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Controllability of Sobolev Type Nonlinear Nonlocal Fractional Functional Integrodifferential Equations

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Abstract: This paper deals with the controllability of mild solution for a class of Sobolev type nonlinear nonlocal fractional order functional integro-differential equations in a general Banach space X. We use fractional calculus, Krasnoselskii's fixed point theorem and semigroup theory for the main results and render the criteria for the complete controllability of considered problem. We also investigate the null controllability. An application is given to illustrate the abstract results.

Keywords: Sobolev-type equations, nonlinear integro-differential equations, fractional calculus, complete controllability, null controllability, analytic semigroup.

1 Introduction

Let $(X,\|.\|)$ and $(Y,\|.\|_Y)$ be the general Banach spaces. For $t \geq 0$, $(C_t,\|.\|_t)$ is a Banach space of all continuous functions from $[-\tau,t]$ into X, where $\|.\|_t$ is defined by

$$\|\psi\|_t := \sup_{-\tau\leq\eta\leq t} \|\psi(\eta)\|, \text{ for } \psi \in C_t.$$

Our main objective in this paper is to establish sufficient conditions for the controllability of the following Sobolev type nonlinear nonlocal fractional order functional integro-differential equation.

$$\begin{cases} {}_0^C D_t^\alpha [Ex(t)] + A(t)x(t) = f(t,x(t),x_t) + \int_0^t g(t,s,x(s),x_s)ds + Bu(t), \quad t \in J = [0,b], \\ h(x_{[-\tau,0]}) = \phi, \end{cases} \tag{1.1}$$

where $\tau > 0$ and $0 < \alpha < 1$. The fractional derivative ${}_0^C D_t^\alpha$ is understood in Caputo's sense. For any $t \in J$, x_t denotes the element in C_0 defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau,0]$. $h : C_0 \to C_0$, $f : J \times X \times C_0 \to Y$, and $g : \Delta \times X \times C_0 \to Y$ $(\Delta = \{(t,s) \in J \times J : t \geq s\})$ are nonlinear maps. The control function $u(.)$ is given in $L^2[J,Z]$ with Z as a Banach space and $B \in \mathbb{BL}(Z,Y)$. $-A(t) : D(A(t)) \subset X \to Y$ and $E : D(E) \subset X \to Y$ are closed linear operators such that

(I) For each $t \geq 0$, the domain $D(A(t))(= D(A))$ is independent of t.
(II) $D(E) \subset D(A)$ and E is bijective.
(III) $E^{-1} : Y \to D(E)$ is continuous.

The assumptions (II), (III) and Closed - Graph theorem imply the boundedness of linear operator $-A(t)E^{-1} : Y \to Y$. We denote the operator $-A(t)E^{-1}$ by $-Q(t)$.

Over the past years, the theory of fractional differential equations attracts many researchers due to their applications in various fields of engineering, physics and economics (see the monographs of Podlubny [1] and Tarasov [2]). In fact, many physical phenomena such as behaviors of viscoelastic materials, electrochemical processes, dielectric polarization, colored noise, chaos and many more, can be modeled more accurately by fractional derivatives or fractional integrals rather than the classical integer order derivatives or integrals, for example, see [3,4,5] and the references therein.

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Equations of the form (1.1) serve as an abstract formulation of partial differential equations which appear in many physical phenomena such as in the flow of fluid through fissured rocks [6], thermodynamics and shear in second order fluids [7], the propagation of long waves of small amplitudes [8], and so on. Moreover, the researchers have found that the introduction of nonlocal condition into the system can essentially improve its qualitative and quantitative characteristics. In fact, in many situations, nonlocal conditions arise more precisely for physical measurements than the classical conditions and therefore the problems with nonlocal conditions look more realistic than the problems with classical conditions in the treatment of physical problems. These facts attract many authors to analyze various types of evolution equations with nonlocal conditions [9,10,11].

Furthermore, there has been a significant development in the study of controllability to Sobolev type nonlinear integrodifferential equations of integer order in Banach spaces, for example, see [12,13], and the references listed therein. The controllability of integer order functional evolution systems of Sobolev type is studied by Balachandran and Dauer [14] by using Schauder fixed point theorem and classical semigroup theory. While in [15], Balachandran and Sakthivel established sufficient conditions for the controllability of Sobolev type semilinear integrodifferential equations in Banach spaces. However, there are only few papers dealing with the controllability of Sobolev type integrodifferential equations of fractional order. The problem of controllability for Sobolev type fractional functional evolution system is studied by Michal et al. [16] via the techniques of fixed point theorem and semigroup theory. Mahmudov [17] investigated sufficient conditions for the approximate controllability of Sobolev type fractional stochastic evolution systems by using the Schauder fixed point theorem. Recently, a class of Sobolev-type semilinear fractional evolution systems in a separable Banach space is studied by Wang et al. [18], where they establish the controllability result by applying techniques of fixed point theorem to an appropriate condensing mapping as well as the theory of propagation families and measure of noncompactness.

Different from these works, we analyze the fractional evolution equations of Sobolev type (1.1) with nonlocal condition and establish sufficient conditions for the complete controllability of considered equation (1.1) without assuming the compactness condition on semigroup or on bounded linear operators B and E^{-1}. We also observe that if the associated semigroup or linear operator B or E^{-1} is compact, then the considered evolution equation is completely controllable only in the translation of finite dimensional subspace of X. Moreover, the sufficient conditions for the exact null controllability to (1.1) are obtained.

The paper is organized as follows. In Section 2, we shall set forth some preliminary facts about the fractional differential equations and introduce the concept of mild solutions to (1.1). Main results concerning the sufficient conditions of controllability of (1.1) are proved in Section 3. Finally, an application is given in which a nonlocal fractional partial differential equation of Sobolev type is discussed to illustrate the abstract results.

2 Preliminaries

For an abstract continuous function f on the interval $[a,b]$, the Caputo derivative of order $0<\alpha<1$ is defined as follows. [19]

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds. $$

Here and hereafter, we assume that the operator $-Q(t)$ satisfies the following assumptions.

(B1) For each $t \in [0,T]$, the operator $[\lambda I + Q(t)]^{-1}$ exists for all λ with $\Re(\lambda) \geq 0$ and

$$ \left\| [\lambda I + Q(t)]^{-1} \right\| \leq \frac{C}{\mid \lambda \mid +1}, \quad (\Re(\lambda) \geq 0). $$

(B2) For any $t,\ s,\ \zeta \in [0,T]$, we have

$$ \left\| [Q(t) - Q(\zeta)] Q^{-1}(s) \right\| \leq C \mid t - \zeta \mid^{\gamma}, \ \ 0<\gamma<1, $$

where the constants $C,\ \gamma$ are independent of $t,\ s,\ \zeta$.

Then, for each $\sigma \in [0,T]$, $-Q(\sigma)$ generates an analytic semigroup $\{T_{\sigma}(t) = e^{-tQ(\sigma)}\}$. Moreover, there exists a positive constant C independent of both t and s such that

$$ \| Q^n(s) exp(-tQ(s)) \| \leq \frac{C}{t^n}, $$

where $n = 0,1,\ t>0,\ s \in [0,T]$. For more details we refer to [20,21].

