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The Inverse Burr Negative Binomial Distribution with Application to Real Data

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Abstract: We introduce in this paper a four-parameter lifetime model, called the inverse burr negative binomial distribution. We derive some statistical properties of the proposed model that includes moments, quantile functions, median, reliability and entropy. The method of maximum likelihood is used for estimating the model parameters and the observed information matrix is obtained. Simulation study is performed to investigate the performance of the estimation of the model parameters. Two real data sets are used to demonstrate the flexibility of the new proposed model in comparison with other popular lifetime models.

Keywords: Negative Binomial Distribution, Inverse Burr Distribution, Maximum likelihood, Observed information matrix.

1 Introduction

The negative binomial distribution has been used in compounding distributions to form another flexible model. A lot of disributions have been introduced and applied in survival analysis. [4] pioneered a family of univariate distributions generated by compouding the negative binomial distribution with any continuos model. [3] introduced a lifetime model called the Burr XII negative binomial distribution with application to Lifetime data. The G-negative binomial is defined as follows: For any baseline cumulative distribution function (cdf) $G(x)$, and $x \in \mathbb{R}$, the G-Negative Binomial (G-NB) family of distributions has probability density function (pdf) $f(x)$ and cumulative density function (cdf) $F(x)$ given by

$$f_{\lambda,k}(x) = \frac{\lambda k}{\left[(1-k)^{-\lambda}-1\right]} g(x)\left\{1-k[1-G(x)]\right\}^{-\lambda-1} \quad x>0 \tag{1}$$

and

$$F_{\lambda,k}(x) = \frac{(1-k)^{-\lambda}-\left\{1-k[1-G(x)]\right\}^{-\lambda}}{\left[(1-k)^{-\lambda}-1\right]} \quad x>0 \tag{2}$$

The hazard function is given by

$$h(x) = \frac{\lambda k g(x)\left\{1-k[1-G(x)]\right\}^{-\lambda-1}}{\left\{1-k[1-G(x)]\right\}^{-\lambda}-1} \quad x>0 \tag{3}$$

where $\lambda > 0$ and $k \in (0,1)$

The G-NB family has the same parameters of the G distribution plus two additional shape parameters $\lambda > 0$ and $k \in (0,1)$. If X is a random variable having pdf in (1), we write $X \sim G-NB(\lambda;k)$. This generalization is obtained by increasing the number of parameters compared to the G model, this increase has added more flexibility to the generated distribution. A significant point of the G-NB model is its ability to comparise G distribution as a sub-model when $\lambda = 1$ and $k \to 0$. However, the inverse burr distribution also known as (BurrIII) has been used in various fields of sciences. In the actuarial literature it is known as the inverse Burr distribution see [9] and as the kappa distribution in the meteorological

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literature see [14] and [13]. It has also been employed in finance, environmental studies, survival analysis and reliability theory see [10], [8] and [6]. Furthermore, [5] proposed an extended BIII distribution in low-flow frequency analysis where its lower tail is of main interest. Recently, [2] introduced the Complementary Burr III Poisson Distribution, [1] proposed the geometric inverse burr distribution . In this paper, we use the Cordeiro G.M and M.Percontini [4] generator to define a model, called the Inverse Burr Negative Binomial (IBNB) Distribution. The main reason for proposing IBNB distribution are: (i) This distribution due to its flexibility became an important model that can be used in a different forms of problems in modeling lifetime data. (ii) It provides a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub-shaped and unimodal failure rates, which are common in reliability and biological studies, unlike the exponetial poison (EP) and generalization of exponential poison (GEP) distributions whose ability's are only in modeling data with increasing or decreasing failure rates. (iii) The IBNB distribution is a suitable model for fitting skewed data that cannot be properly fitted by existing distributions.

The rest of the paper is organized as follows. The immediate section after this introduction is the presence of new model IBNB and some investigation on its properties. Section 3 is the statistical properties of the new model IBNB. In section 4 estimation of the parameters using maximum likelihood method are given. Section 5 is the simulation studies while in section 6 two real data are used to show the flexibility of the proposed model. Finally, the concluding remarks is given in section 7.

2 The IBNB

The cumulative distribution function(cdf) and the probability density function(pdf) of the inverse burr distribution are given by

$$G(x) = \left(\frac{x^{\alpha}}{1+x^{\alpha}}\right)^{\beta} \;\; and \;\; g(x) = \alpha\beta x^{\alpha\beta-1}(1+x^{-\alpha})^{-\beta-1} \tag{4}$$

The inverse burr negative binomial is obtained by substituting cdf and pdf of the inverse burr i.e $G(x)$ and $g(x)$ in (4) in the equations (1) and (2). We therefore have the following results

$$f_{\alpha,\beta}(x;p,\alpha,\beta) = \frac{k\lambda\alpha\beta x^{\alpha\beta-1}(1+x^{\alpha})^{-\beta-1}}{\left[(1-k)^{-\lambda}-1\right]}\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-\lambda-1} \quad x>0 \tag{5}$$

the corresponding cumulative distribution function is

$$F_{\alpha,\beta}(x;p,\alpha,\beta) = \frac{(1-k)^{-\lambda}-\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-\lambda}}{\left[(1-k)^{-\lambda}-1\right]} \quad x>0 \tag{6}$$

the hazard rate function is

$$h(x,p,\alpha,\beta) = \frac{k\lambda\alpha\beta x^{\alpha\beta-1}(1+x^{\alpha})^{-\beta-1}\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-s-1}}{-1-\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-\lambda}} \quad x>0 \tag{7}$$

the survival function is

$$s_{\alpha,\beta}(x;p,\alpha,\beta) = \frac{-1-\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-\lambda}}{\left[(1-k)^{-\lambda}-1\right]} \quad x>0 \tag{8}$$

