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The New Probability Distribution: An Aspect to a Life Time Distribution

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Abstract: In the present article, we have proposed a method to construct a new distribution on the basis of any two baseline independent continuous distributions on the same spectrum and studied some statistical properties of the proposed distribution. Further, for application point of view, we have derived it for Weibull distribution as a baseline distribution and proved its application in comparison to the other well known distributions like gamma, Weibull and exponentiated exponential distribution in terms of fitting a real data through AIC, BIC test and log likelihood (LL) criterion of goodness of fit.

Keywords: Life Time Distribution, Reliability Analysis and Maximum Likelihood Estimation.

1 Introduction

In statistical literature, several life time distributions are available like Weibull, gamma, exponentiated exponential and many more to analyze the data on medical field, engineering and finance sectors etc. No doubt, modeling and analyzing the life time data are crucial. The quality of the outputs of the statistical analysis depends heavily on the assumed life time models or distributions (see, Merovci [1]). Recently applications from environmental sciences, biomediacal sciences, finance and engineering sectors and many others shows that the available data sets following the classical distributions are more often the exception rather than the reality (see, Aryal [3]). Therefore, it becomes very important to find more nearer distribution in comparison to the classical distributions to get more accurate results.

To solve such problems many development has taken place. For example, Gupta et al. [12] proposed the cdf of the new distribution as $F_1(t) = [F(t)]^{\alpha}$, for all $\alpha > 0$, where F(t) is the cdf of any baseline distribution. Many works has been published on the basis of it, using different classical baseline distributions (for example, Gupta and Kundu [10], Seenoi et al. [19], Elbatal and Muhammed [18] etc.). Later on quadratic rank transmuted map (QRTM) has been proposed by Shaw and Buckley [8]. According to the QRTM, the cdf $F_2(x)$ of the new distribution corresponding to the base line distribution having cdf F(x), is given by, $F_2(x) = (1+\lambda)F(x) - \lambda F^2(x)$ for all $|\lambda| \leq 1$. Many researchers have used QRTM to develop new life time distribution (see, Khan and King [5], Aryal and Tsokos [6], Khan et al. [2] etc.).

In the present article, we propose the cdf of new distribution (denoting it by G(x)) by the use of any two (may be the same) cdfs $F_1(x)$ and $F_2(x)$ of baseline continuous distribution(s) with common spectrum; by the transformation,

$$G(x) = \frac{F_1(x) + F_2(x)}{1 + F_1(x)} \tag{1}$$

We will call transformation (1) as M transformation for its frequently used purpose in our work or elsewhere.

Theorem:
G(x) possesses the property of a cdf.

Proof:

1. $F_1(x)$ and $F_2(x)$ are the cdfs of any two independent continuous random variables with common spectrum, which implies that $F_1(x)$, $F_2(x)$ and hence $1 + F_1(x)$

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are continuous functions (see, Rohatgi and Saleh [7]). Again, $1+F_1(x)\neq 0\ \forall\ x$. It proves that $G(x)$ is a continuous function of x (see, Mapa [11]).

2. $0\leq F_1(x)\leq 1, 0\leq F_2(x)\leq 1\ \forall\ x \Rightarrow 0\leq G(x)\leq 1\ \forall\, x\in R$
3. $G^{'}(x)=\frac{f_1(x)[1-F_2(x)]+f_2(x)[1+F_1(x)]}{[1+F_1(x)]^2}\geq 0\forall x\in R$
4. $G(-\infty)=0\ \&\ G(\infty)=1$.