Following Borai [22], we define operators ψ, φ and U as follows.

$$\psi(t,s) = \alpha \int_0^\infty \theta t^{\alpha-1} \xi_\alpha(\theta) exp(-t^\alpha \theta Q(s))\, d\theta,$$

where ξ_α is a probability function defined on $[0,\infty)$ whose Laplace transform is given by [22]

$$\int_0^\infty e^{-\theta x} \xi_\alpha(\theta) d\theta = \sum_{j=0}^{j=\infty} \frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad \alpha \in (0,1], \; x > 0.$$

$$\varphi(t,\eta) = \sum_{j=1}^{j=\infty} \varphi_j(t,\eta),$$

with

$$\begin{cases} \varphi_1(t,\eta) = [Q(t) - Q(\eta)]\psi(t-\eta,\eta), \\ \varphi_{j+1}(t,\eta) = \int_\eta^t \varphi_j(t,s)\varphi_1(s,\eta)ds, \text{ for } j = 1,2,\ldots \end{cases}$$

and

$$U(t) = -Q(t)Q^{-1}(0) - \int_0^t \varphi(t,s)Q(s)Q^{-1}(0)ds.$$

Let $\chi \in C_0$ such that $h(\chi) = \phi$. For each $u \in L^2[J,Z]$, a mild solution of the equation (1.1) (see [22,23]) is a function $x \in C_b$ such that $x(t) = \chi(t)$ for $t \in [-\tau,0]$ and for $t \in J$

$$\begin{aligned} x(t) = \chi(0) + \int_0^t E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta + \int_0^t E^{-1}\psi(t-\eta,\eta)(H(\eta)+Bu(\eta))d\eta \\ + \int_0^t \int_0^\eta E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)(H(s)+Bu(s))dsd\eta, \end{aligned}$$

where $H(t) = f(t,x(t),x_t) + \int_0^t g(t,s,x(s),x_s)ds.$

Lemma 1.*(Bochner's Theorem) A measurable function $S : J \to X$ is Bochner integrable if $| S |$ is Lebesgue integrable.*

Lemma 2.*(see [24]) For $m \in L^1[0,b]$, we have*

$$\int_0^t \int_0^\eta (t-\eta)^{\alpha-1}(\eta-s)^{\alpha-1} m(s)dsd\eta = \mathscr{B}(\alpha,\gamma) \int_0^t (t-s)^{\alpha+\gamma-1} m(s)ds,$$

where $\mathscr{B}(\alpha,\gamma)$ is a Beta function.

Theorem 1(Krasnoselskii's Fixed Point Theorem). *Let X be a Banach space. Let N be a bounded, closed and convex subset of X and let $\Im_1$, $\Im_2$ be maps of N into X such that $\Im_1 x + \Im_2 y \in N$ for every pair $x,y \in N$. If $\Im_1$ is a contraction and $\Im_2$ is completely continuous, then the equation $\Im_1 x + \Im_2 x = x$ has a solution on N.*

3 Main Result

This section comprises the main results concerning the controllability of mild solution for the equation (1.1). We consider the following hypotheses.

(H1) $h : C_0 \to C_0$ and there exists Lipschitz continuous function $\chi \in C_0$ such that $h(\chi) = \phi$ with $\chi(0) \in D(E)$.

(H2) $f(t,x,y)$ and $\int_0^t g(t,s,x,y)ds$ are continuous with respect to first variable. Moreover, there exist constants $q_1, q_2 \in (0,\alpha) \cap (0,\gamma)$ and functions $L_f(.) \in L^{\frac{1}{q_1}}[J,\mathbb{R}^+]$ and $L_g^* : \Delta \to [0,\infty)$ with $\int_0^t L_g^*(t,s)ds = L_g(t) \in L^{\frac{1}{q_2}}[J,\mathbb{R}^+]$ such that

$$\begin{aligned} &\|f(t,u,v) - f(t,w,p)\|_Y \le L_f(t)[\|u-w\| + \|v-p\|_0], \\ &\|g(t,s,u,v) - g(t,s,w,p)\|_Y \le L_g^*(t,s)[\|u-w\| + \|v-p\|_0], \end{aligned}$$

for all $t,s \in J$; $u,w \in X$ and $v,p \in C_0$.

(H3) The linear operator $B : L^2[J,Z] \to L^1[J,Y]$ is bounded. $W : L^2[J,Z] \to X$ defined by

$$Wu = \int_0^b E^{-1}\psi(b-\eta,\eta)Bu(\eta)d\eta + \int_0^b \int_0^\eta E^{-1}\psi(b-\eta,\eta)\varphi(\eta,s)Bu(s)dsd\eta,$$

induces an invertible operator $\widetilde{W} : L^2[J,Z]/\mathrm{Ker}W \to X$ and there exist two positive constants $M_2,\ M_3 > 0$ such that $\|B\|_{\mathbb{BL}[Z,Y]} \le M_2$ and $\left\|\widetilde{W}^{-1}\right\|_{\mathbb{BL}\left[X,L^2[J,Z]/\mathrm{Ker}W\right]} \le M_3$.

(H4) For all bounded subsets D, the set

$$\Lambda_{\varepsilon,\zeta}(t) := \left\{\int_0^{t-\varepsilon}\int_\zeta^\infty \mathscr{P}_E(\theta,t,\eta)H(\eta)d\theta d\eta + \int_0^{t-\varepsilon}\int_0^\eta\int_\zeta^\infty \mathscr{P}_E(\theta,t,\eta)\varphi(\eta,s)H(s)d\theta dsd\eta,\ x\in D\right\}$$

is relatively compact in X for arbitrary $\varepsilon \in (0,t)$ and constant $\zeta > 0$. Here, $\mathscr{P}_E(\theta,t,\eta) = \alpha\theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)E^{-1}exp\left(-(t-\eta)^\alpha\theta Q(\eta)\right)$.

For brevity, let $\widehat{E} = \|E^{-1}\|_{\mathbb{BL}[Y,D(E)]}$, where $\mathbb{BL}[Y,D(E)]$ is the class of all bounded linear operators from Y into $D(E)$.

The following lemma plays an important role in our analysis.

Lemma 3. *The following results hold.*

(i) The operator-valued functions $\psi(t-\eta,\eta)$ and $A(t)\psi(t-\eta,\eta)$ are continuous in the uniform operator topology in the variables $t,\ \eta$, where $0\le\eta\le t-\varepsilon$, $0\le t\le b$, for any $\varepsilon>0$. Moreover,

$$\left\|E^{-1}\psi(t-\eta,\eta)\right\| \le C\widehat{E}(t-\eta)^{\alpha-1}.$$

(ii) The function $\varphi(t,\eta)$ is uniformly continuous in the uniform operator topology in t,η provided $0\le\eta\le t-\varepsilon,\ \varepsilon\le t\le b$ for any $\varepsilon>0$. Moreover,

$$\|\varphi(t,\eta)\| \le C(t-\eta)^{\gamma-1}.$$

(iii) For $t\in J$, $\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta$ is uniformly continuous in the norm of $\mathbb{B}(X)$ and

$$\|U(\eta)\| \le C(1+\eta^\gamma).$$

(iv) For $0<\eta<t_1\le t_2$ and $\alpha\in(0,1]$, there exists $\mu\in(0,1]$ such that

$$\left\|E^{-1}[\psi(t_1-\eta,\eta)-\psi(t_2-\eta,\eta)]\right\| \le C\widehat{E}\left[(t_1-\eta)^{\alpha-1}\{1+(t_2-t_1)^\mu\}-(t_2-\eta)^{\alpha-1}\right].$$

