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Application of the Linear Shallow Water Theory to Problems of Oscillations of a Heavy Liquid in a Container in Presence of Surface Tension

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Abstract: We study the problem of small oscillations of inviscid incompressible fluid with surface tension in partially filled tanks. Two cases are discussed (regular and singular) and the linear shallow water theory is used. For each case, we give the approximations of the asymptotic solution of the spectral problem.

Keywords: Incompressible inviscid fluid, surface tension, small oscillations, variational methods.

1 Introduction

The problem of the small oscillations of a heavy inviscid liquid in a container has been the subject of numerous works [5,4]. This is a well studied field in ocean and architecture engineering, applied mathematics and physics, among other disciplines.

In the special case, when the depth of liquid is considerably smaller than the diameter of its free surface, it is possible to use approximate methods and introduce a small parameter to determine the successive approximations of the asymptotic solution of the spectral problem [8,9,4].

In this aim, we propose here a mathematical analysis of linear standing oscillations of inviscid incompressible liquid in a container, considering effects of surface tension and assuming shallow water dynamics. Such theory can be applied to the oscillations of fuel in tanks when the fuel occupies a small volume.

Restricting ourselves for simplicity to the planar problem, we determine the approximations of orders zero, one and two for the velocity potential and the eigenvalues, distinguishing the regular case and the singular case.

The regular case, where the minimum of the depth of liquid is strictly positive, leads to classical problems of functional analysis.

In the singular case, where this minimum is zero, the mathematical solution is more difficult.
We restrict ourselves to the case of the parabolic container, we determine the first approximation of the eigenvalues by means of the theory of Legendre operators [3], show that it is possible to simplify the solution calling for the theory of degenerate elliptic operators of Baouendi and Goulaouic [1,7] and calculate finally the second approximation of the eigenvalues.

2 Position of the problem

2.1 *Study of the equilibrium of the system*

In the equilibrium position, the inviscid incompressible heavy liquid occupies a domain Ω' of the plane $Ox'_1x'_2$ (Ox'_1 directed upwards) bounded by a rigid wall S and the free line Γ.
The pressure in the equilibrium position is

$$P_{\mathrm{eq}} = -\rho g x'_2 + C_0, \qquad (C_0 = \text{constant}) \tag{1}$$

where ρ is the density of the liquid, and g is the constant acceleration of the gravity.
If P_{e} the external pressure considered as constant, the

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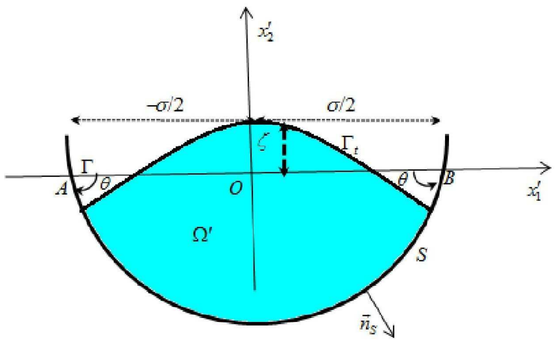

Fig. 1: Model of the system

Laplace law gives

$$P_{\mathrm{eq}} - P_{\mathrm{e}} = -\frac{\tau}{R_0} \qquad \text{on } \Gamma,$$

where τ is the surface tension considered as constant and R_0 the radius of curvature of Γ reckoned as negative if the center of curvature lies on the same side of Γ as the liquid. Then we have

$$-\rho g x_2' + C_0 - P_{\mathrm{e}} = -\frac{\tau}{R_0} \qquad \text{on } \Gamma,$$

In the following, we restrict ourselves to the case R_0 infinite, $C_0 = P_{\mathrm{eq}}$: then, Γ lies on the line $x_2' = 0$ [Fig.1].

2.2 Equations of small oscillations of the liquid

We study the small oscillations of the liquid about its equilibrium position, in the framework of the linear theory.

If $\overrightarrow{V}$ and $\overrightarrow{\gamma}$ are the velocity and acceleration of a particle of the liquid, P' the pressure, $p' = P' - P_{\mathrm{eq}}$ the dynamic pressure, we have

$$\rho\overrightarrow{\gamma} = -\overrightarrow{\mathrm{grad}}P' - \rho g\overrightarrow{x_2} = -\overrightarrow{\mathrm{grad}}p' \quad \text{(Euler's equation)} \tag{2}$$

$$\mathrm{div}\overrightarrow{V} = 0 \quad \text{(incompressibility)} \tag{3}$$

$$\overrightarrow{V}\cdot\overrightarrow{n}_S = 0 \quad \text{(impermeability of } S) \tag{4}$$

where $\overrightarrow{n}_S$ is the normal unit vector to S directed to the exterior of Ω.

Under the condition of the application of Lagrange's theorem, we introduce the velocity potential $\Phi'(x_1', x_2', t)$:

$$\overrightarrow{V} = \overrightarrow{\mathrm{grad}}\Phi' \tag{5}$$

From the equations (3) and (4), we deduce for Φ' the equations

$$\Delta\Phi' \overset{\mathrm{def}}{=} \frac{\partial^2\Phi'}{\partial {x_1'}^2} + \frac{\partial^2\Phi'}{\partial {x_2'}^2} = 0 \tag{6}$$

$$\frac{\partial\Phi'}{\partial n}|_S = 0 \tag{7}$$

where $\frac{\partial}{\partial n}$ is the normal derivative on S.

The Euler's equation (2) can be writen

$$\rho\overrightarrow{\mathrm{grad}}\frac{\partial\Phi'}{\partial t} = -\overrightarrow{\mathrm{grad}}p',$$

so that we have

$$p' = -\rho\frac{\partial\Phi'}{\partial t} + C(t) \tag{8}$$

where $C(t)$ is an arbitrary function of the time.

The Laplace law on the moving free line Γ_t can be written

$$P' - P_{\mathrm{e}} = -\frac{\tau}{R_t} \qquad \text{on } \Gamma_t,$$

where R_t is the radius of curvature of Γ_t, or

$$P_{\mathrm{eq}|\Gamma_t} + p'_{|\Gamma} - P_{\mathrm{e}} = -\frac{\tau}{R_t} \qquad \text{on } \Gamma_t$$

If the equation of Γ_t is $x_2' = \zeta(x_1', t)$, its curvature in linear theory is $\zeta''(x_1', t)$ $\left(\zeta'' = \frac{\partial^2\zeta}{\partial {x_1'}^2}\right)$ and consequently

$$p'_{|\Gamma} = \rho g\zeta - \tau\zeta'' \tag{9}$$

In accordance with the capillarity laws, we must express that the wetting angle between Γ_t and S is constant.

In the following, for simplicity, we restrict ourselves to the case of a symetrical container with respect to Ox_2'. Γ is the segment $AB : x_2' = 0, -\frac{\sigma}{2} \le x_1' \le \frac{\sigma}{2}$ and we denote by θ the angle between S and Ox_1' in A and B [Fig.1].

A classical formula ([6], [5], [4]) or a simple calculation gives

$$\zeta'\left(\pm\frac{\sigma}{2}, t\right) = \pm\frac{1}{R\sin\theta}\zeta\left(\pm\frac{\sigma}{2}, t\right) \tag{10}$$

where R is the radius of curvature of S in A and B.