2.1 *PDF and Hazard Rate Function*

A lot of failure rate fuction have complex expressions because of the integral in the denominator and therefore the determination of the shapes is not explicit. [11] introduced a method to determine the shape of $h(x)$ with at most one turning point. His method uses the density function instead of the failure rate. A turning point of a function is a point at which the function has a local maximum or a local minimum.

$$\eta(x) = -\frac{f^{'}(x)}{f(x)} \tag{9}$$

Theorem 1(*[11]) Let $\eta(x)$ be defined as in (9)*

*(a)if $\eta(x) \in$ **I**(strictly increasing), then $h(x)$ is of type **I***
*(b)if $\eta(x) \in$ **D**(decreasing), then $h(x)$ is of type **D***
*(c)if $\eta(x) \in$ **BT**(bathtube shape), if there exists x_0 such that $h^{'}(x) = 0$, then $h(x)$ is of type **BT**. Otherwise is of type **I***

2.1.1 Probability Density Function(PDF)

The probability density function reported in (5) has the following properties

Theorem 2*The probability density function of IBNB distribution is decreasing for $0 < \alpha < 1$ and unimodal also a constant otherwise.*

Proof. Taking the log of (5) thereafter differentiating the result, we obtain

$$(\log(f(x)))^{'} = n(x) - \frac{k\alpha\beta(\lambda+1)x^{-(\alpha+1)}(1+x^{-\alpha})^{-(\beta+1)}}{(1-k[1-(1+x^{-\alpha})^{-\beta}])}$$

where $n(x) = \frac{\alpha(\beta+1)x^{-(\alpha+1)}}{1+x^{-\alpha}} - \frac{(\alpha+1)}{x}$.

For $0 < \alpha < 1$ the function $n(x)$ is negative. Thus, $f^{'}(x) < 0$ for all $x > 0$. This shows that f is decreasing for $0 < \alpha < 1$. Consider $\alpha > 1$ implies n will have one exact root x_0 and $n(x) > 0$ for $x < x_0$ and $n(x) < 0$ for $x > x_0$. Hence, f is a unimodal function with mode at $x = x_0$. However, for $\alpha > 1$ and some values of β, λ and k, the pdf can be a constant. □

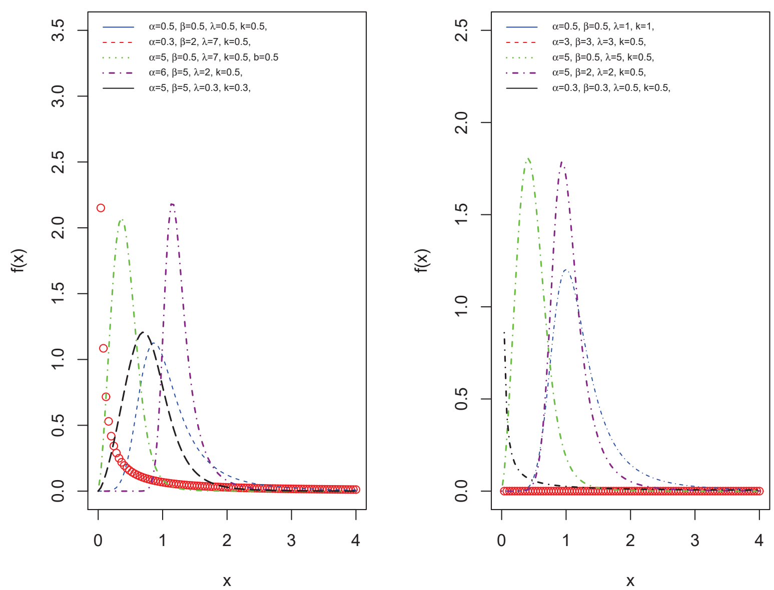

Fig. 1: Probability Density Function of the Proposed Model

2.1.2 Hazard Rate Function

The hazard rate function reported in (7) has the following possible shape for $x > 0, \lambda > 0, k > 0, \alpha > 0, \beta > 0$ in the following theorem.

Theorem 3 *The hazard rate function of the IBNB distribution is decresing for $0 < \alpha \leq 1$, increasing and a bathtube shape otherwise.*

Proof. Consider $\eta(x)$ as it is reported in (9), clearly $\eta(x)$ is positive since $f^{'}(x) < 0$. Taking the first derivative of $\eta(x)$, we have that $\eta^{'}(x) < 0$ i.e it is negative. It follows by [11] that $h(x)$ is decreasing. For $\alpha > 1$, $\eta^{'}$ is positive for all $x > 0$, this shows that by [11] $h(x)$is increasing. For the bathtube shape it is shown graphically. □

