

A Global Error bound for the Feasible Solution of Variational Inequality Problems

Wenling Zhao and Jin Huang

School of Science, Shandong University of Technology, zibo, 255049, China

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Abstract: As for variational inequality problems, we define the optimal value function of trust region subproblems, study the properties of it. And then under strongly monotonic conditions, we use the optimal value function to provide a global error estimate for the feasible solution .

Keywords: variational inequality, optimal value function, strongly monotone, Error bound.

1. Introduction and symbols

The optimization problems are widely used in engineering design, optimal control, information technology, economic equilibrium and other areas [1–3]. In [4] and [5], we know that there is a closed relationship between an optimization problem and a variational inequality problem (*VIP* for short). In special circumstances, the variational inequality problem contains some optimization problems, such as complementarity problems, fixed point problems, and so on. When a function with a symmetric Jacobian, the variational inequality problem can be reformulated as an optimization problem. However, when the symmetry condition and the positive semi-definiteness condition don't hold, it is more difficulty to handle by optimization theory, then the variational inequality is the more general problem in that case [5–8]. The trust region method is a important method for solving an optimization problem [9, 10], which also has wide applications [11, 12].

A variational inequality problem is to find $x^* \in S$ such that:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner products in \mathbb{R}^n , $F(x)$ a mapping from S to \mathbb{R}^n is a continuous, the set

$$S = \{x \in \mathbb{R}^n | c_i(x) \leq 0, i = 1, \dots, m\},$$

$c_i : \mathbb{R}^n \rightarrow \mathbb{R}^n (i = 1, \dots, m)$ are continuously differentiable functions, so S is a closed convex set.

Now we consider the trust region method of the *VIP*, which is a iterative method, and at each iterative point $x \in S$, an often used subproblem is $(QP(x, \Delta))$

$$\max_{y \in S^\Delta(x)} \{ \langle F(x), x - y \rangle - \frac{1}{2} \langle B(x)(y - x), y - x \rangle \},$$

where

$$\begin{aligned} S^\Delta(x) &= S(x) \cap V(x), \\ S(x) &= \{y \in \mathbb{R}^n | c_i(x) \\ &+ \langle \nabla c_i(x), y - x \rangle \leq 0, i = 1, \dots, m\}, \\ V(x) &= \{y \in \mathbb{R}^n | \|y - x\| \leq \Delta\} (\Delta > 0), \end{aligned}$$

∇c_i is gradient of c_i , $B(x)$ is a symmetric positive definite matrix, $V(x)$ is called the trust region, and the positive number Δ is called the trust region radius. The function of form

$$\Phi(x, \Delta) = \max_{y \in S^\Delta(x)} \{ \langle F(x), x - y \rangle - \frac{1}{2} \langle B(x)(y - x), y - x \rangle \},$$

is called the optimal value function of the trust region subproblems.

In this paper, for the matrix $B(x)$, we assume that there are positive numbers λ_{\min} and λ_{\max} such that

$$\lambda_{\min} \leq \|B(x)\| \leq \lambda_{\max}, \quad \forall x \in S.$$

* Corresponding author: e-mail: jin1437@126.com

The point y of the polyhedral set $S(x) = \{y \in \mathbb{R}^n | c_i(x) + \langle \nabla c_i(x), y - x \rangle \leq 0, i = 1, \dots, m\}$ is the intersection of the half-hyperspace $c_i(x) + \langle \nabla c_i(x), y - x \rangle \leq 0$, so the set $S(x)$ is closed. The trust region $V(x)$ is a generalized ball with x is a center and Δ is radius. Thence $S^\Delta(x)$ is a convex set by the operation properties of the convex sets. Thus the constraints set of subproblem $(QP(x, \Delta))$ is a convex set. We can obtain $S \subseteq S(x)$ by convexity of $c_i(\cdot)$.

Let $x \in S$, if there exist $\lambda_i(x) \geq 0 (i = 1, \dots, m)$, such that

$$F(x) + \sum_{i=1}^m \lambda_i(x) \nabla c_i(x) = 0, \tag{1}$$

$$\lambda_i(x) c_i(x) = 0, i = 1, \dots, m, \tag{2}$$

then x is a *KKT* points to problem (VIP) . In this paper, we denote by S^* and \hat{S} the solutions and *KKT* points for the *VIP*, respectively. The optimal solution of the subproblem $(QP(x, \Delta))$ is expressed as $y^*(x, \Delta)$, or simply y^* .