Thus, G(x) satisfying the sufficient conditions for a function to be a cdf and hence so (see, Rohatgi and Saleh [7])

2 A particular case

If we chose, $F_1(x)=F_2(x)=F(x)$, then particularly M transformation (1) reduces to the following form,

$$G(x)=\frac{2F(x)}{1+F(x)} \tag{2}$$

The pdf corresponding to cdf (2) is given by,

$$g(x)=\frac{2f(x)}{(1+F(x))^2} \tag{3}$$

and the corresponding hazard rate function is given by,

$$h(x)=\frac{2f(x)}{1-F^2(x)} \tag{4}$$

2.1 *Order Statistic from pdf (3)*

Let, $X_{(1)}$, $X_{(2)}$, ..., $X_{(n)}$ be the order statistics of a random sample $\underline{\mathbf{X}}$ = (X_1, $X_2,\ldots,$ X_n) of size n from M(.)-distribution having pdf (3), then the pdf of the distribution of first order statistic $X_{(1)}$ is given by (see, Gun, Gupta and Dasgupta [9])

$$\begin{aligned} g_1(X_{(1)}) &= n\, g(x_{(1)})\ [1-G(x_{(1)})]^{n-1} \\ &= n\,\frac{2f(x_{(1)})}{[1+F(x_{(1)})]^2}\left[\frac{1-F(x_{(1)})}{1+F(x_{(1)})}\right]^{n-1} \end{aligned} \tag{5}$$

Similarly, the pdf of the distribution of largest order statistic X(n) is given by,

$$\begin{aligned} g_n(X_{(n)}) &= n\, g((x_{(n)}))\ [G(x_{(n)})]^{n-1} \\ &= n\,\frac{2f(x_{(n)})}{[1+F(x_{(n)})]^2}\left[\frac{2F(x_{(n)})}{1+F(x_{(n)})}\right]^{n-1} \end{aligned} \tag{6}$$

In general, the pdf of the distribution of r^{th} order statistic $x_{(r)}$ is given by,

$$\begin{aligned} g_X(r) &= \frac{n!}{(r-1)!(n-r)!}g(x_{(r)})\,[G(X_{(r)})]^{r-1}\,[1-G(X_{(r)})]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!}\left[\frac{2F(X_{(r)})}{1+F(X_{(r)})}\right]^{r-1}\left[1-\frac{2F(X_{(r)})}{1+F(X_{(r)})}\right]^{n-r}\left[\frac{2f(x_{(r)})}{(1+F(x_{(r)}))^2}\right] \end{aligned} \tag{7}$$

3 M transformation of Weibull distribution using (2)

The most flexible life distribution is Weibull distribution with its pdf given by,

$$f(x)=\left(\frac{k}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1}e^{-(\frac{x}{\lambda})^k}\quad ;\quad x>0 \tag{8}$$

and the cdf of Weibull distribution having pdf (8) is given by,

$$F(x)=1-e^{-(\frac{x}{\lambda})^k} \tag{9}$$

where $k>0$ and $\lambda>0$ are its shape parameter and scale parameters respectively.

For more application related to the Weibull distribution, readers may refer to Mudholkar and Srivastava [15], Mudholkar et al. [14] etc. Mudholkar and Hutson [13] have modeled various failure time data sets with the proposed model with Weibull as the baseline distribution.

Now, using (9) in (2) and (8) in (3), we will get the cdf and pdf of the M transformation of Weibull distribution with parameters k and λ as

$$G(x)=\frac{2\left(1-e^{-(\frac{x}{\lambda})^k}\right)}{2-e^{-(\frac{x}{\lambda})^k}} \tag{10}$$

and

$$g(x)=\frac{2\left(\frac{k}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}}{\left(2-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2} \tag{11}$$

respectively.

We name M transformation (2) of Weibull distribution with the parameters k and λ as M_W(k, λ) distribution for frequently use purpose in the present article or elsewhere. Also, the hazard rate function of M_W(k,λ) distribution having pdf (11) is obtained as follows

$$h(x)=\frac{2\left(\frac{k}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}}{1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2} \tag{12}$$

The plots of g(x) and h(x) are shown in Figures 1 and 2 respectively, for different combinations of the parameters (k, λ).

