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Some Approximation Properties of Baskakov-Szász-Stancu Operators

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Abstract: In this paper, we are dealing with a new type of Baskakov-Szász-Stancu operators $\mathscr{D}_n^{(\alpha,\beta)}(f,x)$ defined by (1.4). First we estimate moments of these operators and also obtain the recurrence relations for the moments. We estimate some approximation properties and asymptotic formulae for these operators. In the last section, we establish some direct results in the polynomial weighted space of continuous functions defined on the interval $[0,\infty)$.

Keywords: Baskakov-Szász type operators, Asymptotic formula, Weighted approximation.

1 Introduction

For $f \in C[0,\infty)$, a new type of Baskakov-Szász operators proposed by Gupta and Srivastava [6] is defined as

$$\mathscr{D}_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt,\ x\in[0,\infty) \quad (1)$$

where $p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$ and $s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$.
In [21] Stancu introduced the following generalization of Bernstein polynomials

$$S_n^{\alpha}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)P_{n,\alpha}^{k}(x),\ \ 0\le x\le 1, \quad (2)$$

where $P_{n,\alpha}^{k}(x) = \binom{n}{k}\frac{\prod_{s=0}^{k-1}(x+\alpha s)\prod_{s=0}^{n-k-1}(1-x+\alpha s)}{\prod_{s=0}^{n-1}(1+\alpha s)}$.
We get the classical Bernstein polynomials by putting $\alpha=0$. Starting with two parameter α,β satisfying the condition $0\le\alpha\le\beta$ in 1983, the other generalization of Stancu operators was given in [22] and studied the linear positive operators $S_n^{\alpha,\beta}: C[0,1]\to C[0,1]$ defined for any $f\in C[0,1]$ as follows:

$$S_n^{\alpha,\beta}(f,x) = \sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k+\alpha}{n+\beta}\right),\ 0\le x\le 1, \quad (3)$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is the Bernstein basis function(cf. [2]).

Recently, Ibrahim [7] introduced Stancu-Chlodowsky polynomial and investigated convergence and approximation properties of these operators. Motivated by such type of operators we introduce Stancu type generalization of the Baskakov-Szász operators (1) as follows:

$$\mathscr{D}_n^{(\alpha,\beta)}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt, \quad (4)$$

where $p_{n,k}(x)$ and $s_{n,k}(t)$ defined as same in (1). The operators $\mathscr{D}_n^{(\alpha,\beta)}(f,x)$ in (4) are called Baskakov-Szász-Stancu operators. For $\alpha=0,\beta=0$ the operators (4) reduce to the operators (1).

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We know that

$$\sum_{k=0}^{\infty} p_{n,k}(x) = 1,\ \int_0^{\infty} p_{n,k}(x)\ dx = \frac{1}{n-1},$$

$$\sum_{k=0}^{\infty} s_{n,k}(t) = 1,\ \int_0^{\infty} s_{n,k}(t)\ dt = \frac{1}{n}.$$

In [16] Moghaddam and Aghili presented a numerical method for solving LNFODE (Linear Non-homogeneous Fractional Ordinary Differential Equation). The method presented is based on Bernstein polynomials approximation.

The aim of the present paper is to study some direct results in terms of the modulus of continuity of second order. We estimate moments for these operators and obtain the recurrence relation for moments. Also, we study direct theorem, Voronovskaja type asymptotic formula and weighted approximation properties for operators (4).

2 Basic Results

Lemma 1. *For $\mathscr{D}_n(t^m;x)$, $m = 0,1,2$, we have*

$$\mathscr{D}_n(1,x) = 1,\ \mathscr{D}_n(t,x) = \frac{nx+1}{n},$$

$$\mathscr{D}_n(t^2,x) = \frac{1}{n^2}[n(n+1)x^2 + 4nx + 2].$$

Lemma 2. *The following equalities hold:*

$$\mathscr{D}_n^{(\alpha,\beta)}(1,x) = 1,\ \ \mathscr{D}_n^{(\alpha,\beta)}(t,x) = \frac{nx+1+\alpha}{n+\beta},$$

$$\mathscr{D}_n^{(\alpha,\beta)}(t^2,x) = \frac{n(n+1)x^2}{(n+\beta)^2} + \frac{(4n+2n\alpha)x}{(n+\beta)^2} + \frac{(2+2\alpha+\alpha^2)}{(n+\beta)^2}.$$