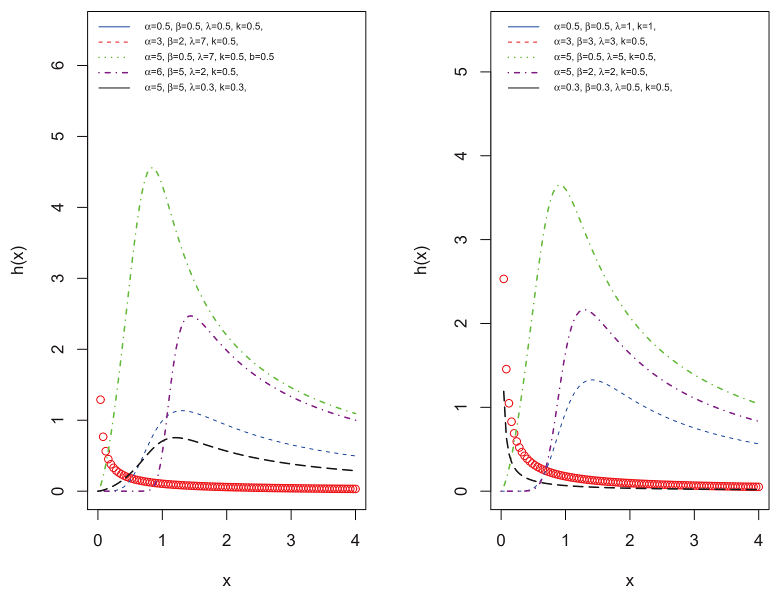

Fig. 2: Hazard Rate Function of the Proposed Model

2.2 Expansions

In this subsection, we present some representations of cdf, pdf of the IBNB distribution. We provide two simple formulae for the IBNB, which may be used for further analytical or numerical analysis. For any real p and $|z| < 1$, we have that

$$(1-z)^{-p} = \sum_{k=0}^{\infty} (p)_k \frac{z^k}{k!} \tag{10}$$

where $(p)_0 = 1$ and $(p)_k = p(p+1)(p+2)...(p+k-1) = \frac{\Gamma(p+k)}{\Gamma(p)}$

and for $|z| < 1$ and $\rho > 0$, the power series expansion is given by

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(p+j)z^j}{\Gamma(p)j!} \tag{11}$$

is the ascending factorial. Using (10) and (11) in (5), we obtain the following

$$f(x;\alpha,\beta,\lambda,k) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda,k)g(x;\alpha,\beta(j+1)) \tag{12}$$

where $w_{j,i}(\lambda,k)=\lambda\frac{(-1)^{i}k^{j+1}(\lambda+1)_{j}\Gamma(j+1)}{[(1-k)^{-\lambda}-1](j+1)j!\Gamma(j+1-i)\Gamma(i+1)}$ and $g(x;\alpha,\beta(j+1))$ is clearly the density function of the inverse burr distribution with parameters α,$\beta(j+1)$ and is given by

$$g(x:\alpha,\beta(j+1))=\alpha\beta(j+1)x^{-\alpha-1}(1+x^{-\alpha})^{-\beta(j+1)-1}$$

After some algebra, we find out that $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{j,i}(\lambda,k)=1$, this shows that IBNB density can be witen as a linear combination of the inverse burr distribution. Taking the integral of (4) we obtain that

$$F(x)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{j,i}(\lambda,k)G(x;\alpha,\beta(j+1)) \tag{13}$$

where $G(x;\alpha,\beta(j+1))$ is the inverse burr cumulative distribution with parameters α and $\beta(j+1)$.

3 Statistical properties

3.1 Moments

Theorem 4 *The rth moment of the inverse burr negative binomial distribution (IBNB) is gven by*

$$E(X^{r})=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{j,i}(\lambda,k)\beta(j+1)B\left(1-(\frac{r+\alpha\beta(j+1)+1}{\alpha}),1-\frac{r}{\alpha}\right)$$

Proof. See Appendix B(1) □

The mean of IBNB is simply obtained by setting $r=1$ in (4) where as the variance take the following form

$$\begin{aligned}Var(X)=&\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{j,i}(\lambda,k)\beta(j+1)B\left(1-(\beta(j+2)+1),1-\frac{2}{\alpha}\right)\\&-\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{j,i}(\lambda,k)\beta(j+1)B\left(1-(\beta(j+1)+1),1-\frac{1}{\alpha}\right)\right\}^{2}\end{aligned} \tag{14}$$

3.2 Quantile and median

By inverting the cdf of the IBNB we obtained the quantile function $(for\ 0<q<1)$ as

$$x_{q}=\left\{\left[\frac{1-\left[(1-k)^{-\lambda}-q\{(1-k)^{-\lambda}-1\}\right]^{-(\frac{1}{\lambda})}-k}{k}\right]^{\frac{1}{\beta}}-1\right\}^{-(\frac{1}{\alpha})} \tag{15}$$

the median is simply obtained by setting $q=0.5$ in (15)

The skewness and kurtosis for IBNB can be obtained from the following equations respectively.

$$\gamma_{3}=\frac{\mu^{(3)}-3\mu\mu^{(2)}+2\mu^{3}}{\left(\mu^{(2)}-\mu^{2}\right)^{\frac{3}{2}}} \tag{16}$$

$$\gamma_{4}=\frac{\mu^{(4)}-4\mu\mu^{(3)}+6\mu^{2}\mu^{(2)}-3\mu^{4}}{\left(\mu^{(2)}-\mu^{2}\right)^{2}} \tag{17}$$

where μ is the mean, $\mu^{(2)}$, $\mu^{(3)}$ and $\mu^{(4)}$ are the second, third, and fourth moment respectively.