Let $A \subset \mathbb{R}^n$ be a nonempty subset, the projection of a point $x \in \mathbb{R}^n$ on A is defined as

$$P(x|A) = \operatorname{argmin}\{\|y - x\| | y \in A\},$$

and the distance from $x \in \mathbb{R}^n$ to A is given by

$$\operatorname{dist}(x, A) = \|x - P(x|A)\|.$$

A mapping $F(x)$ from \mathbb{R}^n into \mathbb{R}^n is Lipschitz continuous at $x^* \in S$, if there exists a positive constant L , for $\forall x \in S$, we have

$$\|F(x) - F(x^*)\| \leq L\|x - x^*\|,$$

or $F(x)$ is *L*-continuous at x^* for short.

A mapping $F(x)$ from \mathbb{R}^n into \mathbb{R}^n is monotone on S , if

$$\langle F(y) - F(x), y - x \rangle \geq 0, \forall x, y \in S.$$

A mapping $F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is strongly monotone at a point $x^* \in S$ if there exists a constant $\alpha > 0$ such that for any $x \in S$,

$$\langle F(x) - F(x^*), x - x^* \rangle \geq \alpha\|x - x^*\|^2.$$

Clearly, if $F(x)$ is strongly monotone at a point x^* , then x^* must be the unique solution of the *VIP*.

2. Properties of the optimal value function

It can be seen $\Phi(x, \Delta) \geq 0$, from the definition of the subproblem $(QP(x, \Delta))$ that for $\forall x \in S$ and $\Delta > 0$. As $S^\Delta(x)$ is the intersection of the polyhedral set $S(x)$ and the trust region ball $V(x)$, the Abadie constraint qualification holds at every point of $S^\Delta(x)$. Then, we have

Lemma 1 For $\forall x \in S$ and $\Delta > 0$, a point $y \in S^\Delta(x)$ is the unique solution y^* of subproblem $(QP(x, \Delta))$ if and

only if there exist Lagrange multipliers $\lambda_0^* \geq 0$ and $\lambda_i^* \geq 0 (i = 1, \dots, m)$ such that

$$F(x) + B(x)(y - x) + \lambda_0^*(y - x) + \sum_{i=1}^m \lambda_i^* \nabla c_i(x) = 0, \tag{3}$$

$$\begin{cases} \lambda_0^* [\|y - x\| - \Delta] = 0, \\ \lambda_i^* [c_i(x) + \langle \nabla c_i(x), y - x \rangle] = 0, \end{cases} \tag{4}$$

where in order to simplicity we use λ_0^* and λ_i^* to denote Lagrange multipliers $\lambda_0^*(x, \Delta)$ and $\lambda_i^*(x, \Delta)$ associated with the unique solution to problem $(QP(x, \Delta))$, respectively.

Theorem 1 For $\forall x \in S$ and $\Delta > 0$,

$$\Phi(x, \Delta) \geq \frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle + \lambda_0^* \|y^* - x\|^2,$$

where λ_0^* is the Lagrange multiplier associated with y^* and satisfies Equation (3) and (4).

Proof. For the optimal solution y^* , by the definition of $\Phi(x, \Delta)$,

$$\Phi(x, \Delta) = \langle F(x), x - y^* \rangle - \frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle.$$

Then multiplying the two sides of (3) by $(x - y^*)^T$, and using the second equality of (4), we have

$$\begin{aligned} & \Phi(x, \Delta) - \frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle - \lambda_0^* \|y^* - x\|^2 \\ &= \langle F(x), x - y^* \rangle - \langle B(x)(y^* - x), y^* - x \rangle \\ & \quad - \lambda_0^* \|y^* - x\|^2 \\ &= \sum_{i=1}^m \lambda_i^* \langle \nabla c_i(x), y^* - x \rangle \\ &= - \sum_{i=1}^m \lambda_i^* c_i(x) \geq 0. \end{aligned}$$