Proof. We observe that,

$$\mathscr{D}_n^{(\alpha,\beta)}(1,x) = \mathscr{D}_n(1,x) = 1.$$

$$\begin{aligned}\mathscr{D}_n^{(\alpha,\beta)}(t,x) &= \frac{n}{n+\beta}\mathscr{D}_n(t,x) + \frac{\alpha}{n+\beta}\mathscr{D}_n(1,x)\\ &= \frac{n}{n+\beta}\left(\frac{nx+1}{n}\right) + \frac{\alpha}{n+\beta}\\ &= \frac{nx+1+\alpha}{n+\beta}.\end{aligned}$$

and

$$\begin{aligned}\mathscr{D}_n^{(\alpha,\beta)}(t^2,x) &= \frac{n^2}{(n+\beta)^2}\mathscr{D}_n(t^2,x) + \frac{2n\alpha}{(n+\beta)^2}\mathscr{D}_n(t,x)\\ &\quad + \frac{\alpha^2}{(n+\beta)^2}\mathscr{D}_n(1,x)\\ &= \frac{n^2}{(n+\beta)^2}\left[\frac{n(n+1)x^2+4nx+2}{n^2}\right]\\ &\quad + \frac{2n\alpha}{(n+\beta)^2}\left(\frac{nx+1}{n}\right) + \frac{\alpha^2}{(n+\beta)^2}\\ &= \frac{n(n+1)}{(n+\beta)^2}x^2 + \frac{(4n+2n\alpha)}{(n+\beta)^2}x\\ &\quad + \frac{(2+2\alpha+\alpha^2)}{(n+\beta)^2}.\end{aligned}$$

3 Moments and recurrence relations

Lemma 3. *If we define the central moments as*

$$\begin{aligned}\mu_{n,m}(x) &= D_n^{(\alpha,\beta)}((t-x)^m,x)\\ &= n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt,\end{aligned}$$

$x \in [0,\infty)$, $m \in \mathbb{N}$.

Then,

$$\mu_{n,0}(x) = 1,\ \mu_{n,1}(x) = \frac{\alpha-\beta x+1}{n+\beta},$$

and for $n > m$, we have the following recurrence relation:

$$\begin{aligned}(n+\beta)\mu_{n,m+1}(x) &= x(1+x)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]\\ &\quad + [m+1+\alpha-\beta x]\mu_{n,m}(x)\\ &\quad - m\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m-1}(x).\end{aligned} \tag{5}$$

Proof. Taking derivative of $\mu_{n,m}(x)$

$$\begin{aligned}\mu'_{n,m}(x) &= -mn\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1} dt\\ &\quad + n\sum_{k=0}^{\infty} p'_{n,k}(x)\int_0^{\infty} s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\ \mu'_{n,m}(x) &= -m\mu_{n,m-1}(x) + n\sum_{k=0}^{\infty} p'_{n,k}(x)\\ &\quad \int_0^{\infty} s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\end{aligned}$$

using $x(1+x)p'_{n,k}(x) = (k-nx)p_{n,k}(x)$, we get

$$\begin{aligned}
&x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&= n\sum_{k=0}^{\infty}(k-nx)p_{n,k}(x)\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&= n\sum_{k=0}^{\infty}kp_{n,k}(x)\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad -n\sum_{k=0}^{\infty}nxp_{n,k}(x)\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&= I - nx\mu_{n,m}(x). \qquad (6)
\end{aligned}$$

We can write I as

$$\begin{aligned}
I &= n\sum_{k=0}^{\infty}kp_{n,k}(x)\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&= \left[n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^\infty [k-nt]s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\right.\\
&\quad \left.+n\left(n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^\infty s_{n,k}(t)t\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\right)\right]\\
&= I_1+I_2,\ (say).
\end{aligned}$$