The condition expressing that the volume of liquid is constant is

$$\int_\Gamma \zeta \,\mathrm{d}\Gamma = 0$$

or

$$\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{\partial\Phi'}{\partial x_2'}\Big|_{x_2'=0} \mathrm{d}x_1' = 0 \tag{11}$$

Finally, setting $C(t)$ in $-\rho\frac{\partial\Phi'}{\partial t}$ and deriving (9) with respect to t, we can write

$$-\rho\frac{\partial^2\Phi'}{\partial t^2} = \rho g\frac{\partial\Phi'}{\partial x_2'} - \tau\frac{\partial^2}{\partial {x_1'}^2}\left(\frac{\partial\Phi'}{\partial x_2'}\right) \qquad \text{on } \Gamma \tag{12}$$

2.3 *Introduction of adimensional variables*

In order to introduce in the sequel a small parameter, we define adimensional variables x_i by setting

$$x_i' = \sigma x_i \qquad (i = 1,2)$$

Writing
$$\tilde{\Phi}\left(x_1,x_2,t\right) = \Phi'\left(x_1',x_2',t\right),$$
we seek solutions of the precedent equations in the form

$$\tilde{\Phi}\left(x_1,x_2,t\right) = e^{i\omega t}\Phi\left(x_1,x_2\right), \qquad \omega \, \text{real}.$$

We obtain easily

$$\frac{\partial^2 \Phi}{\partial {x_1}^2} + \frac{\partial^2 \Phi}{\partial {x_2}^2} = 0 \tag{13}$$

$$\frac{\partial \Phi}{\partial n}|_S = 0 \tag{14}$$

$$\int_\Gamma \frac{\partial \Phi}{\partial x_2}\,\mathrm{d}\Gamma = 0 \tag{15}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi}{\partial x_2}\left(\pm\frac{1}{2},0\right)\right) = -\frac{\sigma}{R\sin\theta}\frac{\partial \Phi}{\partial x_2}\left(\pm\frac{1}{2},0\right) \tag{16}$$

$$\frac{g}{\sigma}\frac{\partial \Phi}{\partial x_2} - \frac{\tau}{\rho g \sigma^3}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \Phi}{\partial x_2}\right) = \omega^2 \Phi \qquad \text{for } x_2 = 0 \tag{17}$$

3 Application of the linear shallow water theory to the regular case

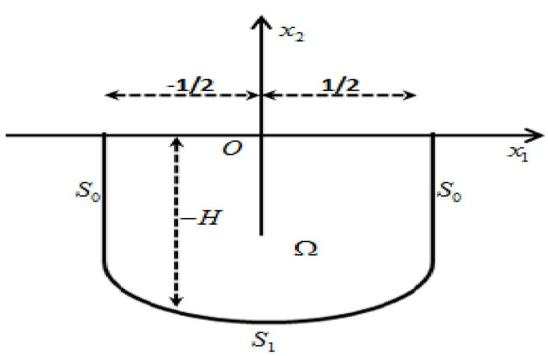

Fig. 2: Model of the system in the regular case

The regular case is the case where the minimum of the depth of liquid is strictly positive.

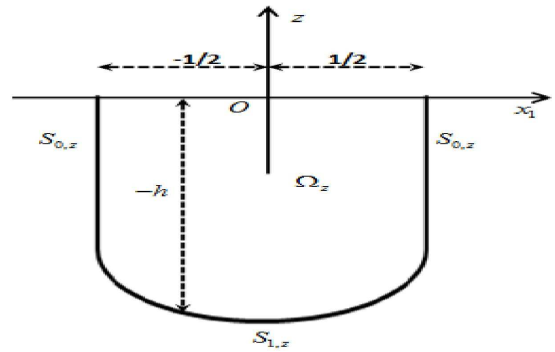

Fig. 3: The transformed figure in the domain Ω_z (regular case)

We suppose that, in the plane Ox_1x_2, the rigid boundary S consists of a vertical side wall S_0 and a bottom S_1 defined by an equation of the form $x_2 = -H(x_1)$, where H is strictly positive in $\left[-\frac{1}{2},\frac{1}{2}\right]$and of the order of $\sqrt{\varepsilon}$, $\varepsilon > 0$ being a small parameter [Fig.2].
Setting

$$x_2 = z\sqrt{\varepsilon} \quad ; \qquad H(x_1) = h(x_1)\sqrt{\varepsilon}$$

we obtain the transformed figure in the plane (x_1,z) [Fig.3] and we consider the transformed problem in the domain Ω_z.
We set

$$\Phi(x_1,x_2) = \Phi(x_1,z\sqrt{\varepsilon}) = \hat{\Phi}(x_1,z)$$

The equation (13) becomes

$$\varepsilon\frac{\partial^2 \hat{\Phi}}{\partial {x_1}^2} + \frac{\partial^2 \hat{\Phi}}{\partial z^2} = 0 \ \text{ in } \ \Omega_z \tag{18}$$

The equation (14) is divided in $\frac{\partial \hat{\Phi}}{\partial x_1} = 0$ on $S_{0,z}$, that we replace by [7]

$$\frac{\partial \hat{\Phi}}{\partial x_1} = 0 \qquad \text{for } z = 0,\ x_1 = \pm\frac{1}{2} \tag{19}$$

and
$$\frac{\partial \hat{\Phi}_1}{\partial n} = 0 \qquad \text{on } S_{1,z}$$

that we can write, since the equation of S_1 is $x_2 + h(x_1)\sqrt{\varepsilon} = 0$:

$$\frac{\partial \hat{\Phi}_1}{\partial z} = -\varepsilon\frac{\partial \hat{\Phi}_1}{\partial x_1}h'(x_1) \qquad \text{for } z = -h(x_1) \tag{20}$$

The equation (15) becomes

$$\int_\Gamma \frac{\partial \hat{\Phi}}{\partial z}\,\mathrm{d}\Gamma = 0 \tag{21}$$

Since R is infinite and $\theta = \frac{\pi}{2}$, the equation (16) can be written

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \hat{\Phi}}{\partial z}|_{z=0}\right) = 0 \ \text{ for } \ x_1 = \pm\frac{1}{2} \qquad (22)$$

Finally, the equation (17) gives

$$\frac{\partial \hat{\Phi}}{\partial z} - \frac{\tau}{\rho g \sigma^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \hat{\Phi}}{\partial z}\right) = \lambda\hat{\Phi} \qquad \text{for} \quad z = 0 \qquad (23)$$

where we have set

$$\lambda = \frac{\sigma\omega^2}{g}\sqrt{\varepsilon} \qquad (24)$$

We seek a solution of the problem (18),..., (24) in the form of an asymptotic expansions of powers of ε

$$\hat{\Phi}(x_1,z,\varepsilon) = \hat{\Phi}_0(x_1,z) + \varepsilon\hat{\Phi}_1(x_1,z) + \varepsilon^2\hat{\Phi}_2(x_1,z) + \cdots \qquad (25)$$

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \qquad (26)$$

3.1 *The approximation of order zero*

Equaling in the equations the terms which do not depend on ε, we obtain

$$\frac{\partial^2\hat{\Phi}}{\partial z^2} = 0 \qquad \text{in} \quad \Omega_z \ ;$$

$$\frac{\partial\hat{\Phi}}{\partial x_1}\left(\pm\frac{1}{2},0\right) = 0 \ ;$$

$$\frac{\partial \hat{\Phi}_0}{\partial z} - \frac{\tau}{\rho g \sigma^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \hat{\Phi}_0}{\partial z}\right) = \lambda_0\hat{\Phi}_0 \qquad \text{for} \quad z = 0 \ ;$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \hat{\Phi}}{\partial z}|_{z=0}\right) = 0 \ \text{ for } \ x_1 = \pm\frac{1}{2} \ ;$$

$$\frac{\partial\hat{\Phi}_0}{\partial z}(x_1,-h(x_1)) = 0 \ ;$$

$$\int_\Gamma \frac{\partial\hat{\Phi}_0}{\partial z}\,\mathrm{d}\Gamma = 0.$$

We find

$$\hat{\Phi}_0 = v(x_1); \quad v'\left(\pm\frac{1}{2}\right) = 0; \quad \lambda_0 v(x_1) = 0 \qquad (27)$$

$\lambda_0 \neq 0$ is impossible, then its gives the trivial solution. Indeed, it is easy to verify that, if $\lambda_0 = 0$, we obtain for $\hat{\Phi}_1$ the same problem as $\hat{\Phi}_0$ with $\lambda_0 \neq 0$, so that $\hat{\Phi}_1 = 0$; continuing, $\hat{\Phi}_n = 0 \ \forall n \geq 0$, and, therefore, $\hat{\Phi} = 0$.
Then, we have $\lambda_0 = 0$ and $v(x_1)$ is to determined later on.

