, which completes the proof.

Corollary 6 *Under the conditions of Theorem 6, we have*

$$\left|\lambda[f(a)+f(b)]+(1-2\lambda)f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le \frac{\|f'\|_\infty}{2(b-a)} \times \begin{cases} 2\lambda(2\lambda-1)(b-a)^2+(b-x)^2+(x-a)^2, \\ \quad 0\le\lambda\le\lambda_m(x;1); \\ (b-a)^2\{2\lambda^2+\lambda_M^2(x;1)-\lambda_m^2(x;1) \\ \quad -2\lambda[\lambda_M(x;1)-\lambda_m(x;1)]\}, \\ \quad \lambda_m(x;1)\le\lambda\le\lambda_M(x;1); \\ 2\lambda(b-a)^2-[(b-x)^2+(x-a)^2], \\ \quad \lambda\ge\lambda_M(x;1). \end{cases} \tag{3.6}$$

Proof. This follows from choosing $n=1$ in the inequality (3.1).

Remark 2 *A simple calculation shows that*

$$\frac{(x-a)^2+(b-x)^2}{2}=\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2.$$

Choosing $\lambda=0$ in (3.6), we obtain Ostrowski's inequality (1.1).

Corollary 7 *Under the conditions of Theorem 3.1, we have*

$$\left|\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le \frac{\|f'\|_\infty(b-a)}{4}\begin{cases} 8\lambda^2+1-4\lambda, & 0\le\lambda\le\frac{1}{2}, \\ 4\lambda-1, & \lambda\ge\frac{1}{2}; \end{cases}$$

$$\left|\lambda[f(a)+f(b)]+(1-2\lambda)f(x)-\frac{(1-2\lambda)(2x-a-b)}{2}f'(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le \frac{\|f''\|_\infty}{6(b-a)} \times \begin{cases} 16[\lambda(b-a)]^3+(b-x)^3+(x-a)^3 \\ \quad -3\lambda(b-a)\left[(b-x)^2+(x-a)^2\right], \\ \quad 0\le\lambda\le\lambda_m(x;2); \\ 8[\lambda(b-a)]^3+8(b-a)^3\left[\lambda_M^3(x;2)-\lambda_m^3(x;2)\right] \\ \quad -12\lambda(b-a)^3\left[\lambda_M^2(x;2)-\lambda_m^2(x;2)\right], \\ \quad \lambda_m(x;2)\le\lambda\le\lambda_M(x;2); \\ 3\lambda(b-a)\left[(b-x)^2+(x-a)^2\right] \\ \quad -\left[(b-x)^3+(x-a)^3\right], \quad \lambda\ge\lambda_M(x;2) \end{cases}$$

$$\le \frac{\|f''\|_\infty(b-a)^2}{6}\begin{cases} 16\lambda^3+1-3\lambda, \quad 0\le\lambda\le\lambda_m(x;2); \\ 8\lambda^3+8\left[\lambda_M^3(x;2)-\lambda_m^3(x;2)\right] \\ \quad -12\lambda\left[\lambda_M^2(x;2)-\lambda_m^2(x;2)\right], \\ \quad \lambda_m(x;2)\le\lambda\le\lambda_M(x;2); \\ 3\lambda-1, \quad \lambda\ge\lambda_M(x;2), \end{cases}$$

and

$$\left|\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \le \frac{\|f''\|_\infty(b-a)^2}{24}\begin{cases} 64\lambda^3+1-6\lambda, & 0\le\lambda\le\frac{1}{4}, \\ 6\lambda-1, & \lambda\ge\frac{1}{4}; \end{cases} \tag{3.7}$$

where $\lambda_m(x;2)$ and $\lambda_M(x;2)$ arc defined in (3.2).

Proof. This follows from taking $n=1,2$ in the inequality (3.5) and $n=2$ in the inequality (3.1) respectively.

Corollary 8 *Under the conditions of Corollary 4, we have*

$$\Bigg|f(y)-f(a)-(y-a)\lambda[f'(a)+f'(y)] - \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\{(x-a)^k[x-a-(k+1)\lambda(y-a)] - (x-y)^k[x-y+(k+1)\lambda(y-a)]\}f^{(k+1)}(x)\Bigg| \le \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \times \begin{cases} 4n^n[\lambda(y-a)]^{n+1}+(y-x)^{n+1}+(x-a)^{n+1} \\ \quad -(n+1)\lambda(y-a)[(y-x)^n+(x-a)^n], \\ \quad 0\le\lambda\le\lambda_m(x;n); \\ 2n^n[\lambda(y-a)]^{n+1} \\ \quad +n^{n+1}(y-a)^{n+1}[\lambda_M^{n+1}(x;n)-\lambda_m^{n+1}(x;n)] \\ \quad -(n+1)\lambda n^n(y-a)^{n+1}[\lambda_M^n(x;n)-\lambda_m^n(x;n)], \\ \quad \lambda_m(x;n)\le\lambda\le\lambda_M(x;n); \\ (n+1)\lambda(y-a)[(y-x)^n+(x-a)^n] \\ \quad -[(y-x)^{n+1}+(x-a)^{n+1}], \\ \quad \lambda\ge\lambda_M(x;n) \end{cases}$$

$$\le \frac{\|f^{(n+1)}\|_\infty (y-a)^{n+1}}{(n+1)!} \times \begin{cases} 4n^n\lambda^{n+1}+1-(n+1)\lambda, & 0\le\lambda\le\lambda_m(x;n); \\ 2n^n\lambda^{n+1}+n^{n+1}[\lambda_M^{n+1}(x;n)-\lambda_m^{n+1}(x;n)] \\ \quad -(n+1)n^n\lambda[\lambda_M^n(x;n)-\lambda_m^n(x;n)], \\ \quad \lambda_m(x;n)\le\lambda\le\lambda_M(x;n); \\ (n+1)\lambda-1, & \lambda\ge\lambda_M(x;n). \end{cases}$$

4 Conclusions

By establishing integral identities for n-time differentiable functions, the authors obtain several integral inequalities in terms of supremum norms of n-time differentiable functions. These newly established inequalities generalize Ostrowski's and Simpson's inequalities.

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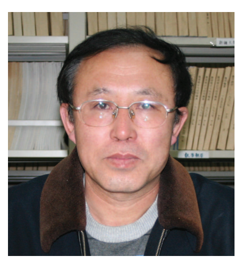

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