To estimate I_2 using $t=\frac{n+\beta}{n}\left[\left(\frac{nt+\alpha}{n+\beta}-x\right)-\left(\frac{\alpha}{n+\beta}-x\right)\right]$, we have

$$\begin{aligned}
I_2 &= \left[n\sum_{k=0}^{\infty}np_{n,k}(x)\int_0^\infty s_{n,k}(t)t\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\right]\\
&= \frac{n+\beta}{n}n\left[\sum_{k=0}^{\infty}np_{n,k}(x)\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt\right.\\
&\quad -\left(\frac{\alpha}{n+\beta}-x\right)\sum_{k=0}^{\infty}np_{n,k}(x)\\
&\quad \left.\int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\right]\\
&= (n+\beta)\left[\mu_{n,m+1}(x)-\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m}\right].
\end{aligned}$$

Next to estimate I_1 using the equality, $ts'_{n,k}(t)=[k-nt]s_{n,k}(t)$

$$I_1 = n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^\infty ts'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt,$$

again putting $t=\frac{n+\beta}{n}\left[\left(\frac{nt+\alpha}{n+\beta}-x\right)-\left(\frac{\alpha}{n+\beta}-x\right)\right]$, we get

$$\begin{aligned}
I_1 &= \frac{n+\beta}{n}\left[\sum_{k=0}^{\infty}np_{n,k}(x)\int_0^\infty s'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt\right.\\
&\quad -\left(\frac{\alpha}{n+\beta}-x\right)\\
&\quad \left.\sum_{k=0}^{\infty}np_{n,k}(x)\int_0^\infty s'_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\right].
\end{aligned}$$

Now integrating by parts, we get

$$\begin{aligned}
I_1 &= \frac{n+\beta}{n}\left[-(m+1)\frac{n}{n+\beta}\sum_{k=0}^{\infty}np_{n,k}(x)\right.\\
&\quad \int_0^\infty s_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad +\frac{mn}{n+\beta}\left(\frac{\alpha}{n+\beta}-x\right)\sum_{k=0}^{\infty}np_{n,k}(x)\int_0^\infty s_{n,k}(t)\\
&\quad \left.\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1}dt\right]\\
&= \left[-(m+1)\mu_{n,m}(x)+m\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m-1}(x)\right].
\end{aligned}$$

Put the values of I_1 and I_2 in I, we get

$$\begin{aligned}
I &= \left[-(m+1)\mu_{n,m}(x)+m\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m-1}(x)\right]\\
&\quad +(n+\beta)\left[\mu_{n,m+1}(x)-\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m}\right].
\end{aligned}$$

Now, put value of I in (6), we get

$$\begin{aligned}
&x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&= -(m+1)\mu_{n,m}(x)+m\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m-1}(x)\\
&\quad +(n+\beta)\left(\mu_{n,m+1}(x)-\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m}\right)-nx\mu_{n,m}(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(n+\beta)\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&+[m+1+\alpha-\beta x]\mu_{n,m}(x)-m\left(\frac{\alpha}{n+\beta}-x\right)\mu_{n,m-1}(x),
\end{aligned}$$

which is the required result.

Remark. For $\alpha=0=\beta$ the relation (5) reduces to

$$\begin{aligned}
n\mu_{n,m+1}(x) &= x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]\\
&\quad +(m+1)\mu_{n,m}(x)+mx\mu_{n,m-1}(x).
\end{aligned}$$

Lemma 4. *For $n\in\mathbb{N}$, we have*

$$\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x) \le \frac{(1+\beta^2)}{n+\beta}\left[\phi^2(x)+\frac{1}{n+\beta}\right],$$

where $\phi(x)=\sqrt{x(1+x)}$, $x\in[0,\infty)$.