Proof. (i), (ii) and (iii) can be deduced by following the similar arguments as in [22]. Now, inequality in (iv) can be proved by using the similar arguments as in (see [25], pp. 437) and the following relation

$$\left\|e^{-t_1Q(s)}v - e^{-t_2Q(s)}\right\| = \left\|\int_{t_2}^{t_1}\frac{d}{dt}\left(e^{-tQ(s)}\right)dt\right\| \le C|\ t_1-t_2\ |^\mu,\ \text{where } \mu\in(0,1].$$

By using Hölder's inequality, we have

$$\begin{aligned}&\int_0^b\left[(b-\eta)^{\alpha-1}+C\mathscr{B}(\alpha,\gamma)(b-\eta)^{\alpha+\gamma-1}\right](L_f(\eta)+L_g(\eta))d\eta\\ &\le L_1\left(\left[\frac{b^{m+1}}{m+1}\right]^{1-q_1}+\left[\frac{b^{p+1}}{p+1}\right]^{1-q_1}\right)+L_2\left(\left[\frac{b^{\beta+1}}{\beta+1}\right]^{1-q_2}+\left[\frac{b^{n+1}}{n+1}\right]^{1-q_2}\right) := M_4,\end{aligned} \tag{3.1}$$

where $L_1 = \left\|L_f\right\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}_+)}$, $L_2 = \left\|L_g\right\|_{L^{\frac{1}{q_2}}(J,\mathbb{R}_+)}$, $\beta = \frac{\alpha-1}{1-q_2}, m = \frac{\alpha-1}{1-q_1}, n = \frac{\alpha+\gamma-1}{1-q_2}, p = \frac{\alpha+\gamma-1}{1-q_1}$ and $\beta,m,n,p\in(-1,0)$. For brevity, let $\widehat{N} = Cb^\alpha\widehat{E}\left[\frac{1}{\alpha}+\frac{C\mathscr{B}(\alpha,\gamma)b^\gamma}{\alpha+\gamma}\right]$.

3.1 Complete Controllability

Definition 31 (Complete Controllability)*The fractional equation* (1.1) *is said to be controllable on interval J if for every initial function $\chi \in C_0$ and $x_1 \in X$ there exists a control $u \in L^2[J,Z]$ such that the mild solution $x(t)$ of the equation satisfies $x(b) = x_1$.*

Theorem 2.*Assume that the hypotheses (H1) - (H4) hold and the linear operator W is not compact. Then, the equation* (1.1) *is completely controllable on J provided $\widehat{C}\left[1+\widehat{N}M_2M_3\right] < 1$, where $\widehat{C} = 2CM_4\widehat{E}$.*

Proof. By (H1), there exists $\chi \in C_0$ such that $h(\chi) = \phi$. We define

$$v(t) = \begin{cases} \chi(t), & \text{if } t \in [-\tau, 0], \\ \chi(0), & \text{if } t \in J. \end{cases}$$

Then, $v_t \in C_0$ and $f(t, v(0), v_0)$, $\int_0^t g(t,s,v(0),v_0)ds$ are continuous functions of t on J. Let

$$N_1 = \sup_{t\in J} \|f(t,v(0),v_0)\|_Y \quad \text{and} \quad N_2 = \sup_{t\in J} \left\| \int_0^t g(t,s,v(0),v_0)ds \right\|_Y. \tag{3.2}$$

For an arbitrary function $x(.) \in C_b$ and $t \in J$, we define the control $u_x(t)$ as follows.

$$\begin{aligned} u_x(t) = \widetilde{W}^{-1} \Big[x_1 - \chi(0) - \int_0^b E^{-1}\psi(b-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta - \int_0^b E^{-1}\psi(b-\eta,\eta)H(\eta)d\eta \\ - \int_0^b \int_0^\eta E^{-1}\psi(b-\eta,\eta)\varphi(\eta,s)H(s)dsd\eta \Big]. \end{aligned} \tag{3.3}$$

Next, we define the operator $\Im$ on C_b by $\Im x = \widetilde{x}$, where $\widetilde{x}(t) = \chi(t)$ for $t \in [-\tau, 0]$ and for $t \in J$

$$\begin{aligned} \widetilde{x}(t) = \chi(0) + \int_0^t E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta + \int_0^t E^{-1}\psi(t-\eta,\eta)(H(\eta)+Bu_x(\eta))d\eta \\ + \int_0^t \int_0^\eta E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)(H(s)+Bu_x(s))dsd\eta. \end{aligned} \tag{3.4}$$

In view of (H2) and equality (3.2), for each $\eta \in J$, we have

$$\begin{aligned} \|H(\eta)\|_Y &= \left\| f(\eta,x(\eta),x_\eta) + \int_0^\eta g(\eta,s,x(s),x_s)ds \right\|_Y \\ &\le \|f(\eta,x(\eta),x_\eta) - f(\eta,v(0),v_0)\|_Y + N_1 + \left\| \int_0^\eta g(\eta,s,x(s),x_s)ds - \int_0^\eta g(\eta,s,v(0),v_0)ds \right\|_Y + N_2 \\ &\le L_f(\eta)\left[\|x(\eta)-v(0)\| + \|x_\eta - v_0\|_0\right] + \int_0^\eta L_g^*(\eta,s)\left[\|x(s)-v(0)\| + \|x_s - v_0\|_0\right]ds + N_1 + N_2 \\ &\le 2(L_f(\eta)+L_g(\eta))[\|x\|_b + \|\chi\|_0] + N_1 + N_2. \end{aligned} \tag{3.5}$$

Now, from (H2), (3.5) and Lemma 3, for each $t, \eta \in J$, we get

$$\left\|E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)\right\| \le C^2\widehat{E}(t-\eta)^{\alpha-1}(1+\eta^\gamma)\|Q(0)E\chi(0)\|_Y,$$

$$\left\|E^{-1}\psi(t-\eta,\eta)H(\eta)\right\| \le C\widehat{E}\left[2(L_f(\eta)+L_g(\eta))[\|x\|_b + \|\chi\|_0] + N_1 + N_2\right](t-\eta)^{\alpha-1},$$

$$\left\|E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)H(s)\right\| \le C^2\widehat{E}\left[2(L_f(s)+L_g(s))[\|x\|_b + \|\chi\|_0] + N_1 + N_2\right](t-\eta)^{\alpha-1}(\eta-s)^{\gamma-1}.$$