3.2 *The first approximation*

i) We have

$$\frac{\partial^2\hat{\Phi}_1}{\partial z^2} = -v''(x_1) \ ;$$

$$\frac{\partial\hat{\Phi}_1}{\partial x_1}\left(\pm\frac{1}{2},0\right) = 0 \ ;$$

$$\frac{\partial \hat{\Phi}_1}{\partial z} - \frac{\tau}{\rho g \sigma^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \hat{\Phi}_1}{\partial z}\right) = \lambda_1 v(x_1) \qquad \text{for} \quad z = 0 \ ;$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \hat{\Phi}_1}{\partial z}|_{z=0}\right) = 0 \ \text{ for } \ x_1 = \pm\frac{1}{2} \ ;$$

$$\frac{\partial\hat{\Phi}_1}{\partial z} = -h'(x_1)v'(x_1) \qquad \text{for} \quad z = -h(x_1) \ ;$$

$$\int_\Gamma \frac{\partial\hat{\Phi}_1}{\partial z}\,\mathrm{d}\Gamma = 0.$$

We obtain successevely

$$\hat{\Phi}_1 = -\frac{1}{2}v''(x_1)z^2 + \alpha(x_1)z + w_1(x_1) \ ;$$

where $\alpha(x_1)$ and $w_1(x_1)$ are functions to determine,

$$w_1'\left(\pm\frac{1}{2}\right) = 0 \ ;$$

$$\alpha(x_1) - \frac{\tau}{\rho g \sigma^2}\alpha''(x_1) = \lambda_1 v(x_1) \ ;$$

$$\alpha'\left(\pm\frac{1}{2}\right) = 0 \ ;$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(h(x_1)v'(x_1)\right) + \alpha(x_1) = 0 \ ;$$

$$\int_{-1/2}^{1/2}\alpha(x_1)\,\mathrm{d}x_1 = 0 \ .$$

Let us notice that, by integrating the third equation, we obtain $\int_{-1/2}^{1/2} v(x_1)\,\mathrm{d}x_1 = 0$.

ii) Instead of consider straight the problem for $v(x_1)$, we are going to solve the problem for $\alpha(x_1)$, λ_1 and $v(x_1)$ being supposed known.
We have

$$\alpha(x_1) - \frac{\tau}{\rho g \sigma^2}\alpha''(x_1) = \lambda_1 v(x_1) \qquad (28)$$

$$\int_{-1/2}^{1/2}\alpha(x_1)\,\mathrm{d}x_1 = 0 \qquad (29)$$

$$\alpha'\left(\pm\frac{1}{2}\right) = 0 \qquad (30)$$

Multiplying (28) by $\overline{\widetilde{\alpha}}(x_1)$ verifing (29), integrating on $\left[-\frac{1}{2},\frac{1}{2}\right]$, we obtain the variational formulation of the problem: To find $\alpha(\cdot)\in\widetilde{H}^1\left(-\frac{1}{2},\frac{1}{2}\right)$, with

$$\widetilde{H}^1\left(-\frac{1}{2},\frac{1}{2}\right)=\left\{\alpha\in H^1\left(-\frac{1}{2},\frac{1}{2}\right);\ \int_{\frac{-1}{2}}^{\frac{1}{2}}\alpha(x_1)\,\mathrm{d}x_1=0\right\},$$

such that

$$\begin{cases} a\left(\alpha,\widetilde{\alpha}\right)\overset{\text{def}}{=}\int_{-1/2}^{1/2}\alpha\overline{\widetilde{\alpha}}\,\mathrm{d}x_1+\frac{\tau}{\rho g\sigma^2}\int_{-1/2}^{1/2}\alpha'\overline{\widetilde{\alpha}'}\,\mathrm{d}x_1 \\ =(\lambda_1 v,\widetilde{\alpha})_{\widetilde{L}^2}\qquad\forall\widetilde{\alpha}\in\widetilde{H}^1\end{cases}\tag{31}$$

where $(\cdot,\cdot)_{\widetilde{L}^2}$ is the scalar product of the space

$$\widetilde{L}^2\left(-\frac{1}{2},\frac{1}{2}\right)=\left\{u\in L^2\left(-\frac{1}{2},\frac{1}{2}\right);\int_{\frac{-1}{2}}^{\frac{1}{2}}u(x_1)\,\mathrm{d}x_1=0\right\}$$

Obviously, $a\left(\alpha,\widetilde{\alpha}\right)$ is coercive, continuous, hermitian, sesquilinear form in $\widetilde{H}^1\times\widetilde{H}^1$ and the embedding of $\widetilde{H}^1$ in $\widetilde{L}^2$ is classically continuous, dense and compact.
Let us call Q_0 the unnbounded operator of $\widetilde{L}^2$ associated to $a(\cdot,\cdot)$ and the pair $\left(\widetilde{H}^1,\widetilde{L}^2\right)$; the equation (31) is equivalent to the equation

$$\alpha=\lambda_1 Q_0^{-1}v\tag{32}$$

iii) Then, we can calcute $v(x_1)$ and λ_1 by solving the eigenvalues problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x_1}\left(hv'\right)=\lambda_1 Q_0^{-1}v\tag{33}$$

$$\int_{-1/2}^{1/2}v(x_1)\,\mathrm{d}x_1=0\tag{34}$$

$$v'\left(\pm\frac{1}{2}\right)=0\tag{35}$$

Its variational formulation is

$$\int_{-1/2}^{1/2}hv'\overline{\widehat{v}'}\,\mathrm{d}x_1=\lambda_1\left(Q_0^{-1}v,\widehat{v}\right)_{\widetilde{L}^2}\quad\forall\widehat{v}\in\widetilde{H}^1\tag{36}$$

Since, in the regular case, $\min h(x_1)>0$, the left hand side of (36) can be considered as a scalar product in $\widetilde{H}^1$ by virtue of the Poincaré inequality.
Then, calling $\widetilde{M}_0$ the unbounded operator of $\widetilde{L}^2$ associated to the form $\int_{\frac{-1}{2}}^{\frac{1}{2}}hv'\overline{\widehat{v}'}\,\mathrm{d}x_1$ and the pair $\left(\widetilde{H}^1,\widetilde{L}^2\right)$, we see that the equation (36) is equivalent to

$$\widetilde{M}_0 v=\lambda_1 Q_0^{-1}v\tag{37}$$

Setting $\widetilde{M}_0^{1/2}v=v^*$ and introducing the operator $B=\widetilde{M}_0^{-1/2}Q_0^{-1}\widetilde{M}_0^{-1/2}$ bounded in $\widetilde{L}^2$, self-adjoint, positive definite and compact, we obtain the equivalent equation

$$Bv^*=\lambda_1^{-1}v^*\tag{38}$$

For the eigenvalues λ_{1j}, inverses of the eigenvalues of B, we have

$$0<\lambda_{11}\le\lambda_{12}\le\cdots\le\lambda_{1n}\le\cdots;\quad\lambda_{1n}\to\infty\ \text{ when }\ n\to\infty$$

The eigenfunction v_i^* of B form an orthogonal basis of $\widetilde{L}^2$. it is easy to see that the eigenfunction v_i of our problem verify

$$\left(Q_0^{-1}v_n,v_m\right)_{\widetilde{L}^2}=\begin{cases}0 & \text{if}\quad m\neq n\\ \lambda_{1n}^{-1} & \text{if}\quad m=n\end{cases}$$