Proof. Using lemma 3 and $\alpha \leq \beta$, we have

$$\begin{aligned}
&\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x)\\
&=\frac{(n+\beta^2)}{(n+\beta)^2}x^2+\frac{x(2n-2\beta-2\alpha\beta)}{(n+\beta)^2}+\frac{2+2\alpha+\alpha^2}{(n+\beta)^2}\\
&=\frac{(n+\beta^2)}{(n+\beta)^2}x^2+\frac{(2n-2(1+\alpha)\beta)}{(n+\beta)^2}x+\frac{2(1+\alpha)+\alpha^2}{(n+\beta)^2}\\
&\leq\frac{(n+\beta^2)}{(n+\beta)^2}x^2+\frac{(2n+2\beta^2)}{(n+\beta)^2}x+\frac{1+\beta^2}{(n+\beta)^2}\\
&=\frac{(n+\beta^2)}{(n+\beta)^2}(x^2+x)+\frac{1+\beta^2}{(n+\beta)^2}.
\end{aligned}$$

Using $(n+\beta^2)\leq(n+\beta)(1+\beta^2)$ for $n\in\mathbb{N}$ and $\beta\geq 0$, we get

$$\begin{aligned}
&\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x)\\
&\leq\frac{(n+\beta)(1+\beta^2)}{(n+\beta)^2}(x^2+x)+\frac{1+\beta^2}{(n+\beta)^2}\\
&=\frac{1}{(n+\beta)}\left[\frac{(n+\beta)(1+\beta^2)}{(n+\beta)}(x^2+x)+\frac{1+\beta^2}{(n+\beta)}\right].
\end{aligned}$$

Thus,

$$\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x)\leq\frac{(1+\beta^2)}{(n+\beta)}\left[\phi^2(x)+\frac{1}{(n+\beta)}\right],$$

which is required.

4 Direct result and asymptotic formula

Let the space $C_B[0,\infty)$ of all continuous and bounded functions be endowed with the norm $\|f\|=sup\{|f(x)|: x\in[0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\}, \tag{7}$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By the method as given [4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}), \tag{8}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)| \tag{9}$$

is the second order modulus of smoothness of $f\in C_B[0,\infty)$. Also we set

$$\omega(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|. \tag{10}$$

We denote the usual modulus of continuity of $f\in C_B[0,\infty)$. In what follows we shall use the notations $\phi(x)=\sqrt{x(x+1)}$, where $x\in[0,\infty)$.

Now, we give local approximation theorems for the operators $\mathscr{D}_n^{(\alpha,\beta)}$.

Theorem 1. *Let $f\in C_B[0,\infty)$. Then, we have following inequality,*

$$\begin{aligned}
&|\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)|\leq\omega_2\left(f,\frac{|1+\alpha-\beta x|}{n+\beta}\right)\\
&\quad+C\omega_2\left(f,\sqrt{\frac{(1+\beta^2)}{n+\beta}\left[\phi^2(x)+\frac{1}{n+\beta}\right]}\right),
\end{aligned}$$

where C is a positive constant.

Proof. Let us define the auxiliary operator $\mathscr{L}_n^{(\alpha,\beta)}$ by

$$\mathscr{L}_n^{(\alpha,\beta)}(f,x)=\mathscr{D}_n^{(\alpha,\beta)}(f,x)+f(x)-f\left(x+\frac{1+\alpha-\beta x}{n+\beta}\right), \tag{11}$$

for every $x\in[0,\infty)$. The operator $\mathscr{L}_n^{(\alpha,\beta)}$ are linear and preserve the linearity properties:

$$\mathscr{L}_n^{(\alpha,\beta)}(t-x,x)=0,\ t\in[0,\infty). \tag{12}$$

Let $g\in W^2$ and $x,t\in[0,\infty)$. By Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du,\ t\in[0,\infty).$$

Applying $\mathscr{L}_n^{(\alpha,\beta)}$ on above and using (12), we get

$$\mathscr{L}_n^{(\alpha,\beta)}(g,x)=g(x)+\mathscr{L}_n^{(\alpha,\beta)}\left(\int_x^t(t-u)g''(u)du,x\right).$$

Hence by Lemma (2) one has

$$\begin{aligned}
&|\mathscr{L}_n^{(\alpha,\beta)}(g,x)-g(x)|\leq\mathscr{L}_n^{(\alpha,\beta)}\left(\left|\int_x^t|t-u|\,|g''(u)|du\right|,x\right)\\
&\leq\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x)\|g''\|\\
&\quad+\left|\int_x^{\left(x+\frac{1+\alpha-\beta x}{n+\beta}\right)}\left(x+\frac{1+\alpha-\beta x}{n+\beta}-u\right)\|g''\|(u)du\right|\\
&\leq\left[\frac{(1+\beta^2)}{n+\beta}\left(\phi^2(x)+\frac{1}{n+\beta}\right)+\left(\frac{1+\alpha-\beta x}{n+\beta}\right)^2\right]\|g''\|\\
&\leq\left[\frac{6(1+\beta^2)}{n+\beta}\left(\phi^2(x)+\frac{1}{n+\beta}\right)\right]\|g''\|.
\end{aligned}$$