3.3 Entropy

The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. A large value of the entropy indicates the greater uncertainty in the data. The Renyi entropy is defined by

$$I_R(r) = \frac{1}{1-r}\log\left[\int_R f^r(x)dx\right] \tag{18}$$

where $r > 0$ and $r \neq 1$

Theorem 5*Let X be distributed according to IBNB, the Renyi entropy of X is given by*

$$I_R(r) = \frac{1}{1-r}\log\left[[\frac{\lambda\beta}{[(1-k)^{-\lambda}-1]}]^r k^{i+r}\alpha^{r-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i k^{j+1}(\lambda+1)_j\Gamma(j+1)}{[(1-k)^{-\lambda}-1]j!\Gamma(i+1-j)\Gamma(i+1)}\right.$$
$$\left.\times B\left(1-(\beta(i+r)+r),1-\frac{r(\alpha\beta+1)+(\alpha+1)}{r}\right)\right]$$

Proof. See Appendix B(2) □

The Shannon entropy is defined by $E[-\log f(x)]$, this is a special case of the Renhi entropy when $r \uparrow 1$.

3.4 Reliability

In the context of reliability, the stress-strength model describes the life of a component which has a random strength $X1$ that is subjected to a random stress $X2$. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X1 > X2$. Hence, $R = P(X2 < X1)$ is a measure of component reliability see [7]. It has many applications especially in the area of engineering. We derive the reliability R when $X1$ and $X2$ have independent $IBNB(\lambda_1;k_1;\alpha;\beta)$ and $IBNB(\lambda_2;k_2;\alpha;\beta)$ distributions. From equations (12) and (13), the reliability reduces to

$$R = P(X1 > X2) = \int_0^\infty f_1(x)F_2(x)dx$$

Substituting for $f_1(x)$ and $F_2(x)$ from the above integral, we obtain that

$$R = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda_1,k_1)w_{j,i}(\lambda_2,k_2)\int_0^\infty g(x;\alpha,\beta(j+1))G(x;\alpha,\beta(j+1))$$

where $g(x;\alpha,\beta(j+1))$, $G(x;\alpha,\beta(j+1))$ is the pdf and cdf of the inverse burr with parameters α and $\beta(j+1)$ respectively. Therefore, by making small algebra we have that

$$R = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda_1,k_1)w_{j,i}(\lambda_2,k_2)\int_0^\infty x^{-(\alpha+1)}(1+x^{-\alpha})^{-2\beta(j+1)-1}$$

now let $u = (1+x^{-\alpha})$, consequently, we obtain the reliability as follows

$$R = \sum_{j,i=0}^{\infty} w_{j,i}(\lambda_1,k_1)w_{j,i}(\lambda_2,k_2)B(2\beta(j+1),\frac{2(\alpha+1)}{\alpha}+1)$$

4 Statistical Inference

In this section, we consider the method of maximum likelihood estimators(MLEs) for the estimation. This is because the MLEs possess under fairly regular conditions with some optimal properties.

4.1 Estimation

Let $X_1,...,X_n$ be a random sample with observed values $x_1,...,x_n$ from the class with parameters α, β, s, k. Let $\Theta = (\alpha,\beta,\lambda,k)^T$ be the parameter vector.The log likelihood function is given by

$$l(\theta) = n\log(\alpha\beta\lambda k) - n\log\left[(1-k)^{-\lambda}-1\right] + (\alpha\beta-1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log(t_i) - (\lambda+1)\sum_{i=1}^{n}\log\left\{1-k\left[1-\frac{x_i^{\alpha}}{t_i}\right]\right\} \tag{19}$$

where $t_i = (1+x_i^{\alpha})$. The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equation obtained by differentiating $l(x;\alpha,\beta,\lambda,k)$ above. The components of the score vector $U = (\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta},\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial k})^T$ are given by

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - \frac{n(1-k)^{-\lambda}log(1-k)}{\left[(1-k)^{-\lambda}-1\right]} - \sum_{i=1}^{n}\log\left\{1-k\left[1-\frac{x_i^{\alpha}}{t_i}\right]\right\} \tag{20}$$

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \beta\sum_{i=1}^{n}\log x_i - \alpha(\beta+1)\sum_{i=1}^{n}\frac{x_i^{(\alpha-1)}}{t_i} - k\beta(\lambda+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i(x_i^{\alpha}t_i^{-1})^{\beta-1}}{\left\{1-k\left[1-\frac{x_i^{\alpha}}{t_i}\right]\right\}} \tag{21}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \alpha\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log t_i - k(\lambda+1)\sum_{i=1}^{n}\frac{(x_i^{\alpha}t_i^{-1})^{\beta}\log(x_i^{\alpha}t_i^{-1})}{\left\{1-k\left[1-\frac{x_i^{\alpha}}{t_i}\right]\right\}} \tag{22}$$

$$\frac{\partial l}{\partial k} = \frac{n}{k} - \frac{n(1-k)^{-\lambda-1}}{\left[(1-k)^{-\lambda}-1\right]} - \sum_{i=1}^{n}\frac{(x_i^{\alpha}t_i^{-1})^{\beta}}{\left\{1-k\left[1-\frac{x_i^{\alpha}}{t_i}\right]\right\}} \tag{23}$$