Theorem 2 For $\forall x \in S$ and $\Delta > 0$, we have

- 1) $\Phi(x, \Delta) \geq [\frac{1}{2} \lambda_{\min} + \lambda_0^*] \cdot \|y^* - x\|^2$;
- 2) $\frac{1}{2} \operatorname{sign}(\lambda_0^*) \Phi(x, \Delta) \geq \lambda_0^* \cdot \Delta$,

where $\operatorname{sign}(\lambda)$ is a sign function, i.e.

$$\operatorname{sign}(\lambda) = \begin{cases} 1, & \lambda > 0; \\ 0, & \lambda = 0; \\ -1, & \lambda < 0. \end{cases}$$

Proof. The conclusion 1) is obtained by the assumption for $B(x)$ and Theorem 1.

When $\lambda_0^* = 0$, the conclusion 2) holds apparently, and when $\lambda_0^* > 0$, from the first equality of (4), we know that $\|y^* - x\| = \Delta$, and hence from Theorem 1 we obtain the conclusion 2) immediately.

By Lemma 1 and the definition of *KKT* point, we obtain the next theorem.

Theorem 3 The following three conclusions are equivalent.

- 1) x is a *KKT* point of the *VIP*.
- 2) $x \in S$, and $\Phi(x, \Delta) = 0$ for $\forall \Delta > 0$.

3) $x \in S$, and $x = y^*$ for $\forall \Delta > 0$.

Proof. If x is a *KKT* point of the *VIP*, $x \in \hat{S}$ satisfy the definition of the *KKT* point, that is exist $\lambda_i(x) \geq 0 (i = 1, \dots, m)$ such that

$$F(x) + \sum_{i=1}^m \lambda_i(x) \nabla c_i(x) = 0,$$

$$\lambda_i(x) c_i(x) = 0, i = 1, \dots, m.$$

Then multiplying the two sides of (1) by $(x - y^*)^T$, we have

$$\langle F(x), x - y^* \rangle + \sum_{i=1}^m \lambda_i(x) \langle \nabla c_i(x), x - y^* \rangle = 0.$$

And using (2),

$$\begin{aligned} & \langle F(x), x - y^* \rangle \\ &= \sum_{i=1}^m \lambda_i(x) \langle \nabla c_i(x), y^* - x \rangle \\ &= \sum_{i=1}^m \lambda_i(x) \langle \nabla c_i(x), y^* - x \rangle + \sum_{i=1}^m \lambda_i(x) c_i(x) \\ &= \sum_{i=1}^m \lambda_i(x) [c_i(x) + \langle \nabla c_i(x), y^* - x \rangle] \\ &\leq 0. \end{aligned}$$

The inequality is obtained by $y^* \in S^\Delta(x) \subseteq S(x)$.

For the positive definite of $B(x)$, we have

$$\Phi(x, \Delta) = \langle F(x), x - y^* \rangle - \frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle \leq 0.$$

It can be seen from subproblem $(QP(x, \Delta))$ that for $\forall x \in S$ and $\Delta > 0$, we have $\Phi(x, \Delta) \geq 0$.

Thus, $\Phi(x, \Delta) = 0$, the conclusion 1) and the conclusion 2) are equivalent.

By Theorem 1, for $\forall x \in S$ and $\Delta > 0$, we have

$$\Phi(x, \Delta) \geq \frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle + \lambda_0^* \|y^* - x\|^2.$$

If the conclusion 2) holds, it implies

$$\frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle + \lambda_0^* \|y^* - x\|^2 \leq 0.$$

Because $B(x)$ is a symmetric and positive semidefinite matrix and $\lambda_0^* \geq 0$, then

$$\frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle + \lambda_0^* \|y^* - x\|^2 \geq 0.$$

Hence,

$$\frac{1}{2} \langle B(x)(y^* - x), y^* - x \rangle + \lambda_0^* \|y^* - x\|^2 = 0,$$

that is $x = y^*$.

Therefore, the conclusion 2) signify the conclusion 3).