Since

$$\left|\mathscr{D}_n^{(\alpha,\beta)}(f,x)\right|\leq n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}s_{n,k}(t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt\leq\|f\|, \tag{13}$$

$$\left|\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)\right| \le \left|\mathscr{L}_n^{(\alpha,\beta)}(f-g,x)-(f-g)(x)\right| + |\mathscr{L}_n^{(\alpha,\beta)}(g,x)-g(x)| + \left|\left(x+\frac{1+\alpha-\beta x}{n+\beta}\right)-f(x)\right| \le 2\|f-g\| + \frac{6(1+\beta^2)}{n+\beta}\left[\phi^2(x)+\frac{1}{n+\beta}\right]\|g''\| + \omega\left(f,\frac{|\alpha-\beta x|}{n+\beta}\right).$$

Taking infimum overall $g \in W^2$, we get

$$| \mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x) | \le K\left(f,\frac{(1+\beta^2)}{n+\beta}\left[\phi^2(x)+\frac{1}{n+\beta}\right]\right) + \omega_2\left(f,\frac{|1+\alpha-\beta x|}{n+\beta}\right). \quad (14)$$

By (8), we get

$$| \mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x) | \quad (15)$$
$$\le C\omega_2\left(f,\sqrt{\frac{(1+\beta^2)}{n+\beta}\left[\phi^2(x)+\frac{1}{n+\beta}\right]}\right) + \omega_2\left(f,\frac{|1+\alpha-\beta x|}{n+\beta}\right), \quad (16)$$

which proves the theorem.

5 Weighted approximation

Let $B_{x^2}[0,\infty)=\{f : for\, every\, x \in [0,\infty), | f(x) |\le M_f(1+x^2)$, where M_f is a constant depending on $f\}$. By $C_{x^2}[0,\infty)$, we denote subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2}=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$.

Now, we discuss the weighted approximation theorem, when the approximation formula holds true on the interval $[0,\infty)$. Several other researchers have studied in this direction and obtained different approximation properties of many operators via summability methods also, we mention some of them as [1], [9]–[15], [17]–[19] etc.

Theorem 2. *For each $f \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}\|\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)\|_{x^2}=0.$$

Proof. Using the theorem in [5] and [18] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|\mathscr{D}_n^{(\alpha,\beta)}(t^r,x)-x^r\|_{x^2}=0,\ r=0,1,2. \quad (17)$$

Since, $\mathscr{D}_n^{(\alpha,\beta)}(1,x)=1$, the first condition of (17) is satisfied for $r=0$. Now,

$$\begin{aligned}\|\mathscr{D}_n^{(\alpha,\beta)}(t,x)-x\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(t,x)-x |}{1+x^2}\\ &\le \sup_{x\in[0,\infty)}\left|\frac{nx+\alpha+1}{n+\beta}-x\right|\times\frac{1}{1+x^2}\\ &\le \left|\frac{n}{(n+\beta)}\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &\quad+\frac{\alpha+1}{n+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}-\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore, condition (17) holds for $r=1$.

Similarly, we can write

$$\|\mathscr{D}_n^{(\alpha,\beta)}(t^2,x)-x^2\|_{x^2} = \sup_{x\in[0,\infty)}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(t^2,x)-x^2 |}{1+x^2}$$
$$\le \left(\frac{n(n+1)}{(n+\beta)^2}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2} + \frac{2n(\alpha+2)}{(n+\beta)^2}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{2+2\alpha+\alpha^2}{(n+\beta)^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}$$
$$\le \frac{n(1-2\beta)-\beta^2}{(n+\beta)^2}+\frac{2n(\alpha+2)}{(n+\beta)^2}+\frac{2+2\alpha+\alpha^2}{(n+\beta)^2},$$

which implies that $\|\mathscr{D}_n^{(\alpha,\beta)}(t^2,x)-x^2\|_{x^2}\to 0$ as $n\to\infty$. Thus the proof is completed.