where $t_i = (1+x_i^{\alpha})$. For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix. The 4×4 unit observed information matrix $J = J(\theta)$ is obtained as

$$\mathbf{J} = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\lambda} & J_{\alpha k} \\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\lambda} & J_{\beta k} \\ J_{\lambda\alpha} & J_{\lambda\beta} & J_{\lambda\lambda} & J_{\lambda k} \\ J_{k\alpha} & J_{k\beta} & J_{k\lambda} & J_{kk} \end{pmatrix}$$

where the expressions for the elements of $\mathbf{J}$ are given in Appendix A.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, asymptotically

$$\sqrt{n}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \sim N_4(0,I^{-1}(\boldsymbol{\theta}))$$

Observe that $\hat{\boldsymbol{\theta}}$ is consistent estimator of $\boldsymbol{\theta}$ and thus, the validity of the asymptotic normality standstill if the fisher information matrix I is replaced by the observed fisher information $\mathbf{J}(\hat{\boldsymbol{\theta}})$: In this case, a $\gamma100\%$ approximate asymtotic interval for each component parameter $\hat{\boldsymbol{\theta}}_l$ of $\hat{\boldsymbol{\theta}}$ is given by

$$\left(\hat{\boldsymbol{\theta}}_l - Z_{\frac{1+\gamma}{2}}\sqrt{\mathbf{J}^{\hat{\theta}_l\hat{\theta}_l}}, \hat{\boldsymbol{\theta}}_l + Z_{\frac{1+\gamma}{2}}\sqrt{\mathbf{J}^{\hat{\theta}_l\hat{\theta}_l}}\right)$$

where $\mathbf{J}^{\hat{\theta}_l\hat{\theta}_l}$ is the diagonal element of $\mathbf{J}(\hat{\boldsymbol{\theta}})^{-1}$ corresponding to each parameter $l = (\alpha,\beta,\lambda,k)$ and $Z_{\frac{1+\gamma}{2}}$ is the quantile $\frac{1+\gamma}{2}$ of the standard normal distribution. The likelihood ration (LR) statistics is used for testing IBNB distribution against some of the existing models. Considering the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T,\boldsymbol{\theta}_2^T)^T$, tests of hypothesis of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ vs $H_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be done by LR statistics which is given by $w = 2\left\{l(\hat{\boldsymbol{\theta}}) - (l\tilde{\boldsymbol{\theta}})\right\}$, where $\hat{\boldsymbol{\theta}}$, $\tilde{\boldsymbol{\theta}}$ are the MLEs of $\boldsymbol{\theta}$ under H_1 and H_0 respectively. Under the null hypothesis, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_q$, where ξ_q is the upper $100_\gamma\%$ point of the χ_q^2 distribution.

5 Simulation

This section provides the outcomes of simulation study. Simulation were performed in order to investigate the proposed estimator of α, β, λ, k of the proposed MLE method. We generate 15000 samples of size n=30, 50, 100, 200, 500, and 800 from IBNB distribution. We assess the correctness of the approximation of the standard error of the MLE determined through the fisher information matrix. The approximate values of $sd(\hat{\alpha})$, $sd(\hat{\beta})$, $sd(\hat{\lambda})$, and $sd(\hat{k})$ are calculated.

Table 1: The average of 15000 MLEs and standard error simulated from IBNB

		AE				SD			
n	(α, β, λ, k)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{k}$	$sd(\hat{\alpha})$	$sd(\hat{\beta})$	$sd(\hat{\lambda})$	$sd(\hat{k})$
30	(0.5, 0.5, 2.0,2.0)	0.628	0.861	2.985	3.103	0.519	0.557	3.518	4.424
	(1.0, 2.0, 1.0,2.0)	2.158	2.058	2.977	3.001	4.632	4.551	3.775	6.223
	(3.0, 0.9, 7.0,5.0)	3.559	3.033	10.267	7.952	6.694	1.677	6.575	7.331
	(7.0, 7.0, 2.0,4.0)	9.566	8.464	3.429	4.311	9.625	8.045	3.205	6.223
50	(0.5, 0.5, 2.0,2.0)	0.611	0.662	2.575	3.120	0.349	0.547	3.118	3.422
	(1.0, 2.0, 1.0,2.0)	1.888	2.355	1.977	2.988	4.437	3.221	2.895	4.222
	(3.0, 0.9, 7.0,5.0)	4.119	4.333	8.777	7.925	5.994	1.566	6.435	7.221
	(7.0, 7.0, 2.0,4.0)	8.552	8.334	2.998	4.291	8.777	8.005	3.122	6.111
100	(0.5, 0.5, 2.0,2.0)	0.588	0.567	2.485	2.994	0.219	0.451	2.544	3.112
	(1.0, 2.0, 1.0,2.0)	1.158	1.958	1.978	2.031	4.326	3.112	2.112	3.343
	(3.0, 0.9, 7.0,5.0)	3.199	3.999	8.208	7.752	4.193	0.977	5.776	6.141
	(7.0, 7.0, 2.0,4.0)	7.544	7.234	2.400	4.111	7.225	7.123	2.146	5.333
200	(0.5, 0.5, 2.0,2.0)	0.528	0.555	2.179	2.419	0.222	0.337	2.423	2.555
	(1.0, 2.0, 1.0,2.0)	1.058	1.558	1.777	2.011	3.992	2.666	1.996	3.112
	(3.0, 0.9, 7.0,5.0)	3.559	3.214	7.287	7.152	4.024	0.167	4.998	5.068
	(7.0, 7.0, 2.0,4.0)	7.152	7.001	2.389	4.119	7.222	6.033	1.999	4.997
500	(0.5, 0.5, 2.0,2.0)	0.522	0.561	2.089	2.103	0.135	0.133	1.555	1.414
	(1.0, 2.0, 1.0,2.0)	1.188	1.358	1.908	2.000	3.113	2.441	1.679	2.113
	(3.0, 0.9, 7.0,5.0)	3.009	3.033	7.208	7.077	3.444	0.077	4.223	4.661
	(7.0, 7.0, 2.0,4.0)	7.552	7.111	2.029	4.333	6.664	5.145	1.200	3.573
800	(0.5, 0.5, 2.0,2.0)	0.518	0.533	2.006	1.999	0.119	0.125	1.239	0.924
	(1.0, 2.0, 1.0,2.0)	1.100	1.051	1.231	1.981	2.632	1.991	0.987	1.983
	(3.0, 0.9, 7.0,5.0)	3.011	3.123	7.009	6.992	2.977	0.045	3.567	3.771
	(7.0, 7.0, 2.0,4.0)	7.211	7.000	2.111	4.033	5.555	4.124	1.102	3.341