We finally show that the conclusion 3) implies the conclusion 1). Let the conclusion 3) holds, then by the definition of $S \subseteq S(x)$ and $V(x)$, for $x \in S$, we have $x \in S(x) \cap V(x) = S^\Delta(x)$.

Using Lemma 1, we obtain

$$F(x) + B(x)(x - x) + \lambda_0^*(x - x) + \sum_{i=1}^m \lambda_i^* \nabla c_i(x) = 0,$$

$$\lambda_i^* [c_i(x) + \langle \nabla c_i(x), x - x \rangle] = 0, i = 1, \dots, m,$$

where λ_i^* is Lagrange multipliers $\lambda_i^*(x, \Delta)$ associated with the optimal solution of subproblem $(QP(x, \Delta))$.

Let $\lambda_i^*(x) = \lambda_i^*(x, \Delta)$, for $\forall \Delta > 0$, then

$$F(x) + \sum_{i=1}^m \lambda_i^*(x) \nabla c_i(x) = 0,$$

$$\lambda_i^*(x) c_i(x) = 0.$$

Thus, x is a *KKT* point of the *VIP*.

From conclusions 1) and 2) of Theorem 3 and the non-negativity of $\Phi(x, \Delta) \geq 0$, it is immediate to have

Corollary 1 Suppose the *VIP* has *KKT* points. Then x^* is a *KKT* point of the *VIP* if and only if for any $\Delta > 0$, x^* is an optimal solution of problem

$$\min\{\Phi(x, \Delta) | x \in S\}.$$

In this case, $\Phi(x^*, \Delta) = 0, x^* = y^*(x^*, \Delta)$.

3. A global error bound

In this section, under the condition that $F(x)$ is strongly monotone, we present a global error bound for $dist(x, S^*)$ by using the property of optimal value function $\Phi(x, \Delta)$. Under this condition, S^* is a singleton set.

Firstly we introduce a lemma.

Lemma 2 For $\forall x \in S, \forall \Delta > 0, \forall z \in S(x)$, we have

$$\langle F(x) + B(x)(y^* - x) + \lambda_0^*(y^* - x), z - y^* \rangle \geq 0. \quad (5)$$

where y^* and λ_0^* are the optimal solution of subproblem $(QP(x, \Delta))$ and an associate Lagrange multiplier, respectively.

Proof. Let $z \in S(x)$, we have

$$c_i(x) + \langle \nabla c_i(x), z - x \rangle \leq 0, i = 1, \dots, m. \quad (6)$$

Multiplying the two sides of (3) by $(z - y^*)^T$, and using (4) and (6), we obtain

$$\begin{aligned} & \langle F(x) + B(x)(y^* - x) + \lambda_0^*(y^* - x), z - y^* \rangle \\ &= \sum_{i=1}^m \lambda_i^* \langle \nabla c_i(x), y^* - z \rangle \\ &= \sum_{i=1}^m \lambda_i^* [\langle \nabla c_i(x), y^* - x \rangle + \langle \nabla c_i(x), x - z \rangle] \\ &\geq \sum_{i=1}^m \lambda_i^* [\langle \nabla c_i(x), y^* - x \rangle + c_i(x)] \\ &= 0. \end{aligned}$$

Lemma 3 If x^* is KKT point of the VIP , then there exist $\lambda_i(x^*) (1 \leq i \leq m)$ that for $\forall x \in S, \Delta > 0$, we have

$$\langle F(x^*), y^* - x^* \rangle + \sum_{i=1}^m \lambda_i(x^*) \langle \nabla c_i(x^*) - \nabla c_i(x), y^* - x \rangle \geq 0.$$

where $\lambda_i(x^*)$ are Lagrange multipliers to meet (1) and (2).