3.1 *Raw Moments and Characteristic Function:*

Raw Moments

The r^{th} moment about origin $\mu_r^{'}$ (raw moment) of M_W(k, λ) distribution having pdf (11) is obtained as follows,

$$\mu_r^{'}=\frac{2k}{\lambda^k}\int_0^\infty\frac{x^{k+r-1}e^{-\left(\frac{x}{\lambda}\right)^k}}{\left(2-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2}\,dx \tag{13}$$

Above integral is not solvable in nice closed form, so we propose to use 19 pt. Gauss Lagurre quadrature formula (GLQF) for its numerical solution or some other technique, like Monte-Carlo simulation etc. may be used.

Characteristic Function

We have derived the characteristic function of $M_W(k, \lambda)$-distribution having pdf (11) and the same is obtained as follows,

$$\phi_X(t) = \frac{2k}{\lambda^k} \int_0^\infty \frac{x^{k-1} e^{itx - \left(\frac{x}{\lambda}\right)^k}}{\left(2 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2} dx \tag{14}$$

where $t \in \mathbf{R}$ is the dummy parameter.

3.2 Random sample Generation:

Using the method of inversion (see, Merovci [1]) we can generate random numbers from the $M_W(k, \lambda)$.

$$\frac{2\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)}{2 - e^{-\left(\frac{x}{\lambda}\right)^k}} = U$$

Solving above equation for x in terms of U, we get

$$x = \lambda \left[-ln \left(\frac{2-2U}{2-U} \right) \right]^{\frac{1}{k}} \tag{15}$$

where U follow $U(0,1)$- distribution.

One can use (15) to generate random samples from $M_W(k, \lambda)$ when parameters λ and k are known.

3.3 Maximum likelihood estimation

Let the lifes of n identical items put on a life testing experiment be
$\underline{\mathbf{X}} = (X_1, X_2, \ldots, X_n)$where each X_i follow $M_W(k, \lambda)$ -distribution. Then the likelihood function for $\underline{\mathbf{X}}$ is given by (see, Potdar and Shirke [4]),

$$\begin{aligned} L &= \prod_{i=1}^{n} g(x_i) \\ &= \prod_{i=1}^{n} \frac{2\left(\frac{k}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-\left(\frac{x_i}{\lambda}\right)^k}}{\left(2 - e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)^2} \\ &= \frac{(2k)^n [\prod_{i=1}^{n} x_i]^{(k-1)} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\lambda}\right)^k}}{\lambda^{nk} \prod_{i=1}^{n} \left(2 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2} \end{aligned} \tag{16}$$

and hence the log- likelihood function is given by:

$$ln\ L = C - nk\ ln\lambda - \sum_{i=1}^{n} \left(\frac{x_i}{\lambda}\right)^k - \sum_{i=1}^{n} ln \left(2 - e^{-\left(\frac{x_i}{\lambda}\right)^k}\right) \tag{17}$$

where C is a constant.

It is obvious that the log-likelihood equations for estimating k and λ are not easily solvable simultaneously for k and λ; therefore we propose to use some numerical iteration method for getting their numerical solutions.

3.4 Order Statistics:

Refer to sub-section 3.1, the pdf of the distribution of first order statistic $X_{(1)}$ of a random sample of size n from $M_W(k, \lambda)$- distribution is given by:

$$g_1(X_{(1)}) = \frac{2\ n\ k\ x_{(1)}^{k-1} \left(e^{-\left(\frac{x_{(1)}}{\lambda}\right)^k}\right)^n}{\lambda^k \left(2 - e^{-\left(\frac{x_{(1)}}{\lambda}\right)^k}\right)^{n+1}} \tag{18}$$