6 Application

In this section, an applications of the IBNB distribution with the estimation of the parameters using the method of maximum likelihood and likelihood ratio (LR) test for comparison of the IBNB distribution with some popular models for given sets of data are presented. The examples has shown the flexibility of the IBNB distribution in comparison with other models including the Exponential Poisson (EP), Generalization of Exponential Poisson (GEP) distributions for data modeling. The MLEs of the IBNB parameters k, λ, α and β are computed by maximizing the objective function using R software. The estimated values of the parameters,log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2log(L)$, Bayesian Information Criterion, $BIC = plog(n) - 2log(L)$, and Consistent Akaike Information Criterion, $CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\boldsymbol{\theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters for the two sets of data are shown in (6.1) and (6.2) respectively. The IBNB distribution is fitted to the data sets and these fits are compared to the fits using the EP, GEP, distributions.

6.1 Data set 1

The source of this data set is the Open University (1993). The following data are the prices of the 31 different childrens wooden toys on sale in a Suffolk craft shop in April 1991: 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 2.85, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

Table 2: MLEs of the prices of the 31 different childrens wooden toys on sale in a Suffolk craft shop in April 1991

	α	β	λ	k	$l(\boldsymbol{\theta})$	AIC	CAIC	BIC
EP(β,λ)	–	0.0077	30.8795	–	-75.9447	155.8894	156.32	158.76
GEP(α,β,λ)	2.3144	0.2369	1.9821	–	-73.6629	153.3258	154.21	157.63
IBNB(α,β,λ,k)	3.10801427	3.13855928	0.74857029	-0.07335403	-68.129	145.3258	146.86	151.06

A comparison of the proposed model with two of the existing model is performed in Table(3). Therefore, considering the significant level of 5%, we reject the null hypothesis.

Table 3: LR statistics for the data

Model	Hypothesis	Statistic w	p-value
IBNB vs *GEP*	$H_0 : \alpha = 1$ vs $H_1 : \alpha \neq 1$	11.0678	8.784×10^{-4}
IBNB vs *EP*	$H_0 : \beta = \lambda = 1$ vs $H_1 : \beta \neq \lambda \neq 1$	15.6314	4.034×10^{-4}

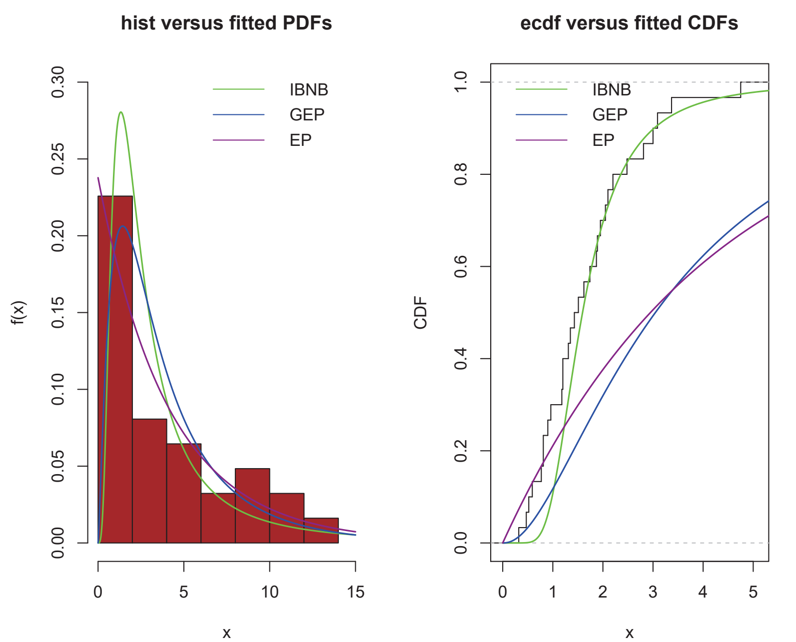

Fig. 3: fitted densities and cdf for the first data

6.2 Data set 2

This data set is given by [12] and consists of thirty sucessive values of March precipitation (in inches) in Minneapolis/St Paul. The data are 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.