Proof. Multiplying both sides of (1) by $(y^* - x^*)^T$, noticing that $y^* \in S^\Delta(x)$. Using convexity of $c_i(x)$ and (2), we obtain

$$\begin{aligned} & \langle F(x^*), y^* - x^* \rangle \\ &= \sum_{i=1}^m \lambda_i(x^*) \langle \nabla c_i(x^*), x^* - y^* \rangle \\ &= \sum_{i=1}^m \lambda_i(x^*) [\langle \nabla c_i(x^*), x^* - x \rangle + \langle \nabla c_i(x^*), x - y^* \rangle] \\ &\geq \sum_{i=1}^m \lambda_i(x^*) [c_i(x^*) - c_i(x) + \langle \nabla c_i(x^*), x - y^* \rangle] \\ &\geq \sum_{i=1}^m \lambda_i(x^*) [\langle \nabla c_i(x^*) - \nabla c_i(x), x - y^* \rangle \\ &\quad + \langle \nabla c_i(x), x - y^* \rangle - c_i(x)] \\ &\geq \sum_{i=1}^m \lambda_i(x^*) [\langle \nabla c_i(x^*) - \nabla c_i(x), x - y^* \rangle]. \end{aligned}$$

Lemma 4 Suppose that x^* is KKT point of the $VIP, F(\cdot)$ and $\nabla c_i(\cdot) (i = 1, \dots, m)$ are L_1 - continuous and L_2 - continuous at x^* , respectively. Then, for $\forall x \in S$ and $\forall \Delta > 0$, we obtain

$$\begin{aligned} & \langle F(x) - F(x^*), x - x^* \rangle + [\lambda_0^* + \lambda_{\min}] \|y^* - x\|^2 \\ & \leq [\lambda_0^* + \eta(x^*)] \|y^* - x\| \cdot \|x - x^*\|, \end{aligned}$$

where $\eta(x^*) = \lambda_{\max} + L_1 + L_2 \sum_{i=1}^m \lambda_i(x^*)$.

Proof. Due to $S \subseteq S(x)$, we can choose $z = x^*$ in (5). Hence we have

$$\langle F(x) + B(x)(y^* - x) + \lambda_0^*(y^* - x), x^* - y^* \rangle \geq 0,$$

that is

$$\begin{aligned} & \langle F(x), x^* - y^* \rangle + \langle B(x)(y^* - x), x^* - y^* \rangle \\ & + \langle \lambda_0^*(y^* - x), x^* - y^* \rangle \geq 0. \end{aligned} \tag{7}$$

By Lemma 3 and Lipschitz continuity of $\nabla c_i(x)$, we obtain

$$\begin{aligned} 0 & \leq \langle F(x^*), y^* - x^* \rangle \\ & + \sum_{i=1}^m \lambda_i(x^*) \langle \nabla c_i(x^*) - \nabla c_i(x), y^* - x \rangle \\ & \leq \langle F(x^*), y^* - x^* \rangle \\ & + \sum_{i=1}^m \lambda_i(x^*) \|\nabla c_i(x^*) - \nabla c_i(x)\| \cdot \|y^* - x\| \\ & \leq \langle F(x^*), y^* - x^* \rangle + L_2 \sum_{i=1}^m \lambda_i(x^*) \|x^* - x\| \cdot \|y^* - x\|. \end{aligned} \tag{8}$$

Adding (7) and (8), and using the L_1 - continuity of $F(x)$, we have

$$\begin{aligned} 0 & \leq \langle F(x) - F(x^*), x^* - y^* \rangle + L_2 \sum_{i=1}^m \lambda_i(x^*) \|x^* - x\| \cdot \\ & \|y^* - x\| + \langle B(x)(y^* - x), x^* - y^* \rangle \\ & + \langle \lambda_0^*(y^* - x), x^* - y^* \rangle \\ & = \langle F(x) - F(x^*), x^* - x \rangle + \langle F(x) - F(x^*), x - y^* \rangle \\ & + L_2 \sum_{i=1}^m \lambda_i(x^*) \|x^* - x\| \cdot \|y^* - x\| \\ & + \langle B(x)(y^* - x), x^* - x \rangle - \langle B(x)(y^* - x), y^* - x \rangle \\ & + \lambda_0^* \langle (y^* - x), x^* - x \rangle \lambda_0^* \|y^* - x\|^2 \\ & \leq \langle F(x) - F(x^*), x^* - x \rangle + \|F(x) - F(x^*)\| \cdot \|x - y^*\| \\ & + L_2 \sum_{i=1}^m \lambda_i(x^*) \|x^* - x\| \cdot \|y^* - x\| + \lambda_{\max} \|y^* - x\| \cdot \\ & \|x^* - x\| - \lambda_{\min} \|y^* - x\|^2 + \lambda_0^* \|y^* - x\| \cdot \|x^* - x\| \\ & - \lambda_0^* \|y^* - x\|^2 \\ & \leq \langle F(x) - F(x^*), x^* - x \rangle + [L_1 + L_2 \sum_{i=1}^m \lambda_i(x^*) \\ & + \lambda_{\max} + \lambda_0^*] \|y^* - x\| \cdot \|x - x^*\| \\ & - [\lambda_0^* + \lambda_{\min}] \|y^* - x\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} & \langle F(x) - F(x^*), x^* - x \rangle + [\lambda_0^* + \lambda_{\min}] \|y^* - x\|^2 \\ & \leq [\lambda_0^* + \eta(x^*)] \|y^* - x\| \cdot \|x - x^*\|, \end{aligned}$$