Therefore, using above inequalities, (3.1) and Lemma 2, we have

$$\int_0^t \left\|E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)\right\| d\eta \leq \frac{C^2}{\alpha}\widehat{E}b^{\alpha}(1+b^{\gamma})\left\|Q(0)E\chi(0)\right\|_Y,$$

$$\begin{aligned}
&\int_0^t \left\|E^{-1}\psi(t-\eta,\eta)H(\eta)\right\| \\
&\quad\leq C\widehat{E}\left[\frac{b^{\alpha}(N_1+N_2)}{\alpha}+2(\|x\|_b+\|\chi\|_0)\int_0^b (b-\eta)^{\alpha-1}(L_f(\eta)+L_g(\eta))d\eta\right] \\
&\quad\leq C\widehat{E}\left[\frac{b^{\alpha}(N_1+N_2)}{\alpha}+2(\|x\|_b+\|\chi\|_0)\left(\frac{L_1 b^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}}+\frac{L_2 b^{(\beta+1)(1-q_2)}}{(\beta+1)^{(1-q_2)}}\right)\right],
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_0^{\eta} \left\|E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)H(s)\right\| dsd\eta \\
&\quad\leq C^2\widehat{E}\mathscr{B}(\alpha,\gamma)\left[\frac{b^{\alpha+\gamma}(N_1+N_2)}{\alpha+\gamma}+2(\|x\|_b+\|\chi\|_0)\int_0^b (b-\eta)^{\alpha+\gamma-1}(L_f(\eta)+L_g(\eta))d\eta\right] \\
&\quad\leq C^2\widehat{E}\mathscr{B}(\alpha,\gamma)\left[\frac{b^{\alpha+\gamma}(N_1+N_2)}{\alpha+\gamma}+2(\|x\|_b+\|\chi\|_0)\left(\frac{L_1 b^{(p+1)(1-q_1)}}{(p+1)^{(1-q_1)}}+\frac{L_2 b^{(n+1)(1-q_2)}}{(n+1)^{(1-q_2)}}\right)\right].
\end{aligned}$$

Now, for each $t\in J$ and for any $x\in C_b$, we have

$$\begin{aligned}
\|u_x(t)\| &\leq M_3\Bigg[\|x_1\|+\|\chi(0)\|+\frac{C^2}{\alpha}\widehat{E}b^{\alpha}(1+b^{\gamma})\left\|Q(0)E\chi(0)\right\|_Y+2C\widehat{E}(\|x\|_b+\|\chi\|_0) \\
&\qquad\times\int_0^b\left[(b-\eta)^{\alpha-1}+C\mathscr{B}(\alpha,\gamma)(b-\eta)^{\alpha+\gamma-1}\right](L_f(\eta)+L_g(\eta))d\eta \\
&\qquad+\ Cb^{\alpha}\widehat{E}\left[\frac{1}{\alpha}+\frac{C\mathscr{B}(\alpha,\gamma)b^{\gamma}}{\alpha+\gamma}\right](N_1+N_2)\Bigg] \\
&\leq M_3[\|x_1\|+a+\widehat{C}\|x\|_b],
\end{aligned}$$

where $a=(1+\widehat{C})\|\chi\|_0+\frac{C^2}{\alpha}\widehat{E}b^{\alpha}(1+b^{\gamma})\left\|Q(0)E\chi(0)\right\|_Y+\widehat{N}(N_1+N_2)$.

Thus, from above inequality, (H3) and Lemma 3, we have

$$\int_0^t \left\|E^{-1}\psi(t-\eta,\eta)Bu_x(\eta)\right\| d\eta \leq \frac{Cb^{\alpha}}{\alpha}\widehat{E}M_2M_3\left[\|x_1\|+a+\widehat{C}\|x\|_b\right],$$

$$\int_0^t\int_0^{\eta} \left\|E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)Bu_x(s)\right\| dsd\eta \leq \frac{C^2b^{\alpha+\gamma}}{\alpha+\gamma}\widehat{E}M_2M_3\mathscr{B}(\alpha,\gamma)\left[\|x_1\|+a+\widehat{C}\|x\|_b\right].$$

Now, it is clear from above inequalities that the integral terms $\left\|E^{-1}\psi(t-\eta,\eta)H(\eta)\right\|$, $\left\|E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)Bu_x(s)\right\|$, $\left\|E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)\right\|$, $\left\|E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)H(s)\right\|$ and $\left\|E^{-1}\psi(t-\eta,\eta)Bu_x(\eta)\right\|$ are Lebesgue integrable with respect to $\eta,\ s\in[0,t]$ for all $t\in J$. Therefore, from Lemma 1, it follows that all integral terms in $u_x(t)$ and (3.4) exist in Bochner sense.

Step1: First we claim that $\Im(C_b)\subseteq C_b$. For this, let $x\in C_b$, then, $\Im x(t)=\chi(t),$ for $t\in[-\tau,0]$ and for $0\leq t_1\leq t_2\leq b$, we have

$$\begin{aligned}
\Im x(t_2)-\Im x(t_1) &= \int_0^{t_1}E^{-1}\left[\psi(t_2-\eta,\eta)-\psi(t_1-\eta,\eta)\right]U(\eta)Q(0)E\chi(0)d\eta \\
&\quad+\int_{t_1}^{t_2}E^{-1}\psi(t_2-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta \\
&\quad+\int_0^{t_1}E^{-1}\left[\psi(t_2-\eta,\eta)-\psi(t_1-\eta,\eta)\right](H(\eta)+Bu_x(\eta))d\eta \\
&\quad+\int_{t_1}^{t_2}E^{-1}\psi(t_2-\eta,\eta)(H(\eta)+Bu_x(\eta))d\eta \\
&\quad+\int_0^{t_1}\int_0^{\eta}E^{-1}\left[\psi(t_2-\eta,\eta)-\psi(t_1-\eta,\eta)\right]\varphi(\eta,s)(H(s)+Bu_x(s))dsd\eta \\
&\quad+\int_{t_1}^{t_2}\int_0^{\eta}E^{-1}\psi(t_2-\eta,\eta)\varphi(\eta,s)(H(s)+Bu_x(s))dsd\eta.
\end{aligned}$$

In view of (H1), it is easy to see that $\Im x$ is continuous on $[-\tau,0]$. Also, from hypotheses (H2), (H3), Lemma 3 and $x\in C_b$, it can be seen that the nonlinear map H and the operators ψ,φ,B are continuous on J. Therefore, $\Im(C_b)\subseteq C_b$. Now, for each $t\in J$, we have

$$\begin{aligned}\|\Im x(t)\| &\leq \|\chi\|_0+\frac{C^2}{\alpha}\widehat{E}b^{\alpha}(1+b^{\gamma})\|Q(0)E\chi(0)\|_Y+\widehat{N}(N_1+N_2)+\widehat{C}(\|x\|_b+\|\chi\|_0)+\widehat{N}M_2M_3(\|x_1\|+a+\widehat{C}\|x\|_b)\\ &= aM_6+(M_6-1)\|x_1\|+\widehat{C}M_6\|x\|_b,\end{aligned}$$

where $M_6=1+\widehat{N}M_2M_3$. We choose $r\geq\frac{aM_6+(M_6-1)\|x_1\|}{1-\widehat{C}M_6}$ and define $\mathbb{B}_r:=\{x\in C_b: x(0)=\chi(0) \text{ and } \|x\|_b\leq r\}$. Then, for each $r\geq 0$, $\mathbb{B}_r$ is a closed, bounded and convex subset of C_b and from above inequality, it follows that $\Im(\mathbb{B}_r)\subseteq\mathbb{B}_r$.