We give the following theorem to approximate all functions in $C_{x^2}[0,\infty)$. This types of results are given in [5] for locally integrable functions.

Theorem 3. *For each $f \in C_{x^2}[0,\infty)$ and $\xi>0$, we have* $\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)|}{(1+x^2)^{1+\xi}}=0$.

Proof. For any fixed $x_0>0$,

$$\sup_{x\in[0,\infty)}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x) |}{(1+x^2)^{1+\xi}} \le$$
$$\sup_{x\le x_0}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x) |}{(1+x^2)^{1+\xi}}+\sup_{x\ge x_0}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x) |}{(1+x^2)^{1+\xi}}$$
$$\le \|\mathscr{D}_n^{(\alpha,\beta)}(f)-f\|_{C[0,x_0]} + \|f\|_{x^2}\sup_{x\ge x_0}\frac{| \mathscr{D}_n^{(\alpha,\beta)}(1+t^2,x) |}{(1+x^2)^{1+\xi}}+\sup_{x\ge x_0}\frac{| f(x) |}{(1+x^2)^{1+\xi}}.$$

The first term of the above inequality tends to zero from Theorem 2 of [20]. By Lemma 4 for any fixed $x_0>0$ it is easily seen that $\sup_{x\geq x_0}\frac{|\mathscr{D}_n^{(\alpha,\beta)}(1+t^2,x)|}{(1+x^2)^{1+\xi}}$ tends to zero as $n\to\infty$. We can choose $x_0>0$ so large that the last part of the above inequality can be made small enough. Thus the proof is completed.

6 Voronovskaja type theorem

In this section we establish a Voronovskaja type asymptotic formula for the operators $\mathscr{D}_n^{(\alpha,\beta)}$.

Lemma 5. *For every $x\in[0,\infty)$, we have*

$$\lim_{n\to\infty} n\mathscr{D}_n^{(\alpha,\beta)}(t-x,x)=(1+\alpha-\beta x), \tag{18}$$

$$\lim_{n\to\infty} n\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x)=x(2+x). \tag{19}$$

Theorem 4. *If any $f\in C_{x^2}[0,\infty)$ such that $f',f''\in C_{x^2}[0,\infty)$ and $x\in[0,\infty)$ then, we have*

$$\lim_{n\to\infty} n(\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)) = (1+\alpha-\beta x)f'(x) + \frac{x(2+x)}{2}f''(x),$$

for every $x\geq 0$.

Proof. Let $f,f',f''\in C_{x^2}[0,\infty)$ and $x\in[0,\infty)$. By Taylor's expansion we can write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2!}f''(x)(t-x)^2+r(x,t)(t-x)^2, \tag{20}$$

where $r(t,x)$ is Peano form of the remainder, $r(\cdot,x)\in C^*_{x^2}[0,\infty)$ and $\lim_{t\to x} r(t,x)=0$. Applying $\mathscr{D}_n^{(\alpha,\beta)}$ to above, we obtain

$$\begin{aligned} n[\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)] &= f'(x)n\mathscr{D}_n^{(\alpha,\beta)}(t-x,x) \\ &+\frac{n}{2!}f''(x)\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x) \\ &+n\mathscr{D}_n^{(\alpha,\beta)}(r(t,x)(t-x)^2,x). \end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\mathscr{D}_n^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)\leq \\ &\sqrt{\mathscr{D}_n^{(\alpha,\beta)}(r^2(t,x),x)}\sqrt{\mathscr{D}_n^{(\alpha,\beta)}((t-x)^4,x)}. \end{aligned} \tag{21}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C_{x^2}[0,\infty)$. Then, we have

$$\lim_{n\to\infty} n\mathscr{D}_n^{(\alpha,\beta)}(r^2(t,x),x)=r^2(x,x)=0, \tag{22}$$

uniformly with respect to $x\in[0,A]$, where $A>0$. Now from (21) and (22) and Lemma 5, we obtain

$$\lim_{n\to\infty} n\mathscr{D}_n^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)=0.$$