3.3 The second order approximation

i) We have

$$\frac{\partial^2\hat{\Phi}_2}{\partial z^2}=\frac{1}{2}v^{IV}z^2+\frac{\mathrm{d}^3}{\mathrm{d}x_1^3}(hv')z-w_1'\,;$$

$$\frac{\partial\hat{\Phi}_2}{\partial x_1}=0\,;\ \text{for}\quad z=0\,,\ x_1=\pm\frac{1}{2};$$

$$\frac{\partial\hat{\Phi}_2}{\partial z}-\frac{\tau}{\rho g\sigma^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial\hat{\Phi}_2}{\partial z}\right)=\lambda_2\hat{\Phi}_0+\lambda_1\hat{\Phi}_1\qquad\text{for}\quad z=0\,;$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\hat{\Phi}_2}{\partial z}|_{z=0}\right)=0\ \text{ for }\ x_1=\pm\frac{1}{2}\,;$$

$$\frac{\partial\hat{\Phi}_2}{\partial z}=-h'(x_1)\frac{\partial\hat{\Phi}_1}{\partial x_1}\qquad\text{for}\quad z=-h(x_1)\,;$$

$$\int_\Gamma\frac{\partial\hat{\Phi}_2}{\partial z}\,\mathrm{d}\Gamma=0.$$

These equations give successively

$$\hat{\Phi}_2=\frac{1}{24}v^{IV}z^4+\frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}x_1^3}(hv')z^3-\frac{1}{2}w_1''z^2+\beta(x_1)z+w_2(x_1)\,;$$

$$w_2'\left(\pm\frac{1}{2}\right)=0\,;$$

$$\beta(x_1)-\frac{\tau}{\rho g\sigma^2}\beta''(x_1)=\lambda_2 v(x_1)+\lambda_1 w_1(x_1)$$

$$\beta'\left(\pm\frac{1}{2}\right)=0\,;$$

$$\beta(x_1)=\frac{\mathrm{d}}{\mathrm{d}x_1}\left[\frac{1}{6}h^3v''-\frac{1}{2}h^2\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}(hv')-h'w_1\right]\,;$$

$$\int_{-1/2}^{1/2}\beta(x_1)\,\mathrm{d}x_1=0.$$

Let us notice that, by integrating the third equation, we obtain $\int_{-1/2}^{1/2} w_1(x_1)\,\mathrm{d}x_1 = 0$.
ii) Instead of consider straight the problem for $w_1(x_1)$, we are going to solve the problem for $\beta(x_1)$, λ_2 and $w_1(x_1)$ being supposed known.
We have

$$\beta(x_1) - \frac{\tau}{\rho g \sigma^2}\beta''(x_1) = \lambda_2 v(x_1) + \lambda_1 w_1(x_1) \tag{39}$$

$$\int_{-1/2}^{1/2} \beta(x_1)\,\mathrm{d}x_1 = 0 \tag{40}$$

$$\beta'\left(\pm\frac{1}{2}\right) = 0 \tag{41}$$

This problem is analogous to the problem for $\alpha(x_1)$; therfore, we have

$$\beta(x_1) = \lambda_2 Q_0^{-1} v + \lambda_1 Q_0^{-1} w_1$$

iii) Now, we can calculate $w_1(x_1)$ and λ_2 by solving the eigenvalues problem

$$-\frac{\mathrm{d}}{\mathrm{d}x_1}(hw_1') - \lambda_1 Q_0^{-1} w_1 = f(v) + \lambda_2 Q_0^{-1} v \tag{42}$$

$$\int_{-1/2}^{1/2} w_1(x_1)\,\mathrm{d}x_1 = 0 \tag{43}$$

$$w_1'\left(\pm\frac{1}{2}\right) = 0 \tag{44}$$

with

$$f(v) = \frac{\mathrm{d}}{\mathrm{d}x_1}\left[\frac{1}{6}h^3 v''' - \frac{1}{2}h^2 \frac{\mathrm{d}^2}{\mathrm{d}x_1^2}(hv')\right]$$

Anyway, it is a matter of a countable infinity to problems, obtained by replacing λ_1 by λ_{1k} and v by v_k $(k = 1, 2, \cdots)$. If the right-hand side of the equation (42) was equal to zero, we would have for w_1 the same problems as for v. consequently, for each k, the homogeneous problem has solutions different from the trivial solution. Classically, our problem is pssible only if the right-hand side of (42) is orthogonal in $\widetilde{L}^2$ to the eigenlements v_k correspoding to λ_{1k}.
For instance, let us suppose that the λ_{1k} are simple eigenvalues; we have the only condition

$$\left(f(v_k) + \lambda_2 Q_0^{-1} v_k, v_k\right)_{\widetilde{L}^2} = 0 \quad \text{(without summation in } k)$$

and the second approximation of the eigenvalues is

$$\lambda_{2k} = -\lambda_{1k}\left(f(v_k), v_k\right)_{\widetilde{L}^2} \tag{45}$$

The problem for w_1 can be written, with the precedent notations

$$\widetilde{M}_0 w_1 = \lambda_{1k} Q_0^{-1} w_1 + f(v_k) + \lambda_{2k} Q_0^{-1} v_k$$

Setting $\widetilde{M}_0^{1/2} w_1 = w_1^*$, we obtain the equation

$$w_1^* = \lambda_{1k} B w_1^* + \widetilde{M}_0^{-1/2} f(v_k) + \lambda_{2k} B v_k^*$$

Let us seek w_1^* in the form

$$w_1^* \sim \sum_n c_n v_n^* \,,$$

$\sim$ denoting the convergence in $\widetilde{L}^2$ We have

$$\sum_n c_n v_n^* \sim \lambda_{1k} \sum_n c_n B v_n^* + \widetilde{M}_0^{-1/2} f(v_k) + \lambda_{2k} B v_k^*$$

Replacing Bv_n^* by $\lambda_{1n}^{-1} v_n^*$ and setting $\widetilde{M}_0^{-1/2} f(v_k) \sim \sum_n d_n v_n^*$, where the d_n are known, we obtain the relations

$$\begin{cases} c_n(1 - \lambda_{1k}\lambda_{1n}^{-1}) - d_n = 0 & \text{for } n \neq k \ ; \\ -d_k - \lambda_{2k}\lambda_{1k}^{-1} = 0 & \text{for } n = k \end{cases}$$

The last equation gives (45); the others give the c_n for $n \neq k$, c_k remaining indeterminate.
We find finally

$$w_1 \sim c_k v_k + \sum_{n \neq k} d_n (1 - \lambda_{1k}\lambda_{1n}^{-1}) v_n$$

Consequently, we must calculate the third approximation in order to determine c_k and then w_1.

4 Application of the linear shallow water theory to the singular case

Now, we are going to consider the case where the minimum of the depth of liquid is equal to zero, restricting ourselves to the problem for a parabolic container.

4.1 Position of the problem

In the equilibrium position, the domain Ω' occupied by the liquid [Fig.4] is bounded by the parabolic wall S.

$$x_2' = X\left(\frac{x_1'^2}{d} - d\right); \quad x_2' \leq 0$$

and the free line

$$x_2' = 0; \quad -d \leq x_1' \leq d$$

X is an adimensional coefficient considered as small. If θ is the angle between S and Ox_1', we have $2X = \tan\theta$, so

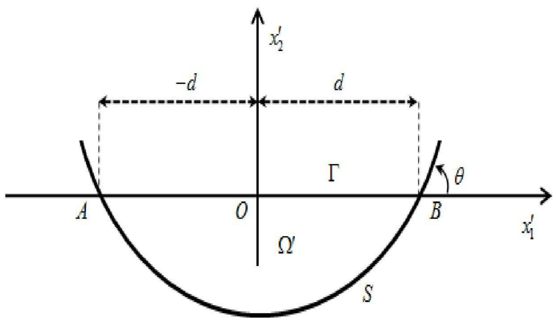

Fig. 4: Model of the system in the singular case

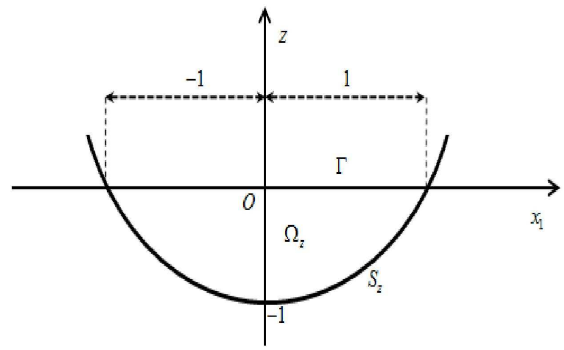

Fig. 5: The transformed figure in the domain Ω_z (singular case)

that θ is small and of the order of X.
The equations (1), $\cdots$, (9), (11) and (12) of the regular case unchanged. The calculation of the radius of curvature of Γ_t, gives easily $R\sin\theta = d$ at the first order, so that the equation (10) becomes

$$\zeta'(\pm d,t) = \pm d^{-1}\zeta(\pm d,t) \tag{10'}$$

4.2 Introduction of adimensional variables

We set

$$x_1' = dx_i \qquad (i = 1,2)$$

The equations (13), (14), (15) for $\Phi(x_1,x_2)$ are unchanged and the equations (16) and (17) become