For $x\in C_b$, we define operators $\Im_1$ and $\Im_2$ such that $\Im_1 x(t)=\chi(t)$ and $\Im_2 x(t)=0$ for $t\in[-\tau,0]$, and for $t\in J$,

$$\begin{aligned}(\Im_1 x)(t)&=\chi(0)+\int_0^t E^{-1}\psi(t-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta+\int_0^t E^{-1}\psi(t-\eta,\eta)Bu_x(\eta)d\eta\\ &\quad+\int_0^t\int_0^{\eta}E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)Bu_x(s)dsd\eta,\end{aligned}\tag{3.6}$$

$$(\Im_2 x)(t)=\int_0^t E^{-1}\psi(t-\eta,\eta)H(\eta)d\eta+\int_0^t\int_0^{\eta}E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)H(s)dsd\eta.\tag{3.7}$$

Then, $(\Im x)(t)=(\Im_1 x)(t)+(\Im_2 x)(t)$ for $t\in[-\tau,b]$. Moreover, for $x,y\in\mathbb{B}_r$ and $t\in J$, we have

$$\|(\Im_1 x+\Im_2 y)(t)\|\leq aM_6+(M_6-1)\|x_1\|+\widehat{C}M_6 r.$$

Since $r\geq\frac{aM_6+(M_6-1)\|x_1\|}{1-\widehat{C}M_6}$, therefore, $aM_6+(M_6-1)\|x_1\|\leq r(1-\widehat{C}M_6)$. Hence, from above inequality and $(\Im_1 x+\Im_2 y)(0)=\chi(0)$, it is clear that $\Im_1 x+\Im_2 y\in\mathbb{B}_r$ for every $x,y\in\mathbb{B}_r$.

Step2: Next, we will show that $\Im_1$ is contraction. For this, let $x,\ y\in\mathbb{B}_r$ and define

$$\widetilde{H}(t):=\{f(t,x(t),x_t)-f(t,y(t),y_t)\}+\int_0^t\{g(t,s,x(s),x_s)-g(t,s,y(s),y_s)\}ds.$$

In view of (H2), Lemmas 2 and 3, for each $t\in J$, we have

$$\begin{aligned}\|u_x(t)-u_y(t)\| &\leq M_3\left[\int_0^b\left\|E^{-1}\psi(b-\eta,\eta)\widetilde{H}(\eta)\right\|d\eta+\int_0^b\left\|E^{-1}\psi(b-\eta,\eta)\varphi(\eta,s)\widetilde{H}(s)\right\|dsd\eta\right]\\ &\leq 2CM_3\widehat{E}\|x-y\|_b\left[\int_0^b\left\{(b-\eta)^{\alpha-1}+C\mathscr{B}(\alpha,\gamma)(b-\eta)^{\alpha+\gamma-1}\right\}(L_f(\eta)+L_g(\eta))d\eta\right]\\ &\leq 2C\widehat{E}M_4M_3\|x-y\|_b=\widehat{C}M_3\|x-y\|_b.\end{aligned}$$

Therefore, from (3.6), Lemmas 2, 3 and above inequality, for each $t\in J$, we get

$$\begin{aligned}\|\Im_1 x(t)-\Im_1 y(t)\| &\leq C\widehat{E}\int_0^t\left[(t-\eta)^{\alpha-1}+C\mathscr{B}(\alpha,\gamma)(t-\eta)^{\alpha+\gamma-1}\right]\left\|Bu_x(\eta)-Bu_y(\eta)\right\|d\eta\\ &\leq C\widehat{E}M_2M_3\widehat{C}b^{\alpha}\left(\frac{1}{\alpha}+\frac{C\mathscr{B}(\alpha,\gamma)b^{\gamma}}{\alpha+\gamma}\right)\|x-y\|_b=\widehat{N}M_2M_3\widehat{C}\|x-y\|_b.\end{aligned}$$

Since $\widehat{C}\left[1+\widehat{N}M_2M_3\right]<1$, therefore, $\widehat{N}M_2M_3\widehat{C}<1$. Also, $\|(\Im_1 x)(t)-(\Im_1 y)(t)\|=0$ for $t\in[-\tau,0]$. Hence, $\Im_1$ is a contraction on $\mathbb{B}_r$.

Step3: Next, we will show that the map $\Im_2$ is completely continuous on $\mathbb{B}_r$. For this, first we will prove that the map $\Im_2$ is continuous on $\mathbb{B}_r$. Then, we show that $\Im_2(\mathbb{B}_r)\subseteq C_b$ is equicontinuous and $\Im_2(\mathbb{B}_r)(t)$ is relatively compact for each $t\in J$ and then the compactness of $\Im_2$ follows from the Ascoli-Arzela theorem.

Let us consider a sequence $\{x^{(n)}\}\subseteq\mathbb{B}_r$ with $x^{(n)}\to x$ in $\mathbb{B}_r$. We denote $H_n(t)=f(t,x^{(n)}(t),x_t^{(n)})+\int_0^t g(t,s,x^{(n)}(s),x_s^{(n)})ds$. From (H2) and Lemma 3, it is easy to see that

$$\begin{aligned}&E^{-1}\psi(.-s,s)H_n(s)\to E^{-1}\psi(.-s,s)H(s), \qquad \text{a.e. } s\in J,\\ &E^{-1}\psi(.-\eta,\eta)\varphi(\eta,s)H_n(s)\to E^{-1}\psi(.-\eta,\eta)\varphi(\eta,s)H(s), \qquad \text{a.e. } s\in J,\\ &\left\|E^{-1}\psi(.-\eta,\eta)(H_n(\eta)-H(\eta))\right\|\leq 4rC\widehat{E}(.-\eta)^{\alpha-1}(L_f(\eta)+L_g(\eta))\in L^1(J,\mathbb{R}^+),\\ &\left\|E^{-1}\psi(.-\eta,\eta)\varphi(\eta,s)(H_n(s)-H(s))\right\|\leq 4rC^2\widehat{E}(L_f(\eta)+L_g(\eta))(.-\eta)^{\alpha-1}(\eta-s)^{\gamma-1}\in L^1(J,\mathbb{R}^+).\end{aligned}$$

Then, from the Lebesgue dominated convergence theorem, we have

$$\int_0^t \left\| E^{-1}\psi(t-\eta,\eta)\,(H_n(\eta)-H(\eta))\right\| d\eta \to 0 \text{ as } n\to\infty,$$
$$\int_0^t\int_0^\eta \left\| E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)\,(H_n(s)-H(s))\right\| ds d\eta \to 0 \text{ as } n\to\infty.$$

Hence, from the above result and hypothesis (H2), we get

$$\left\|\Im_2 x^{(n)} - \Im_2 x\right\|_b \le \sup_{t\in J}\left[\int_0^t \left\| E^{-1}\psi(t-\eta,\eta)\,(H_n(\eta)-H(\eta))\right\| d\eta + \int_0^t\int_0^\eta \left\| E^{-1}\psi(t-\eta,\eta)\varphi(\eta,s)\,(H_n(s)-H(s))\right\| ds d\eta\right] \to 0 \text{ as } n\to\infty,$$

which implies that $\Im_2$ is continuous on $\mathbb{B}_r$. Next we claim that $\Im_2(\mathbb{B}_r)\subseteq C_b$ is equicontinuous. For this, let $0<t<t+\varepsilon\le b$ and $x\in\mathbb{B}_r$. Then, we have

$$(\Im_2 x)(t+\varepsilon) - (\Im_2 x)(t) = I_1+I_2+I_3+I_4, \tag{3.8}$$

where

$$I_1 = \int_0^t E^{-1}\left[\psi(t+\varepsilon-\eta,\eta)-\psi(t-\eta,\eta)\right]H(\eta)d\eta,$$
$$I_2 = \int_0^t\int_0^\eta E^{-1}\left[\psi(t+\varepsilon-\eta,\eta)-\psi(t-\eta,\eta)\right]\varphi(\eta,s)H(s)dsd\eta,$$
$$I_3 = \int_t^{t+\varepsilon} E^{-1}\psi(t+\varepsilon-\eta,\eta)H(\eta)d\eta,$$
$$I_4 = \int_t^{t+\varepsilon}\int_0^\eta E^{-1}\psi(t+\varepsilon-\eta,\eta)\varphi(\eta,s)H(s)d\eta.$$