Table 4: MLEs of the prices of the 31 different childrens wooden toys on sale in a Suffolk craft shop in April 1991

	α	β	λ	k	$l(\boldsymbol{\theta})$	AIC	CAIC	BIC
EP(β,λ)	–	0.0186	31.9785	–	−45.7935	95.587	96.03	98.39
GEP(α,β,λ)	2.7329	0.7336	0.8003	–	−39.7229	85.4458	86.37	89.65
IBNB(α,β,λ,k)	3.3591898	3.2513528	1.0284761	−0.2839195	−36.8475	81.695	83.295	87.299

A comparison of the proposed model with two of the existing model is performed in Table (5). Therefore, considering the significant level of 5%, we reject the null hypothesis.

Table 5: LR statistics for the data

Model	Hypothesis	Statistic w	p-value
IBNB vs *GEP*	$H_0 : \alpha = 1$ vs $H_1 : \alpha \neq 1$	5.7508	0.0164811
IBNB vs *EP*	$H_0 : \beta = \lambda = 1$ vs $H_1 : \beta \neq \lambda \neq 1$	17.6534	1.468×10^{-4}

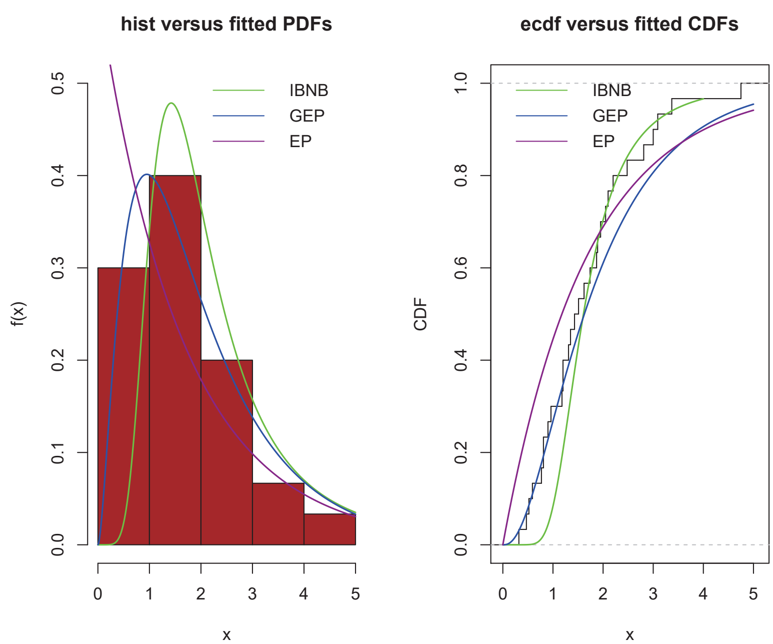

Fig. 4: fitted pdf and cdf for the second data

7 Conclusion

We introduce the inverse burr negative binomial distribution IBNB. The new distribution brings a very vital result as it fits some real data better than some existing model. The pdf of the new model is decreasing, a constant and unimodal depending on the values of the parameter. The hazard rate function is also decreasing, increasing and a bathtube shape. Statistical properties are investigated and the parameters of the IBNB are estimated using the method of maximum likelihood and the information matrix is obtained . We test the hypothesis using LR test and the simulation study have shown that, the model parameters performed very well. The usefulness of the IBNB distribution is enunciated in two application to a real data sets. The new proposed model gives a more flexible result for fitting lifetime data in reliability, biology and other areas.

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Appendix A

The elements of the 4×4 information matrix are given by

$$J_{\lambda\lambda} = \frac{n}{\lambda^2} - n\frac{\left\{[(1-k)^{\lambda}-1]^2 - \lambda^2(1-k)^{\lambda}\log^2(1-k)\right\}}{[(1-k)^{\lambda}-1]^2}$$

$$J_{\lambda k} = n\frac{\left\{1-(1-k)^{\lambda}+\lambda(1-k)^{\lambda}\log(1-k)\right\}}{[(1-k)^{\lambda}-1]^2} + \sum_{i=1}^{n}\frac{t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{\lambda\alpha} = -\sum_{i=1}^{n}\frac{\alpha k x_i^{\alpha}\log x_i t_i^{-\alpha-1}}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{kk} = \frac{n}{k^2} - \frac{\left\{[(1-k)^{\lambda}-1]n(\lambda+1)(1-k)^{-\lambda-2}+n\lambda(1-k)^{-2\lambda-2}\right\}}{[(1-k)^{\lambda}-1]^2} - \sum_{i=1}^{n}\frac{\left(x_i^{\alpha}t_i^{-1}[1-\frac{x_i^{\alpha}}{t_i}]\right)}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2}$$

$$J_{k\lambda} = \frac{n(1-k)^{-\lambda-1}\log(1-k)}{[(1-k)^{-\lambda}-1]}$$

$$J_{k\beta} = -(\lambda+1)\sum_{i=1}^{n}\left\{\frac{\alpha x_i^{\alpha}\log x_i t_i^{-(\alpha+1)}}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)} + \frac{k\alpha x_i^{\alpha}\log x_i t_i^{-(2\alpha+1)}}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2}\right\}$$