Similarly, pdf of the distribution of the largest order statistics X(n) is given by,

$$g_n(X_{(n)}) = \frac{n\ k\ 2^n \left(1 - e^{-\left(\frac{x_{(n)}}{\lambda}\right)^k}\right)^{n-1} e^{-\left(\frac{x_{(n)}}{\lambda}\right)^k}}{\lambda^k \left(2 - e^{-\left(\frac{x_{(1)}}{\lambda}\right)^k}\right)^{n+1}} \tag{19}$$

Finally, the pdf of the distribution of the r^{th} order statistic $X_(r)$ is given by,

$$g_r(X_{(r)}) = \left(\frac{n!}{r!(n-r)!}\right) \left(\frac{r\ k\ 2^r\ x_{(r)}^{k-1} \left(e^{-\left(\frac{x_{(r)}}{\lambda}\right)^k}\right)^{n-r+1} \left(1 - e^{-\left(\frac{x_{(r)}}{\lambda}\right)^k}\right)^{r-1}}{\lambda^k \left(2 - e^{-\left(\frac{x_{(r)}}{\lambda}\right)^k}\right)^{n+1}} \right) \tag{20}$$

4 A particular case of $M_W(k, \lambda)$ when $k = 1$

In particular, if k=1, then $M_W(k, \lambda)$- distribution shall be treated as $M_E(\lambda)$; as the M transformation through (2) of $Exp(\lambda)$- distribution. By the use of (4), we get the hazard rate function of $M_E(\lambda)$- distribution as follows,

$$h(x) = \frac{2}{\lambda\left(2 - e^{-\frac{x}{\lambda}}\right)} \tag{21}$$

It may be recalled that the hazard rate function is constant for Exp (λ)- distribution. Now, we want to show the shape of hazard rate function of $M_E(\lambda)$- distribution.

Differentiating (21) partially w.r.t. x, we get

$$h'(x) = -\frac{2\, e^{-\frac{x}{\lambda}}}{\lambda^2 \left(2 - e^{-\frac{x}{\lambda}}\right)^2} \leq 0 \,\forall\, x \geq 0, \lambda \geq 0 \qquad (22)$$

Thus, we can say that hazard rate function of $M_E(\lambda)$- distribution is always decreasing.

The pdfs and hazard rate functions of $M_E(\lambda)$- distribution for different values of λ are shown in the Figures 3 and 4 respectively.

5 Real Data Application:

Here, we have shown the applicability of $M_W(k,\lambda)$- distribution on the real life data by showing that it is better fit in comparison to some well known and exploited existing distributions. For the purpose, we considered the following data of the failure times of the air conditioning system of an air plane (see, Linhart and Zucchini [17]). The considered data set is used by many authors such as Gupta and Kundu [10], and Singh et al. [21] etc.

$23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.$

Gupta and Kundu [10] considered above data set and showed that Weibull distribution fits the best in terms of likelihood and in terms of Chi- square as compared to EED and gamma distribution and they conclude that in certain circumstances Weibull distribution might work better than EED or gamma distribution. In our case, we have derived AIC (Akaike information criterion), BIC (Bayesian information criterion) and LL criterion values of above data set for $M_W(k,\lambda)$- distribution, Weibull distribution, EED and gamma distribution

$$\text{AIC} = 2k - 2m \text{ and BIC} = k\ \ln(n) - 2m$$

where k denotes the number of unknown parameters in the model, n is the sample size and the maximized value of the log-likelihood function(LL) under the considered model is m. The results are shown in Table 1.

AIC , BIC and LL criterion of fitting is used by several author authors, for example Gupta and Kundu [10], and Singh et al. [20] etc.

From Table 1, it is quite clear that $M_W(k,\lambda)$- distribution dominates Weibull distribution, gamma distribution and EED in terms of AIC, BIC and LL test values and we may, therefore conclude that in certain circumstances $M_W(k,\lambda)$)- distribution might work better than Webull distribution, gamma distribution and EED.