Now, we claim that $\|I_i\|\to 0$ as $\varepsilon\to 0$, for $i=1,2,3,4$. From (H2), (3.5) and Lemma 3, we have

$$\left\|E^{-1}\left[\psi(t+\varepsilon-\eta,\eta)-\psi(t-\eta,\eta)\right]H(\eta)\right\|$$
$$\le C\widehat{E}\left[(t-\eta)^{\alpha-1}(1+\varepsilon^\mu)-(t+\varepsilon-\eta)^{\alpha-1}\right]\left[2(L_f(\eta)+L_g(\eta))[r+\|\chi\|_0]+N_1+N_2\right],$$
$$\left\|E^{-1}\left[\psi(t+\varepsilon-\eta,\eta)-\psi(t-\eta,\eta)\right]\varphi(\eta,s)H(s)\right\|$$
$$\le C^2\widehat{E}\left[(t-\eta)^{\alpha-1}(1+\varepsilon^\mu)-(t+\varepsilon-\eta)^{\alpha-1}\right](\eta-s)^{\gamma-1}\left[2(L_f(s)+L_g(s))[r+\|\chi\|_0]+N_1+N_2\right],$$
$$\left\|E^{-1}\psi(t+\varepsilon-\eta,\eta)H(\eta)\right\|$$
$$\le C\widehat{E}(t+\varepsilon-\eta)^{\alpha-1}\left[2(L_f(\eta)+L_g(\eta))[r+\|\chi\|_0]+N_1+N_2\right],$$
$$\left\|E^{-1}\psi(t+\varepsilon-\eta,\eta)\varphi(\eta,s)H(s)\right\|$$
$$\le C^2\widehat{E}(t+\varepsilon-\eta)^{\alpha-1}(\eta-s)^{\gamma-1}\left[2(L_f(s)+L_g(s))[r+\|\chi\|_0]+N_1+N_2\right].$$

Also, from (H2) and Hölder's inequality, for each $\eta\in(0,t)$ and $t\in[0,b]$, we get

$$\int_0^\eta (\eta-s)^{\gamma-1}(L_f(s)+L_g(s))ds \le \frac{L_1 b^{(m_1+1)(1-q_1)}}{(m_1+1)^{(1-q_1)}} + \frac{L_2 b^{(m_2+1)(1-q_2)}}{(m_2+1)^{(1-q_2)}} := M_5,$$

where $m_1 = \frac{\gamma-1}{1-q_1}$, and $m_2 = \frac{\gamma-1}{1-q_2} \in (-1,0)$. Hence, using Lemma 3, (H2), and the fact $| a^\sigma - b^\sigma | \le (b-a)^\sigma$ for $\sigma \in (0,1]$, $0 < a \le b$, we have

$$\|I_1\| \le 2C\widehat{E}(r+\|\chi\|_0)\Bigg[\frac{L_1\{(2\varepsilon^{(m+1)})^{(1-q_1)} + \varepsilon^\mu b^{(m+1)(1-q_1)}\}}{(m+1)^{(1-q_1)}} + \frac{L_2\{(2\varepsilon^{(\beta+1)})^{(1-q_2)} + \varepsilon^\mu b^{(\beta+1)(1-q_2)}\}}{(\beta+1)^{(1-q_2)}}\Bigg] + \frac{C}{\alpha}\widehat{E}(N_1+N_2)[b^\alpha\varepsilon^\mu + 2\varepsilon^\alpha],$$

$$\|I_2\| \le C^2\widehat{E}(2\varepsilon^\alpha + b^\alpha\varepsilon^\mu)\left[\frac{2M_5(r+\|\chi\|_0)}{\alpha} + \frac{b^\gamma(N_1+N_2)}{\alpha\gamma}\right],$$

$$\|I_3\| \le 2C\widehat{E}(r+\|\chi\|_0)\left[\frac{L_1\varepsilon^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}} + \frac{L_2\varepsilon^{(\beta+1)(1-q_2)}}{(\beta+1)^{(1-q_2)}}\right] + \frac{C\varepsilon^\alpha}{\alpha}\widehat{E}(N_1+N_2),$$

$$\|I_4\| \le \frac{C^2\widehat{E}\varepsilon^\alpha}{\alpha}\left[2M_5(r+\|\chi\|_0) + \frac{b^\gamma(N_1+N_2)}{\gamma}\right].$$

Now, from above inequalities, it is easy to see that $\|I_i\| \to 0$ as $\varepsilon \to 0$, for $i = 1,2,3,4$. Hence, from (3.7), $\|(\Im_2 x)(t+\varepsilon) - (\Im_2 x)(t)\| \to 0$ as $\varepsilon \to 0$, which implies the equicontinuity of $\Im_2$ on $\mathbb{B}_r$.

Next we claim that, for each $t \in J$, the set $\Pi(t) := \{(\Im_2 x)(t) : x \in \mathbb{B}_r\}$ is relatively compact in X. Clearly, $\Pi(0) = \{(\Im_2 x)(0) : x \in \mathbb{B}_r\} = \{0\}$ is compact. Hence, it is only necessary to consider $t > 0$. Now, for each $\varepsilon \in (0,t)$, $t \in (0,b]$, constant $\zeta > 0$ and $x \in \mathbb{B}_r$, we define $\Pi_{\varepsilon,\zeta}(t) = \big\{(\Im_{2,\varepsilon,\zeta} x)(t) : x \in \mathbb{B}_r\big\}$, where

$$\Im_{2,\varepsilon,\zeta} x(t) = \int_0^{t-\varepsilon}\int_\zeta^\infty \mathscr{P}_E(\theta,t,\eta)H(\eta)d\theta d\eta + \int_0^{t-\varepsilon}\int_0^\eta\int_\zeta^\infty \mathscr{P}_E(\theta,t,\eta)\varphi(\eta,s)H(s)d\theta ds d\eta.$$

Since, $\mathbb{B}_r$ is a bounded subset of C_b, therefore, from (H4), $(\Im_{2,\varepsilon,\zeta} x)(t)$ is relatively compact for arbitrary $\varepsilon \in (0,t)$ and $\zeta > 0$. Also, we have

$$(\Im_2 x)(t) - (\Im_{2,\varepsilon,\zeta} x)(t) = P_1 + P_2 + P_3 + P_4, \tag{3.9}$$

where

$$P_1 = \int_0^{t-\varepsilon}\int_0^\zeta \mathscr{P}_E(\theta,t,\eta)H(\eta)d\theta d\eta,$$

$$P_2 = \int_0^{t-\varepsilon}\int_0^\eta\int_0^\zeta \mathscr{P}_E(\theta,t,\eta)\varphi(\eta,s)H(s)d\theta ds d\eta,$$

$$P_3 = \int_{t-\varepsilon}^t\int_0^\infty \mathscr{P}_E(\theta,t,\eta)H(\eta)d\theta d\eta,$$

$$P_4 = \int_{t-\varepsilon}^t\int_0^\eta\int_0^\infty \mathscr{P}_E(\theta,t,\eta)\varphi(\eta,s)H(s)d\theta ds d\eta.$$