$$J_{k\alpha} = -(\lambda+1)\sum_{i=1}^{n}\left\{\frac{k t_i^{-2\alpha}\log t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2} + \frac{t_i^{-\alpha}\log t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}\right\}$$

$$J_{\alpha\beta} = \sum_{i=1}^{n}\log x_i - \alpha\sum_{i=1}^{n}\frac{x_i^{\alpha-1}}{t_i} - k(\lambda+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i(x_i^{\alpha}t_i^{-1})^{\beta-1}\log(x_i^{\alpha}t_i^{-1})}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{\alpha\lambda} = k\beta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i(x_i^{\alpha}t_i^{-1})^{\beta-1}}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{\alpha k} = \beta(\lambda+1)\sum_{i=1}^{n}\frac{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2 x_i^{\alpha}\log x_i(x_i^{\alpha}t_i^{-1})^{\beta-1} - x_i^{\alpha}\log x_i(x_i^{\alpha}t_i^{-1})^{\beta-1}(1-x_i^{\alpha}t_i^{-1})}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2}$$

$$J_{\beta\beta} = \frac{n}{\beta^2} - k(\lambda+1)\sum_{i=1}^{n}\frac{(x_i^{\alpha}t_i^{-1})^{\beta}\log^2(x_i^{\alpha}t_i^{-1})}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{\beta\beta} = k\sum_{i=1}^{n} \frac{(x_i^{\alpha} t_i^{-1})^{\beta} \log(x_i^{\alpha} t_i^{-1})}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$J_{\alpha k} = -(\lambda+1)\sum_{i=1}^{n} \frac{k t_i^{-2\alpha}\log t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2} + \frac{t_i^{-\alpha}\log t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)}$$

$$\begin{aligned} J_{\alpha\alpha} = {} & \frac{n}{\alpha^2} - (\beta+1)\sum_{i=1}^{n} \frac{t_i x_i^{\alpha-1}\log x_i + \alpha x_i^{2\alpha-1}\log x_i}{t_i^2} \\ & - k\beta(\lambda+1)\sum_{i=1}^{n} \frac{x_i^{\alpha}\log^2 x_i (x_i^{\alpha} t_i^{-1})^{\beta-1} + x_i^{\alpha}\log x_i (x_i^{\alpha} t_i^{-1})\log(x_i^{\alpha} t_i^{-1})}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)} \\ & - k^2\beta(\lambda+1)\sum_{i=1}^{n} \frac{x_i^{2\alpha}\log^2 x_i (x_i^{\alpha} t_i^{-1})^{\beta-1} t_i^{-1} - x_i^{2\alpha}\log x_i (x_i^{\alpha} t_i^{-1})^{\beta-1} t_i^{-1}\log t_i}{\left(1-k[1-\frac{x_i^{\alpha}}{t_i}]\right)^2} \end{aligned} \tag{24}$$

Appendix B

1.**Proof.** of (4). The rth moment is given by

$$E(X^r) = \int_0^{\infty} x^r f(x)dx = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda,k)\int_0^{\infty} x^r g(x;\alpha,\beta(j+1))$$

where $g(x;\alpha,\beta(j+1))$ is the pdf of the inverse burr with parameters α and $\beta(j+1)$. We therefore obtain the following

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda,k)\alpha\beta(j+1)\int_0^{\infty} x^{r+\alpha\beta(j+1)-1}(1+x^{-\alpha})^{-\beta(j+1)-1}$$

let $u = 1+x^{-\alpha}$ implies $-(1-u) = x^{-\alpha}$ which shows that $x = -(1-u)^{-\frac{1}{\alpha}}$ and also $dx = -\frac{du}{\alpha x^{-\alpha-1}}$ and after some algebra, we obtain

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{j,i}(\lambda,k)\beta(j+1)B\left(1-(\frac{r+\alpha\beta(j+1)+1}{\alpha}),1-\frac{r}{\alpha}\right)$$

□

2.**Proof.** of (5). The Renhi entropy is given by

$$I_R(r) = \frac{1}{1-r}\log\left[\int_R f^r(x)dx\right]$$

taking the integral we have that,

$$\int_0^{\infty} f^r(x)dx = \frac{k\lambda\alpha\beta}{[(1-k)^{-\lambda}-1]}\int_0^{\infty} x^{r(\alpha\beta-1)}(1+x^{\alpha})^{-r(\beta+1)}\left\{1-k[1-(\frac{x^{\alpha}}{1+x^{\alpha}})^{\beta}]\right\}^{-r(\lambda+1)} dx.$$

using (10) and (11), we obatain

$$\begin{aligned} I_R(r) = \frac{1}{1-r}\log\Bigg[& [\frac{\lambda\beta}{[(1-k)^{-\lambda}-1]}]^r k^{i+r}\alpha^{r-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i k^{j+1}(\lambda+1)_j\Gamma(j+1)}{[(1-k)^{-\lambda}-1]j!\Gamma(i+1-j)\Gamma(i+1)} \\ & \times B\left(1-(\beta(i+r)+r),1-\frac{r(\alpha\beta+1)+(\alpha+1)}{r}\right)\Bigg] \end{aligned}$$

□

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