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi}{\partial x_2}(\pm 1,0)\right) = \pm\frac{\partial \Phi}{\partial x_2}(\pm 1,0) \tag{16'}$$

$$\frac{g}{d}\frac{\partial \Phi}{\partial x_2} - \frac{\tau}{\rho g d^3}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \Phi}{\partial x_2}\right) = \omega^2\Phi \qquad \text{for } x_2 = 0 \tag{17'}$$

4.3. Application of the linear shallow water theory
We introduce a small parameter ε, setting

$$X = \sqrt{\varepsilon}$$

Setting $X_2 = z\sqrt{\varepsilon}$, we obtain the equation of S_z:

$$z = x_1^2 - 1$$

and, like in the regular case, we solve the problem in the transformed domain Ω_z [Fig.5]. Setting

$$\Phi(x_1,x_2) = \Phi(x_1, z\sqrt{\varepsilon}) = \hat{\Phi}_1(x_1,z)$$

We obtain the equations

$$\varepsilon\frac{\partial^2 \hat{\Phi}}{\partial {x_1}^2} + \frac{\partial^2 \hat{\Phi}}{\partial z^2} = 0 \ \text{ in } \ \Omega_z \tag{46}$$

$$\frac{\partial \hat{\Phi}}{\partial z} = -2\varepsilon x_1\frac{\partial \hat{\Phi}}{\partial x_1} \qquad \text{for} \qquad z = x_1^2 - 1 \tag{47}$$

$$\int_{-1}^{1}\frac{\partial \hat{\Phi}}{\partial z}\,|_{z=0}\,\mathrm{d}x_1 = 0 \tag{48}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \hat{\Phi}}{\partial z}(\pm 1,0)\right) = \pm\frac{\partial \hat{\Phi}}{\partial z}(\pm 1,0) \tag{49}$$

$$\frac{\partial \hat{\Phi}}{\partial z} - \frac{\tau}{\rho g d^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \hat{\Phi}}{\partial z}\right) = \lambda\hat{\Phi} \qquad \text{for } z = 0 \tag{50}$$

with

$$\lambda = \frac{d\omega^2}{g}\sqrt{\varepsilon} \tag{51}$$

We remark that the impermeability condition of the vertical wall S_0 of the regular case disapears .

We seek a solution of the problem in the form of the asymptotic expantion of powers of ε, (25), (26) and we use the notations of the regular case.

4.3 The approximation of order zero

We obtain easily

$$\lambda_0 = 0; \qquad \hat{\Phi}_0 = v(x_1),$$

but without condition for $v(x_1)$, that is to determined later on.

i) The first approximation
We have

$$\frac{\partial^2\hat{\Phi}_1}{\partial z^2} = -v''(x_1) \qquad \text{in} \ \ \Omega_z \ ;$$

$$\frac{\partial\hat{\Phi}_1}{\partial x_1} = 2x_1 v'(x_1) \qquad \text{for} \ \ z = x_1^2 - 1 \ ;$$

$$\int_{-1}^{1} \frac{\partial\hat{\Phi}_1}{\partial z}|_{z=0} \ \mathrm{d}x_1 = 0 \ ;$$

$$\frac{\partial\hat{\Phi}_1}{\partial z} - \frac{\tau}{\rho g d^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial\hat{\Phi}_1}{\partial z}\right) = \lambda_1 v(x_1) \qquad \text{for} \quad z = 0 \ ;$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\hat{\Phi}_1}{\partial z}|_{z=0}\right) = \pm\frac{\partial\hat{\Phi}_1}{\partial z}|_{z=0} \quad \text{for} \quad x_1 = \pm 1 \ ;$$

We obtain successively

$$\hat{\Phi}_1 = -\frac{z^2}{2}v''(x_1) + z\alpha(x_1) + w_1(x_1) \ ;$$

where $\alpha(x_1)$ and $w_1(x_1)$ are functions to determine,

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)v'(x_1)\right] = -\alpha(x_1) \ ;$$

$$\int_{-1}^{1}\alpha(x_1)\,\mathrm{d}x_1 = 0$$

$$\alpha(x_1) - \frac{\tau}{\rho g d^2}\alpha''(x_1) = \lambda_1 v(x_1)$$

$$\alpha'(\pm 1) = \pm\alpha(\pm 1)$$

ii) Instead of consider straight the problem for $v(x_1)$, we are going to solve the problem for $\alpha(x_1)$, λ_1 and $v(x_1)$ being supposed known.
We have

$$\alpha(x_1) - \frac{\tau}{\rho g d^2}\alpha''(x_1) = \lambda_1 v(x_1) \tag{52}$$

$$\int_{-1}^{1}\alpha(x_1)\,\mathrm{d}x_1 = 0 \tag{53}$$

$$\alpha'(\pm 1) = \pm\alpha(\pm 1) \tag{54}$$

Like in the regular case, we obtain the variational equation of this problem:

$$\begin{cases} a_0(\alpha,\hat{\alpha}) \overset{\text{def}}{=} \int_{-1}^{1}\alpha\overline{\hat{\alpha}}\,\mathrm{d}x_1 + \frac{\tau}{\rho g d^2}\int_{-1}^{1}\alpha'\overline{\hat{\alpha}'}\,\mathrm{d}x_1 \\ -\frac{\tau}{\rho g d^2}\left[\alpha(1)\overline{\hat{\alpha}}(1) + \alpha(-1)\overline{\hat{\alpha}}(-1)\right] \\ = (\lambda_1 v, \hat{\alpha})_{\widetilde{L}^2} \qquad \forall\hat{\alpha}\in\widetilde{H}^1 \end{cases} \tag{55}$$

The imbedding $\widetilde{H}^1(-1,1) \subset \widetilde{L}^2(-1,1)$ is classically continuous, dense and compact,.
The sesquilinear form $a_0(\cdot,\cdot)$ is hermitian and continuous, we are going to prove that it is coercive in $\widetilde{H}^1\times\widetilde{H}^1$ by using an auxiliary problem. Let us show that

$$\inf_{\alpha\in\widetilde{H}^1}\frac{\int_{-1}^{1}\alpha'^2(x_1)\,\mathrm{d}x_1}{\alpha^2(1)+\alpha^2(-1)} = 1$$

At first, we seek the inf in

$$\widetilde{C}^2[-1,1] = \left\{\alpha\in C^2[-1,1]; \ \int_{-1}^{1}\alpha(x_1)\,\mathrm{d}x_1 = 0\right\}$$

In amounts to the same thing to seek the inf of $\int_{-1}^{1}\alpha'^2(x_1)\,\mathrm{d}x_1$ in $C^2[-1,1]$ under the conditions

$$\alpha^2(1)+\alpha^2(-1) = 1; \qquad \int_{-1}^{1}\alpha(x_1)\,\mathrm{d}x_1 = 0.$$