Now, from Hölder's inequality and Lemma 3, we have

$$\|P_1\| \le \alpha C\widehat{E}\left(\int_0^\zeta \theta\xi_\alpha(\theta)d\theta\right)\left[2(r+\|\chi\|_0)\left(\frac{L_1 b^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}} + \frac{L_2 b^{(\beta+1)(1-q_2)}}{(\beta+1)^{(1-q_2)}}\right) + \frac{(N_1+N_2)b^\alpha}{\alpha}\right],$$

$$\|P_2\| \le \alpha C^2\widehat{E}\left(\int_0^\zeta \theta\xi_\alpha(\theta)d\theta\right)\left[\frac{2b^\alpha}{\alpha}M_5(r+\|\chi\|_0) + \frac{(N_1+N_2)b^{\alpha+\gamma}}{\alpha\gamma}\right],$$

$$\|P_3\| \le \alpha C\widehat{E}\left(\int_0^\infty \theta\xi_\alpha(\theta)d\theta\right)\left[2(r+\|\chi\|_0)\left(\frac{L_1\varepsilon^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}} + \frac{L_2\varepsilon^{(\beta+1)(1-q_2)}}{(\beta+1)^{(1-q_2)}}\right) + \frac{(N_1+N_2)\varepsilon^\alpha}{\alpha}\right],$$

$$\|P_4\| \le \alpha C^2\widehat{E}\left(\int_0^\infty \theta\xi_\alpha(\theta)d\theta\right)\left[\frac{2\varepsilon^\alpha}{\alpha}M_5(r+\|\chi\|_0) + \frac{(N_1+N_2)b^\gamma\varepsilon^\alpha}{\alpha\gamma}\right].$$

Now, from above inequalities, $\|P_i\| \to 0$ as $\varepsilon,\zeta \to 0$, for each $i = 1,2,3,4$. Hence, from (3.9), $\big\|(\Im_2 x)(t) - (\Im_{2,\varepsilon,\zeta} x)(t)\big\| \to 0$ as $\varepsilon,\zeta \to 0$, which implies that the set $\Pi(t)$ can be arbitrarily approximated by the relatively compact sets $\Pi_{\varepsilon,\zeta}(t)$. Therefore, $(\Im_2\mathbb{B}_r)(t) \subseteq X$ is relatively compact in X. Hence, the continuity of $\Im_2$ and relatively compactness of $\{\Im_2 x : x \in$

$\mathbb{B}_r\}$ imply that $\Im_2$ is completely continuous. Hence, using Krasnoselskii's Fixed Point theorem $\Im$ has a fixed point x on $\mathbb{B}_r$. It is easy to see that x is a mild solution of the equation (1.1), satisfying $x(b) = x_1$. Therefore, the fractional equation (1.1) is completely controllable on J.

Theorem 3. *Assume that the hypotheses (H1) - (H4) hold and the linear operator W is compact. For $x(.) \in C_b$ satisfying (1.1), we define*

$$z_x = \chi(0) + \int_0^b E^{-1}\psi(b-\eta,\eta)U(\eta)Q(0)E\chi(0)d\eta + \int_0^b E^{-1}\psi(b-\eta,\eta)H(\eta)d\eta$$
$$+ \int_0^b \int_0^\eta E^{-1}\psi(b-\eta,\eta)\varphi(\eta,s)H(s)dsd\eta.$$

Then, for all $x_1 \in X$ and $\chi \in C_0$, the equation (1.1) is controllable on J if and only if $x_1 \in R(\widetilde{W}) + z_x$ and $\chi \in C_0$ provided $\widehat{C}\left[1 + \widehat{N}M_2M_3\right] < 1$, where $\widehat{C} = 2CM_4\widehat{E}$.

Proof. Since, we assume that the linear operator W is compact and it induces an invertible operator $\widetilde{W}$ with $\left\|\widetilde{W}^{-1}\right\| \leq M_3$, therefore, $R(\widetilde{W})(\neq X)$ is a finite dimensional closed subspace of X. Also, there exists a closed subspace Z of X such that $X = R(\widetilde{W}) \bigoplus Z$, i.e., for every $x \in X$ there exist unique $y \in R(\widetilde{W})$ and $z \in Z$ such that $x = y + z$.

Now, if $x_1 \in R(\widetilde{W}) + z_x$, then, by following the similar steps of Theorem 2, one can prove that the equation (1.1) is controllable with control $u_x = \widetilde{W}^{-1}[x_1 - z_x]$.

Conversely, let $x_1 \in X$ and the equation (1.1) is controllable in J. Since, $x_1 \in X$, therefore, there exist unique $v \in R(\widetilde{W})$ and $z \in Z$ such that $x_1 = v + z$. Also, for $v \in R(\widetilde{W})$, there exists unique $u \in D(\widetilde{W})$ such that $\widetilde{W}u = v$. Therefore, $x_1 = \widetilde{W}u + z$.

Now, the equation (1.1) is controllable iff $x_1 = x(b) = \widetilde{W}u + z_x$ iff $\widetilde{W}u + z = \widetilde{W}u + z_x$ iff $z_x = z$ iff $x_1 \in R(\widetilde{W}) + z_x$. Hence our claim.

3.2 Exact Null Controllability

Definition 32 (Exact Null Controllability) *The fractional equation (1.1) is said to be exactly null controllable on interval J if for every initial function $\chi \in C_0$ there exists a control $u \in L^2[J, Z]$ such that the mild solution $x(t)$ of the equation satisfies $x(b) = 0$.*

Theorem 4. *Assume that the hypotheses (H1) - (H4) hold and the linear operator W is not compact. Then, the equation (1.1) is exactly null controllable on J provided $\widehat{C}\left[1 + \widehat{N}M_2M_3\right] < 1$, where $\widehat{C} = 2CM_4\widehat{E}$.*

Proof. For proving the exact null controllability of the equation (1.1), we follow the similar proof of Theorem 2 after replacing $x_1 = 0$. Then, the exact null controllability of the equation (1.1) on J follows with control $u_x(t)$, which is defined by (3.3) with $x_1 = 0$.

Moreover, if W is a compact linear operator, then Theorem 3 implies the null controllability of the equation (1.1) on interval J provided $z_x \in R(\widetilde{W})$.

4 Example

Consider the following nonlocal Sobolev type functional differential equation of fractional order

$$\begin{cases} \partial_t^\alpha \left(z(t,\xi) - z_{\xi\xi}(t,\xi)\right) - a(t,\xi)\frac{\partial^2}{\partial\xi^2}z(t,\xi) = \mathscr{F}(t, z(t,\xi), z_t) + \int_0^t a_1(t,s)e^{-z(s,\xi)}ds + w\mu(t,\xi), \\ z(t,0) = z(t,\pi) = 0, \\ h_0(z(\theta,\xi)) = \phi_0(\xi), \quad \theta \in [-\tau, 0], \quad t \in [0,1] = J, \quad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where ∂_t^α is Caputo partial derivative of order $\alpha \in (0,1)$, $w > 0$ is a constant and

(i) $a(t,\xi)$ is a continuous function and $a(t,\xi) \geq \delta_0$ $(\delta_0 > 0)$, for all $\xi \in [0,\pi]$.