Hence,

$$\begin{aligned} &\lim_{n\to\infty} n[\mathscr{D}_n^{(\alpha,\beta)}(f,x)-f(x)] \\ &=\lim_{n\to\infty}\Big(f'(x)n\mathscr{D}_n^{(\alpha,\beta)}(t-x,x)+\frac{n}{2}f''(x)\mathscr{D}_n^{(\alpha,\beta)}((t-x)^2,x) \\ &+n\mathscr{D}_n^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)\Big) \\ &=(1+\alpha-\beta x)f'(x)+x(x+2)/2f''(x), \end{aligned}$$

which completes the proof.

7 Better estimation

It is well know that the operators preserve constant as well as linear functions. To make the convergence faster, King [8] proposed an approach to modify the classical Bernstein polynomials, so that this sequence preserves two test functions e_0 and e_1. After this several researchers have studied that many approximating operators L, possess these properties i.e. $L(e_i,x)=e_i(x)$ where $e_i(x)=x^i(i=0,1)$, for examples Bernstein, Baskakov and Baskakov-Durrmeyer-Stancu operators.
In 2012 [3] Braica et al. find some properties of a King-type operator and gave an approximation theorem and a Voronovskaja type theorem for this operator.

As the operators $\mathscr{D}_n^{(\alpha,\beta)}$ introduced in (4) preserve only the constant functions so further modification of said operators is proposed to be made so that the modified operators preserve the constant as well as linear functions, for this purpose the modification of $\mathscr{D}_n^{(\alpha,\beta)}$ as follows:

$$\mathscr{D}_n^{*(\alpha,\beta)}(f,x)=n\sum_{k=0}^{\infty}p_{n,k}(r_n(x))\int_0^{\infty}s_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt, \tag{23}$$

where $r_n(x)=\frac{(n+\beta)x-(\alpha+1)}{n}$ and $x\in I_n=\left[\frac{\alpha+1}{n+\beta},\infty\right)$.

Lemma 6. *For each $x\in I_n$, we have*

$$\begin{aligned} &\mathscr{D}^{*(\alpha,\beta)}_n(1,x)=1, \\ &\mathscr{D}^{*(\alpha,\beta)}_n(t,x)=x, \\ &\mathscr{D}^{*(\alpha,\beta)}_n(t^2,x)=\left(\frac{(n+1)}{n}\right)x^2+\left(\frac{2[n-(\alpha+1)]}{n(n+\beta)}\right)x \\ &\quad+\left\{\frac{(n+1)}{n}\frac{(\alpha+1)^2}{(n+\beta)^2}-\frac{(4+2\alpha)(\alpha+1)}{(n+\beta)^2}+\frac{(2+2\alpha+\alpha^2)}{(n+\beta)^2}\right\}. \end{aligned}$$

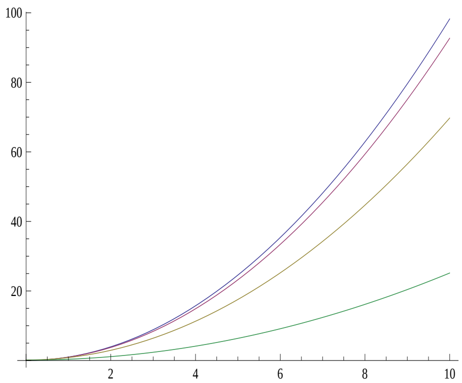

Fig. 1: Curves for $f(x)=x^2$, $\mathscr{D}_n^{(0.5,5)}(t^2,x)$, $\mathscr{D}_n^{(10,20)}(t^2,x)$, $\mathscr{D}_n^{(50,100)}(t^2,x)$, $\mathscr{D}_n^{(100,500)}(t^2,x)$ at $n=500$.