Using the methods of the classical calculus of variations, we write, λ and 2μ being the multipliers associated to both conditions

$$\begin{cases} \delta\left(\int_{-1}^{1}\alpha'^2(x_1)\,\mathrm{d}x_1\right) - \lambda\,\delta\left[\alpha^2(1)+\alpha^2(-1)\right] \\ -2\mu\int_{-1}^{1}\delta\alpha\,\mathrm{d}x_1 = 0 \end{cases}$$

i.e

$$\begin{cases} -\int_{-1}^{1}(\alpha''+\mu)\delta\alpha\,\mathrm{d}x_1 + [\alpha'(1)-\lambda\,\alpha(1)]\,\delta\alpha(1) \\ -[\alpha'(-1)+\lambda\,\alpha(-1)]\,\delta\alpha(-1) = 0 \end{cases}$$

Therfore, we have

$$\alpha''+\mu = 0 \ ;$$

$$\alpha'(1) - \lambda\,\alpha(1) = 0 \ ;$$

$$\alpha'(-1) + \lambda\,\alpha(-1) = 0 \ ,$$

with

$$\int_{-1}^{1}\alpha(x_1)\,\mathrm{d}x_1 = 0$$

It is a Steklov's problem that is easy to solve; we find $\lambda = 1$ and $\lambda = 3$, so that the inf is 1. Then, we have

$$\int_{-1}^{1}\alpha'^2(x_1)\,\mathrm{d}x_1 \geq \alpha^2(1)+\alpha^2(-1); \qquad \forall\alpha\in\widetilde{C}^2[-1,1]$$

Now we use the following theorem([2], p 127): Let $\alpha\in H^1(-1,1)$. There exists a sequence $\alpha_n\in D(\mathbb{R})$ such that the restriction $\hat{\alpha}_n$ of α_n to the interval $[-1,1]$ tends to α in $H^1(-1,1)$, $\hat{\alpha}_n\in C^2[-1,1]$, so that $C^2[-1,1]$ is dense in $H^1(-1,1)$.
Then, $\widetilde{C}^2[-1,1]$ is dense in $\widetilde{H}^1(-1,1)$ and the inequality is valid for $\alpha\in\widetilde{H}^1(-1,1)$.
We denote that, if α is cmplex ($\alpha = \alpha_1 + i\alpha_2$, α_1, α_2 reals), we have easily:

$$\int_{-1}^{1}|\alpha'|^2(x_1)\,\mathrm{d}x_1 \geq |\alpha(1)|^2 + |\alpha(-1)|^2; \qquad \forall\alpha\in\widetilde{H}^1$$

Now, we can prove that $[a_0(\alpha,\alpha)]^{1/2}$ defines on $\widetilde{H}^1(-1,1)$ a norm which is equivalent to the classical norm $\|\alpha\|_1$ of $H^1(-1,1)$.
We set

$$b(\alpha,\hat{\alpha}) = \int_{-1}^{1} \alpha\overline{\hat{\alpha}'}\,\mathrm{d}x_1 - \left[\alpha(1)\overline{\widetilde{\alpha}}(1) + \alpha(-1)\overline{\widetilde{\alpha}}(-1)\right]$$

By virtue of the precedent result, we have

$$b(\alpha,\alpha) \geq 0$$

Consequently, we have

$$a_0(\alpha,\alpha) = \int_{-1}^{1} |\alpha|^2\,\mathrm{d}x_1 + \frac{\tau}{\rho g d^2} b(\alpha,\alpha) \geq 0$$

Let us prove that there exists a constant such that $c > 0$

$$\frac{a_0(\alpha,\alpha)}{\|\alpha\|_1^2} \geq c\,; \qquad \forall\alpha \in \widetilde{H}^1(-1,1)$$

If c doesn't exist, there is a sequence $\{\alpha_n\} \in \widetilde{H}^1(-1,1)$ such that

$$\frac{a_0(\alpha_n,\alpha_n)}{\|\alpha_n\|_1^2} \to 0 \qquad n \to +\infty$$

By homogeneity, we can suppose $\|\alpha_n\|_1^2 = 1$, and then $a_0(\alpha_n,\alpha_n) \to 0$.
From the sequence $\{\alpha_n\}$ bounded in $\widetilde{H}^1(-1,1)$, we can extract a subsequence, still denoted $\{\alpha_n\}$, that is weakly convergent in $\widetilde{H}^1(-1,1)$, thus strongly convergent in $\widetilde{L}^2(-1,1)$ to a limit $\alpha^* \in \widetilde{H}^1(-1,1) \subset \widetilde{L}^2(-1,1)$.
From $a_0(\alpha_n,\alpha_n) \to 0$, we deduce

$$\|\alpha_n - \alpha_m\|_{L^2}^2 + \frac{\tau}{\rho g d^2} b(\alpha_n - \alpha_m, \alpha_n - \alpha_m) \to 0;$$

when $n,\ m \to +\infty$,

so that

$$b(\alpha_n - \alpha_m, \alpha_n - \alpha_m) \to 0$$

or

$$\begin{cases} \|\alpha_n' - \alpha_m'\|_{L^2}^2 \\ -\left[(\alpha_n(1) - \alpha_m(1))^2 + (\alpha_n(-1) - \alpha_m(-1))^2\right] \end{cases} \to 0$$

Since $\{\alpha_n\}$ is weakly convergent in $\widetilde{H}^1$, the sequences of traces $\{\alpha_n(1)\}$ and $\{\alpha_n(-1)\}$ are strongly convergent in $\mathbb{C}$, so that

$$\left\|\alpha_n' - \alpha_m'\right\|_{L^2} \to 0$$

Therefore, from

$$\|\alpha_n - \alpha_m\|_1^2 = \left\|\alpha_n' - \alpha_m'\right\|_{L^2}^2 + \|\alpha_n - \alpha_m\|_{L^2}^2\,,$$

we deduce

$$\|\alpha_n - \alpha_m\|_1 \to 0$$

and the sequence $\{\alpha_n\}$ is strongly convergent to α^* in $\widetilde{H}^1$. Then, from $\|\alpha_n\|_1 = 1$, we deduce $\|\alpha^*\|_1 = 1$.
On the other hand, from $a_0(\alpha_n,\alpha_n) \to 0$ we deduce $\|\alpha_n\|_{L^2}^2 \to 0$ and consequently $\alpha^* = 0$, that contradicts the precedent result.
Finally, c exists and the form $a_0(\alpha,\widehat{\alpha})$ is coercive in $\widetilde{H}^1 \times \widetilde{H}^1$.
Let us return to the variational equation (55).
Contrary to α and $\hat{\alpha}$, v does not verify $\int_{-1}^{1} v\,\mathrm{d}x_1 = 0$. But $w = v - \frac{1}{2}\int_{-1}^{1} v\,\mathrm{d}x_1$ verify

$$\int_{-1}^{1} w\,\mathrm{d}x_1 = 0; \qquad w' = v' \text{ and } \int_{-1}^{1} w\overline{\widehat{\alpha}}\,\mathrm{d}x_1 = \int_{-1}^{1} v\overline{\widehat{\alpha}}\,\mathrm{d}x_1.$$

Therefore, the variational formulation of the problem for $\alpha(x_1)$ is:
To find $\alpha(\cdot) \in \widetilde{H}^1(-1,1)$ such that

$$a_0(\alpha,\widehat{\alpha}) = (\lambda_1 w, \widehat{\alpha})_{\widetilde{L}^2} \quad \forall\widehat{\alpha} \in \widetilde{H}^1 \tag{56}$$