(ii) For all $t,s \in J$ and $\xi \in [0,\pi]$ there exist constants $C > 0$ and $\gamma \in (0,1)$ such that $| a(t,\xi) - a(s,\xi) | \leq C | t-s |^{\gamma}$, where C,γ are independent in t.
(iii) For each $t \in J$, $a_1(t,.) \in L^1(J)$ and $\int_0^t a_1(t,s)ds := \widetilde{a}_1(t) \in L^{\frac{1}{q_2}}[J,\mathbb{R}^+]$, where $q_2 \in (0,\alpha) \cap (0,\gamma)$.
(iv) $\mu : J \times (0,\pi) \to (0,\pi)$ is continuous in t.

Let $X = Y = L^2([0,\pi])$ be the space of functions which are square integrable. We define the operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ as follows.

$$Aw = -w'' \quad \text{and} \quad Ew = w - w'',$$

where each domain $D(A)$ and $D(E)$ is given by

$$\left\{ w \in X : w, w' \text{ are absolutely continuous}, w'' \in X \text{ and } w(0) = w(\pi) = 0 \right\}.$$

Then, A and E can be written respectively as [26]

$$Aw = \sum_{n=1}^{\infty} n^2 < w, w_n > w_n, \;\; w \in D(A),$$
$$Ew = \sum_{n=1}^{\infty} (1+n^2) < w, w_n > w_n, \;\; w \in D(E),$$

where $w_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} sin(n\xi),\; n = 1,2,......,$ is the orthogonal set of eigenvectors of A. Also, for $w \in X$ [26],

$$E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2} < w, w_n > w_n,$$
$$-AE^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} < w, w_n > w_n,$$
$$T(t)w = \sum_{n=1}^{\infty} e^{\left(\frac{-n^2}{1+n^2}\right)t} < w, w_n > w_n.$$

It is easy to see that $-AE^{-1}$ is a bounded linear operator from Y to Y, $\left\|E^{-1}\right\| \leq \frac{1}{4}$ and $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$. Moreover, by using the fact that $R(\lambda, -AE^{-1})w = \int_0^{\infty} e^{-\lambda t} T(t)w$ for $w \in Y$ (see [21]), one can prove that $\left\|R(\lambda, -AE^{-1})\right\| \leq \frac{1}{1+\lambda}$, where $\lambda = \Re(\lambda) = \frac{n^2}{1+n^2} > 0$. Hence, the semigroup $T(t)$ generated by linear operator $-AE^{-1}$ is an analytic semigroup. Since the eigenvalues of E^{-1} are $\lambda_n = \frac{1}{1+n^2}, n = 1,2,.....,$ and $\lim_{n\to\infty} \lambda_n = 0$, therefore, the linear operator E^{-1} is compact [26].

Now, we define $x(t)(\xi) = z(t,\xi)$ and $x_t = z_t(\theta,.)$, that is $(x(t+\theta))(\xi) = z(t+\theta,\xi)$ for $t \in J,\; \xi \in [0,\pi]$ and $\theta \in [-\tau,0]$. Also, we define $f : J \times X \times C_0 \to Y$, $g : \Delta \times X \times C_0 \to Y$ $(\Delta = \{(t,s) \in J \times J : t \geq s\})$, $Bu : J \to X$, $A(t) : D(A(t)) \subset X \to Y$, $Q(t) : Y \to Y$ and $\tilde{h} : C_0 \to X$ by

$$\begin{aligned}
&f(t,x(t),x_t)(\xi) = \mathscr{F}(t,z(t,\xi),z_t(\theta,\xi)),\\
&g(t,s,x(s),x_s)(\xi) = a_1(t,s)e^{-x(s)(\xi)},\\
&(Bu)(t)(\xi) = w\mu(t,\xi),\\
&A(t)x(\xi) = a(t,\xi)Ax(\xi),\\
&-Q(t)x(\xi) = -A(t)E^{-1}x(\xi) = -a(t,\xi)AE^{-1}x(\xi),\\
&\tilde{h}(\psi)(\xi) = h_0(\psi(t,\xi)).
\end{aligned}$$

for $t \in J$ and $\xi \in [0,\pi]$. Also, here $D(A(t)) = D(A)$, for $t \in J$.

Now, we can write (4.1) in an abstract form

$$\begin{cases} {}_0^C D_t^{\alpha}[Ex(t)] + A(t)x(t) = f(t,x(t),x_t) + \int_0^t g(t,s,x(s),x_s)ds + Bu(t), \;\; t \in J, \\ \tilde{h}(x_0) = \phi_0. \end{cases}$$

Note that boundary condition is absorbed into the definition of domain of operator $A(t)$ and the requirement that $x(t) \in D(A)$, for all $t \in J$. Moreover, under the points (i) - (ii) and due to the definitions of operators $-AE^{-1}, -Q(t)$ and $T(t)$, it can be proved that the conditions (B1) - (B2) hold for $-Q(t)$.

Let $\|B\| \leq M_2$ and the linear operator W defined by

$$(Wu)(\xi) = w\left[\int_0^1 E^{-1}\psi(1-\eta,\eta)\mu(\eta,\xi)d\eta + \int_0^1\int_0^\eta E^{-1}\psi(1-\eta,\eta)\varphi(\eta,s)\mu(s,\xi)\eta\right], \text{ for } \xi \in (0,\pi),$$

induces an invertible operator $\widetilde{W}$ such that $\left\|\widetilde{W}^{-1}\right\|_{\mathbb{BL}\left[R(\widetilde{W}),L^2[J,Z]/\mathrm{Ker}W\right]} \leq M_3$.

Next, we consider $f(t,x(t),x_t) = x(t) + sin(x_t)$. Then, it is easy to see that the nonlinear maps f, g are satisfying hypotheses (H2) along with $L_f(t) = 1, L_g^*(t,s) = a_1(t,s)$ and $L_g(t) = \widetilde{a}_1(t)$. Now, it is easy to compute the constants $\widehat{N}, M_4$ and $\widehat{C}$. Let $\tilde{h}$ be defined by $\tilde{h}(\psi)(x) = \int_{-\tau}^0 l(s)\psi(s)(x)ds,\ l \in L^1([-\tau,0])$. Then, we can write (4.1) as a fractional delay differential equation of the form (1.1), where $h(x_0)(\theta) \equiv \tilde{h}(x_0)$ for $x_0 \in C_0,\ \theta \in [-\tau,0]$ and $\phi(\theta) \equiv \phi_0$ for $\theta \in [-\tau,0]$. Now we can take $\chi(t) = \frac{1}{k}\phi_0$ on $[-\tau,0]$ with $k = \int_{-\tau}^0 l(s)ds \neq 0$. Hence, the hypotheses (H1), (H2) and (H3) hold. Since E^{-1} is compact, therefore, the hypothesis (H4) holds and the linear operator W is compact. Now, if $\widehat{C}\left[1+\widehat{N}M_2M_3\right] < 1$, then one can apply the Theorem 3 to see the controllability of considered fractional evolution equation of Sobolev type.

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