Table 1: LL, AIC and BIC values of Gamma, Weibull, EED and $M_W(k,\lambda)$- distribution for the data of the failure times of the air conditioning system of an air plane

Distributions	$\hat{\lambda}$	$\hat{k}$	LL	AIC	BIC
$M_W(k,\lambda)$	84.1122	0.9872	-151.498	306.996	309.267
EED	0.0145	0.8130	-152.264	308.528	311.330
Weibull	54.6448	0.8554	-152.007	307.874	310.676
Gamma	0.0136	0.8134	-152.231	308.462	311.265

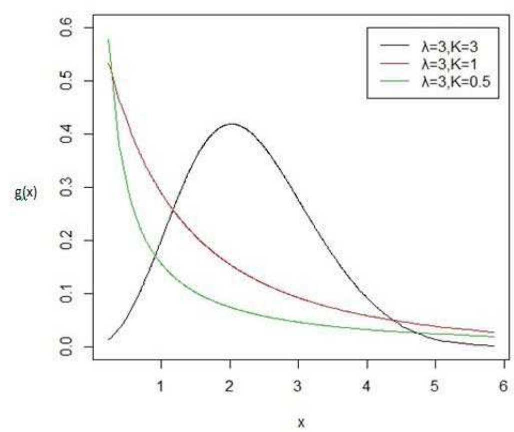

Fig. 1: Plots of pdf of M_W(k,λ)-distribution

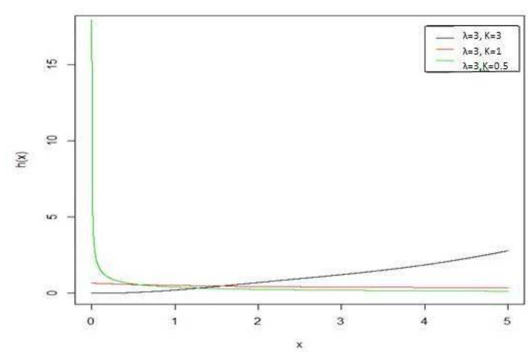

Fig. 2: Plots of hazard rate function of M_W(k,λ)-distribution

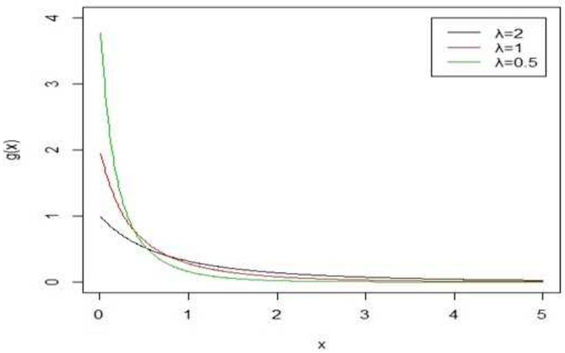

Fig. 3: Plots of pdfs of $M_E(\lambda)$- distribution for different values of λ

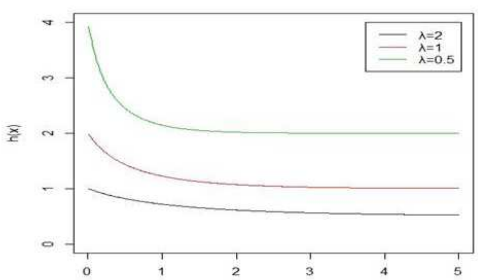

Fig. 4: Plots of Hazard rate function of$M_E(\lambda)$- distribution for different values of λ

6 Conclusion:

In the present article, we have proposed a new distribution with the help of M transformation that uses any two arbitrary continuous distributions, with common spectrum, as baseline distribution(s). As an application part, we have derived it by assuming Weibull distribution as a baseline distribution. The new distribution, thus obtained is much more flexible as its hazard rate function covers different shapes. By considering a real data, we proved its applicability in comparison to other existing exploited lifetime distributions; like gamma, Weibull and Exponentiated Exponential distributions. Thus, we recommend the use of M transformation to get new life time distributions.

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