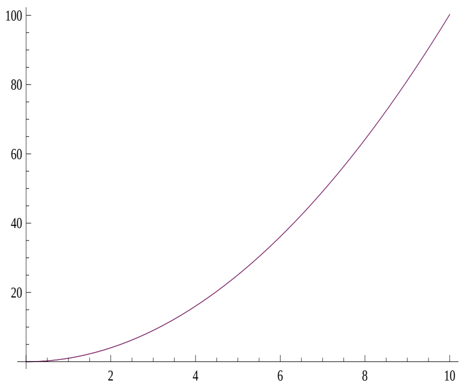

Fig. 2: Curves for $f(x)=x^2$, $\mathscr{D}_n^{*(0.5,5)}(t^2,x)$, $\mathscr{D}_n^{*(10,20)}(t^2,x)$, $\mathscr{D}_n^{*(50,100)}(t^2,x)$, $\mathscr{D}_n^{*(100,500)}(t^2,x)$ at $n=500$.

Lemma 7. *For $x\in I_n$, the following holds,*

$$\tilde{\mu}_{n,1}(x)=\mathscr{D}^{*(\alpha,\beta)}_n(t-x,x)=0,$$

$$\tilde{\mu}_{n,2}(x)=\mathscr{D}^{*(\alpha,\beta)}_n((t-x)^2,x)=\left(\frac{1}{n}\right)x^2+\left(\frac{2[n-(\alpha+1)]}{n(n+\beta)}\right)x +\left\{\frac{(n+1)}{n}\frac{(\alpha+1)^2}{(n+\beta)^2}-\frac{(4+2\alpha)(\alpha+1)}{(n+\beta)^2} +\frac{(2+2\alpha+\alpha^2)}{(n+\beta)^2}\right\}.$$

Theorem 5. *Let $f\in C_B(I_n)$, $x\in I_n$. Then, for $n>1$, there exist an absolute constant $C>0$ such that*

$$\left|\mathscr{D}^{*(\alpha,\beta)}_n(f,x)-f(x)\right|\leq C\omega_2\left(f,\sqrt{\tilde{\mu}_{n,2}(x)}\right).$$

Proof. Let $g\in C_B(I_n)$ and $x,t\in I_n$. By Taylor's expansion we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du. \quad (24)$$

Applying $\mathscr{D}^{*(\alpha,\beta)}_n$ on (24), we get

$$\mathscr{D}^{*(\alpha,\beta)}_n(g,x)-g(x)=g'(x)\mathscr{D}^{*(\alpha,\beta)}_n((t-x),x) +\mathscr{D}^{*(\alpha,\beta)}_n\left(\int_x^t(t-u)g''(u)du,x\right).$$

Obviously, we have $\left|\int_x^t(t-x)g''(u)du\right|\leq(t-x)^2\|g''\|$,

$$\left|\mathscr{D}^{*(\alpha,\beta)}_n(g,x)-g(x)\right|\leq\mathscr{D}^{*(\alpha,\beta)}_n((t-x)^2,x)\|g''\|=\tilde{\mu}_{n,2}\|g''\|.$$

Since $\left|\mathscr{D}^{*(\alpha,\beta)}_n(f,x)\right|\leq\|f\|$,

$$\left|\mathscr{D}^{*(\alpha,\beta)}_n(f,x)-f(x)\right|\leq\left|\mathscr{D}^{*(\alpha,\beta)}_n(f-g,x)-(f-g)(x)\right| +\left|\mathscr{D}^{*(\alpha,\beta)}_n(g,x)-g(x)\right| \leq 2\|f-g\|+\tilde{\mu}_{n,2}\|g''\|.$$

Taking infimum overall $g\in C^2(I_n)$, we obtain

$$\left|\mathscr{D}^{*(\alpha,\beta)}_n(f,x)-f(x)\right|\leq K_2(f,\tilde{\mu}_{n,2}).$$

By (8), we have

$$\left|\mathscr{D}^{*(\alpha,\beta)}_n(f,x)-f(x)\right|\leq C\omega_2\left(f,\sqrt{\tilde{\mu}_{n,2}}\right),$$

which proves the theorem.

Theorem 6. *For any $f\in C^*_{x^2}(I_n)$ such that $f',f''\in C^*_{x^2}(I_n)$, we have*

$$\lim_{n\to\infty}n\left[\mathscr{D}^{*(\alpha,\beta)}_n(f,x)-f(x)\right]=(x(x+2)/2)f''(x)$$

for every $x\in I_n$.

Proof: The proof of above Theorem is in similar manner as Theorem 4.

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