If we call Q_0 the unbounded operator of $\widetilde{L}^2(-1,1)$ associated to $a_0(.,.)$ and the pair $\left(\widetilde{H}^1, \widetilde{L}^2\right)$, the equation (56) is equivalent to

$$\alpha = \lambda_1 Q_0^{-1} w \tag{57}$$

iii) Then, we can calculate $w(x_1)$ and λ_1 by solving the eigenvalues problem (P) :

$$-\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1 - x_1^2)w'\right] = \lambda_1 Q_0^{-1} w \tag{58}$$

$$\int_{-1}^{1} w(x_1)\,\mathrm{d}x_1 = 0 \tag{59}$$

and there are not boundary conditions.
If λ_1 is an eigenvalue and and w a corresponding eigenelement, the eigenelement v is given by the equation (52):

$$v = Q_0^{-1} w - \frac{\tau}{\rho g d^2}\left(Q_0^{-1} w\right)''$$

(We remark that $Q_0^{-1} w$ belongin to $D(Q_0)$, belongs to $\widetilde{H}^1(-1,1)$).
In order to solve the problem (P), we use results obtained in [3] on the Legendre's operators.
We introduce the spaces

$$V = \left\{u \in L^2(-1,1);\ \sqrt{1 - x_1^2}\,u' \in L^2(-1,1)\right\};$$

$$\widetilde{V} = \left\{u \in V; \int_{-1}^{1} u(x_1)\,\mathrm{d}x_1 = 0\right\}$$

equipped with the scalar product

$$a_0(u,\widetilde{u}) = \int_{-1}^{1}\left\{(1 - x_1^2)u'\overline{\widetilde{u}'} + u\overline{\widetilde{u}}\right\}\mathrm{d}x_1$$

V and $\widetilde{V}$ are Hilbert spaces and the embedding $V \subset L^2$ and $\widetilde{V} \subset L^2$ are continuous, dense and compact [3].
Let consider the problem (P).
Since $\lambda Q_0^{-1} w \in \widetilde{L}^2$, $\sqrt{1-x_1^2}\, w' \in H^1$ and, consequently [3] $\sqrt{1-x_1^2}\, w' = 0$ for $x_1 = \pm 1$.
We have

$$\int_{-1}^{1} -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)w'\right]\cdot \overline{W} \mathrm{d}x_1 = \left(\lambda_1 Q_0^{-1} w, W\right)_{\widetilde{L}^2} \quad \forall W \in \widetilde{V}$$

Writing $W = \widetilde{W} - \frac{1}{2}\int_{-1}^{1} \widetilde{W}\, \mathrm{d}x_1$ with $\widetilde{W}$ arbitrary in V, we have

$$\int_{-1}^{1} -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)w'\right]\cdot \overline{\widetilde{W}} \mathrm{d}x_1 = \left(\lambda_1 Q_0^{-1} w, \widetilde{W}\right)_{L^2} \quad \forall \widetilde{W} \in V.$$

But $\mathscr{D}(-1,1)$ is dense in V [3]. Taking $\hat{W} \in \mathscr{D}(-1,1)$ and integrating by parts, we have

$$\begin{cases} \left[-(1-x_1^2)w'\overline{\hat{W}}\right]_{-1}^{1} + \int_{-1}^{1}(1-x_1^2)w'\cdot \overline{\hat{W}}'\, \mathrm{d}x_1 \\ = \lambda_1 \left(Q_0^{-1} w, \hat{W}\right)_{L^2}; \qquad \forall \hat{W} \in \mathscr{D}(-1,1) \end{cases}$$

The first term of the left-hand side has sense and it is equal to zero, so that we have

$$\int_{-1}^{1}(1-x_1^2)w'\cdot \overline{\hat{W}}'\, \mathrm{d}x_1 = \lambda_1 \left(Q_0^{-1} w, \hat{W}\right)_{L^2}; \ \forall \hat{W} \in \mathscr{D}(-1,1) \tag{60}$$

therefore by density for each $\hat{W} \in V$ and finally

$$\int_{-1}^{1}(1-x_1^2)w'\cdot \overline{\widetilde{W}}'\, \mathrm{d}x_1 = \lambda_1 \left(Q_0^{-1} w, \widetilde{W}\right)_{L^2}; \ \forall \widetilde{W} \in \widetilde{V} \tag{61}$$

Reciprocally, from (61), we deduce (60) and consequently

$$-\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)w'\right] = \lambda_1 Q_0^{-1} w \ \text{ in } \ \mathscr{D}'(-1,1)$$

Therefore, the variational formulation of the problem (P) is : to find $w \in \widetilde{V}$ verifying (61).
The sesquilinear form of the left-hand side of (61) is hermitian and continuous on $\widetilde{V} \times \widetilde{V}$. We are going to prove that is coercive, i.e there exists a constant $c_0 > 0$ such that

$$\int_{-1}^{1}(1-x_1^2)w'^2 \mathrm{d}x_1 \geq c_0 \|w\|_{\widetilde{V}}^2; \ \forall w \in \widetilde{V}.$$

If $c_0 > 0$ does not exist, there exists a sequence $\{w_n\} \in \widetilde{V}$ such that

$$\|w_n\|_{\widetilde{V}} = 1, \qquad \int_{-1}^{1}(1-x_1^2)w_n'^2 \mathrm{d}x_1 \to 0 \ \text{ when } \ n \to \infty.$$

From the sequence $\{w_n\}$, we can extract a subsequence, still denoted $\{w_n\}$ that is strongly convergent in $\widetilde{L}^2$ to a limit $\widetilde{w} \in \widetilde{V} \subset \widetilde{L}^2$:

$$\|w_n - \widetilde{w}\|_{\widetilde{L}^2} \to 0$$

From

$$\|w_n - w_m\|_{\widetilde{V}}^2 = \int_{-1}^{1}\left[(1-x_1^2)\left(w_n' - w_m'\right)^2 + (w_n - w_m)^2\right]\mathrm{d}x_1$$

and the precedent result, we deduce

$$\|w_n - w_m\|_{\widetilde{V}} \to 0 \ \text{ when } \ n,\ m \ \to \infty.$$

The sequence $\{w_n\}$ is convergent in $\widetilde{V}$ to $\widetilde{w}$ and consequently

$$\|\widetilde{w}\|_{\widetilde{V}} = 1.$$

On the other hand, from $\int_{-1}^{1}(1-x_1^2)\widetilde{w}'^2 \mathrm{d}x_1 = 0$, we deduce $\widetilde{w}' = 0$, then $\widetilde{w} = cte$, and, since $\widetilde{w} \in \widetilde{V}$, $\widetilde{w} = 0$, that contradicts the precedent result.
There, we can finish like in the regular case.
let us call $\widetilde{L}_0$, the unbounded opertor of $\widetilde{L}^2$ associated to the form $\int_{-1}^{1}(1-x_1^2)w'\cdot \overline{W}' \mathrm{d}x_1$ and the pair $\left(\widetilde{V}, \widetilde{L}^2\right)$.
The variational equation (61) is equivalent to the operatorial equation

$$\widetilde{L}_0 w = \lambda_1 Q_0^{-1} w \tag{62}$$

Setting $\widetilde{L}_0^{1/2} w = w^*$ and and introducing the operator $C = \widetilde{L}_0^{-1/2} Q_0^{-1} \widetilde{L}_0^{-1/2}$ bounded in $\widetilde{L}^2$, self-adjoint, positive definite and compact, we obtain the equivalent equation

$$C w^* = \lambda_1^{-1} w^* \tag{63}$$

The eigenvalues λ_{1j} of the problem are the inverse of the eigenvalues of C and we have

$$0 < \lambda_{11} \leq \lambda_{12} \leq \cdots \leq \lambda_{1n} \leq \cdots; \quad \lambda_{1n} \to \infty \ \text{ when } \ n \to \infty$$

The eigenfunction w_n of the problem verify

$$\left(Q_0^{-1} w_n, w_m\right)_{\widetilde{L}^2} = \begin{cases} 0 & \text{if } \ m \neq n \\ \lambda_{1n}^{-1} & \text{if } \ m = n \end{cases}$$

The v_n are given by

$$v_n = Q_0^{-1} w_n - \frac{\tau}{\rho g d^2}\left(Q_0^{-1} w_n\right)''$$

Remark: It was possible to avoid a few precedent calculations by using the general theory of the degenerate elliptic operators of Baouendi and Goulaouic ([1], [7]); this theory must be used in the case of an arbitrary container.□