where $\eta(x^*) = \lambda_{\max} + L_1 + L_2 \sum_{i=1}^m \lambda_i(x^*)$.

Theorem 4. Suppose that $F(x)$ is strongly monotone and L_1 - continuous at $x^* \in S$, and $\nabla c_i(x) (1 \leq i \leq m)$ is L_2 - continuous at x^* , and the point x^* is a KKT point of the problem (VIP). Then x^* is the unique solution of the VIP , and there exist constants $\eta_1(x^*) > 0, \eta_2 > 0$, such that for $\forall x \in S$ and $\Delta > 0$, we have

$$\|x - x^*\| \leq \eta_1(x^*) \Phi(x, \Delta)^{\frac{1}{2}} + \eta_2 \text{sign}(\lambda_0^*) \cdot \frac{\Phi(x, \Delta)}{\Delta}.$$

Proof. The uniqueness is obvious due to strongly monotonicity of $F(x)$. From strongly monotonicity of $F(x)$ and Lemma 4, there exist constants $\alpha > 0$, and we have

$$\begin{aligned} \sigma \|x - x^*\|^2 & \leq \langle F(x) - F(x^*), x - x^* \rangle \\ & \leq \langle F(x) - F(x^*), x - x^* \rangle + [\lambda_0^* + \lambda_{\min}] \|y^* - x\| \\ & \leq [\lambda_0^* + \eta(x^*)] \|y^* - x\| \cdot \|x - x^*\|. \end{aligned}$$

Therefore, by Theorem 2, we obtain

$$\begin{aligned} \|x - x^*\| & \leq \frac{\eta(x^*)}{\alpha} \|y^* - x\| + \frac{\lambda_0^*}{\alpha} \|y^* - x\| \\ & \leq \frac{\eta(x^*)}{\alpha} \left(\frac{2}{\lambda_{\min}}\right)^{\frac{1}{2}} \Phi(x, \Delta)^{\frac{1}{2}} + \frac{1}{\alpha} \text{sign}(\lambda_0^*) \frac{\Phi(x, \Delta)}{\Delta} \\ & = \eta_1(x^*) \Phi(x, \Delta)^{\frac{1}{2}} + \eta_2 \text{sign}(\lambda_0^*) \cdot \frac{\Phi(x, \Delta)}{\Delta}, \end{aligned}$$

where, $\eta_1(x^*) = \frac{\eta(x^*)}{\alpha} \left(\frac{2}{\lambda_{\min}}\right)^{\frac{1}{2}}, \eta_2 = \frac{1}{\alpha}$.

4. Conclusion

In this paper, we employ the trust region methods to analyze the variational inequality problems, and structure the optimal value function of it. By the properties of the optimal value function, we obtain some lemmas and theorems. And the last Theorem provide a global error estimate between the feasible solution and the optimal solution by using the value function $\Phi(x, \Delta)$.



Wenling Zhao received her BS and MS from Qufu Normal University in 1985 and 1991, respectively. Since 2005 she has been at Dalian University of Technology for Ph.D. Her research interests focus on optimization theory and method, OR methods and applications.

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Jin Huang is a master student of Shandong University of Technology. Now Ms Zhao is her trainer. Her MB was been get from this university.

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