In our case, we consider the operator (of Legendre)

$$\widetilde{L}_0 = -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)\frac{\mathrm{d}}{\mathrm{d}x_1}\right]$$

without boundary condition.
It can be proved that
1) $\widetilde{L}_0$ is topological isomorphism of

$$D(\widetilde{L}_0) = \left\{ w \in \widetilde{H}^1(-1,1);\ \ (1-x_1^2)\,w \in H^2(-1,1) \right\}$$

onto $\widetilde{L}^2(-1,1)$
2) $\widetilde{L}_0$, considered as an unbounded operator of $\widetilde{L}^2(-1,1)$, has an inverse $\widetilde{L}_0^{-1}$ bounded from $\widetilde{L}^2(-1,1)$ in $\widetilde{L}^2(-1,1)$, self-adjoint, definite positive and compact.
Then the problem (P) is :
To find $w \in D(\widetilde{L}_0)$ and $\lambda_1 \in \mathbb{R}$ such that

$$\widetilde{L}_0 w = \lambda_1 Q_0^{-1} w$$

It is the equation (62)

4.4 The second order approximation

i) we have

$$\frac{\partial^2 \hat{\Phi}_2}{\partial z^2} = \frac{z^2}{2} v^{IV}(x_1) - z\alpha''(x_1) - w''(x_1) \ \text{ in } \ \Omega_z;$$

$$\begin{cases} \frac{\partial \hat{\Phi}_2}{\partial z}\Big|_{z=x_1^2-1} \\ = 2x_1 \left[-\frac{(x_1^2-1)^2}{2} v'''(x_1) + (x_1^2-1)\alpha'(x_1) + w_1'(x_1) \right]; \end{cases}$$

$$\int_{-1}^{1} \frac{\partial \hat{\Phi}_2}{\partial z}|_{z=0}\ \mathrm{d}x_1 = 0\ ;$$

$$\frac{\partial \hat{\Phi}_2}{\partial z}|_{z=0} - \frac{\tau}{\rho g d^2} \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial \hat{\Phi}_2}{\partial z}|_{z=0} \right) = \lambda_2 v(x_1) + \lambda_1 w_1(x_1);$$

$$\frac{\partial}{\partial x_1} \left(\frac{\partial \hat{\Phi}_2}{\partial z} \right) = \pm \frac{\partial \hat{\Phi}_1}{\partial z} \ \text{ for } \ z=0; \ \ x_1 = \pm 1.$$

These equations give

$$\hat{\Phi}_2 = \frac{z^4}{24} v^{IV}(x_1) - \frac{z^3}{6}\alpha''(x_1) - \frac{z^2}{2} w_1''(x_1) + z\beta(x_1) + w_2(x_1),$$

where $\beta(x_1)$ and $w_2(x_1)$ are to determine,

$$\beta(x_1) = \frac{\mathrm{d}}{\mathrm{d}x_1} \left[-\frac{(x_1^2-1)^3}{6} v''' + \frac{(x_1^2-1)^2}{2}\alpha' + (x_1^2-1) w_1' \right];$$

$$\int_{-1}^{1} \beta(x_1)\,\mathrm{d}x_1 = 0\ ;$$

$$\beta(x_1) - \frac{\tau}{\rho g d^2}\beta''(x_1) = \lambda_2 v(x_1) + \lambda_1 w_2(x_1)\ ;$$

$$\beta'(\pm 1) = \pm \beta(\pm 1)\ .$$

ii) Instead of consider straight the problem for $w_1(x_1)$, we are going to solve the problem for $\beta(x_1),\ v(x_1),\ w_1(x_1),\ \lambda_1,\ \lambda_2$ being supposed known.
We have

$$\beta(x_1) - \frac{\tau}{\rho g d^2}\beta''(x_1) = \lambda_2 v(x_1) + \lambda_1 w_2(x_1) \tag{64}$$

$$\int_{-1}^{1} \beta(x_1)\,\mathrm{d}x_1 = 0 \tag{65}$$

$$\beta'(\pm 1) = \pm \beta(\pm 1) \tag{66}$$

This problem is analogous to the problem for $\alpha(x_1)$ and we have with the precedent notations

$$a_0\left(\beta, \hat{\beta}\right) = \lambda_2 \int_{-1}^{1} v\overline{\hat{\beta}}\,\mathrm{d}x_1 + \lambda_1 \int_{-1}^{1} w_1 \overline{\hat{\beta}}\,\mathrm{d}x_1 \qquad \forall \beta \in \widetilde{H}^1.$$

Since v (resp w_1) does not verify $\int_{-1}^{1} v(x_1)\,\mathrm{d}x_1 = 0$ (resp $\int_{-1}^{1} w_1(x_1)\,\mathrm{d}x_1 = 0$), we introduce

$$w = v - \frac{1}{2}\int_{-1}^{1} v(x_1)\,\mathrm{d}x_1\ ; \qquad W_1 = w_1 - \frac{1}{2}\int_{-1}^{1} w_1(x_1)\,\mathrm{d}x_1.$$

The variational formulation of the problem is:
To find $\beta \in \widetilde{H}^1$ such that

$$a_0\left(\beta, \hat{\beta}\right) = \left(\lambda_2 w_0 + \lambda_1 W_1, \widehat{\beta}\right)_{\widetilde{L}^2} \quad \forall \widehat{\beta} \in \widetilde{H}^1.$$

and we obtain, Q_0 being the unbounded operator wich we have used:

$$\beta(x_1) = \lambda_2 Q_0^{-1} w + \lambda_1 Q_0^{-1} W_1.$$

iii) Now, we can calculate $w_1(x_1)$ and λ_2 by solving the eigenvalues problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x_1}\left[(1-x_1^2)W_1'\right] - \lambda_1 Q_0^{-1} W_1 = \hat{f}(w) + \lambda_2 Q_0^{-1} w \tag{67}$$

$$\int_{-1}^{1} W_1\,\mathrm{d}x_1 = 0 \tag{68}$$

where

$$\hat{f}(w) = -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[\frac{(1-x_1^2)^6}{6} w'' - \frac{(1-x_1^2)^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}x_1^2}(1-x_1^2) w' \right]$$

Anyway, it is a matter of a countable infinity of problems, obtained by replacing λ_1 by λ_{1k} and w by $w_k\ (k = 1, 2\cdots)$. Like in the regular case, the problem is possible only if the right-hand side of (67) is orthogonal in $\widetilde{L}^2$ to the eigenelements w_k corresponding to the λ_{1k}.
Supposing still that the λ_{1k} are simple eigenvalues, we obtain

$$\lambda_{2k} = -\lambda_{1k}\left(\hat{f}(w_k), w_k\right)_{\widetilde{L}^2} = 0 \tag{69}$$

The problem for W_1 can be written

$$\widetilde{L}_0 W_1 = \lambda_{1k} Q_0^{-1} W_1 + \hat{f}(w_k) + \lambda_{2k} Q_0^{-1} w_k$$

Setting

$$\widetilde{L}_0^{1/2} W_1 = W_1^*, \quad C = \widetilde{L}_0^{-1/2} Q_0^{-1} \widetilde{L}_0^{-1/2}$$

we obtain

$$W_1^* = \lambda_{1k} C W_1^* + \widetilde{L}_0^{-1/2} \hat{f}(w_k) + \lambda_{2k} C w_k^*$$

Let us seek W_1^* in the form

$$W_1^* \sim \sum_n e_n w_n^*;$$

We obtain, by seeking

$$\widetilde{L}_0^{-1/2} \hat{f}(w_k) \sim \sum_n g_n w_n^*,$$

where the g_n are known:

$$\begin{cases} e_n(1-\lambda_{1k}\lambda_{1n}^{-1}) - g_n = 0 & \text{for } n \neq k \ ; \\ -g_k - \lambda_{2k}\lambda_{1k}^{-1} = 0 & \text{for } n = k \end{cases}$$

The last equation gives (69); the others give the e_n for $n \neq k$, e_k remaining indeterminate.
We find finally

$$W_1 \sim e_k w_k + \sum_{n \neq k} g_n (1 - \lambda_{1k}\lambda_{1n}^{-1}) w_n$$

Consequently, we must calculate the third approximation in ordrer to determine e_k and then W_1.

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