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Intuitionistic Fuzzy Einstein Prioritized Weighted Average Operators and their Application to Multiple Attribute Group Decision Making

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Abstract: This study investigates the multiple attribute decision making under intuitionistic fuzzy environment in which the attributes and the decision makers are in different priority levels. In this paper, we first propose two new intuitionistic fuzzy aggregation operators such as the intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator and the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator for aggregating intuitionistic fuzzy information. These proposed operators can capture the prioritization phenomenon among the aggregated arguments. The properties of the new aggregation operators are studied in details and their special cases are examined. Furthermore, based on the proposed operators, an approach to deal with multiple attribute group decision making problems under intuitionistic fuzzy environment is developed. Finally, a practical example is provided to illustrate the multiple attribute group decision making process with intuitionistic fuzzy information.

Keywords: Intuitionistic fuzzy sets, intuitionistic fuzzy numbers, prioritized weighted average operator, intuitionistic fuzzy prioritized weighted average operator, intuitionistic fuzzy prioritized weighted geometric operator

1 Introduction

Intuitionistic fuzzy set (IFS), as a generalized form of fuzzy set [48], was introduced by Atanassov [1]. It is characterized by three functions expressing the degree of membership, the degree of non-membership and the degree of hesitancy, respectively. Fuzzy sets are IFSs, but the converse is not necessarily true. Over the last few decades, IFS theory has been extensively investigated by many researchers and applied in a variety of fields including decision making [2,10,12,16,17],[19]-[38],[42,44,45] and [52,53] medical diagnosis [5,18] and pattern recognition [3,6,8,9,11,13] etc.

Information aggregation [15] is an important and useful research topic in intuitionistic fuzzy set theory, which has received quite some attention from researchers and practitioners in the last couple of years. To aggregate intuitionistic fuzzy information, Xu and Yager [32] proposed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, the intuitionistic fuzzy hybrid geometric (IFHG) operator and developed an application of IFHG operator to multiple attribute decision making with intuitionistic fuzzy information. Xu [33] developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and the intuitionistic fuzzy hybrid averaging (IFHWA) operator. Motivated by Yager [41], Zhao et al. [51] proposed the generalized intuitionistic fuzzy weighted averaging (GIFWA) operator, the generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator and the generalized intuitionistic fuzzy hybrid averaging (GIFHA) operator. Further, Xia and Xu [30] developed a number of generalized intuitionistic fuzzy point aggregation operators such as the generalized

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intuitionistic fuzzy point weighted averaging (GIFPWA) operators, the generalized intuitionistic fuzzy point ordered weighted averaging (GIFPOWA) operator, and the generalized intuitionistic fuzzy point hybrid averaging (GIFPHA) operator and studied their properties with some special cases. Wei [26] proposed some induced intuitionistic fuzzy geometric aggregation operators and studied their applications in group decision making under intuitionistic fuzzy environment. Based on the ordered weighted distance (OWD) operator [38], Zeng and Su [49] proposed the intuitionistic fuzzy ordered weighted distance (IFOWD) operator to aggregate the intuitionistic fuzzy information. Xu and Yager [36] investigated the dynamic intuitionistic fuzzy multiple-attribute decision making problems and developed the dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator to aggregate dynamic intuitionistic fuzzy information. Also, Wei [26] proposed the dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and applied it to dynamic multiple attribute decision making with intuitionistic fuzzy information.

Recently, Wang and Liu [19] introduced some new operations on IFSs, such as Einstein sum, Einstein product, Einstein exponentiation etc. and developed some new intuitionistic fuzzy aggregation operators such as the intuitionistic fuzzy Einstein weighted average (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted average (IFEOWA) operator. Wang and Liu [20] further proposed some new geometric intuitionistic fuzzy aggregation operators, such as the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator. They also established some useful properties of these operators such as commutativity, idempotency and monotonicity, and developed a decision making method for solving multiple attribute decision making problem under intuitionistic fuzzy environment. Zhao and Wei [50] introduced some intuitionistic fuzzy Einstein hybrid aggregation operators and discussed their applications in multiple attribute decision making. Xu et al. [31] introduced a new aggregation operator called induced intuitionistic fuzzy Einstein ordered weighted averaging (I-IFEOWA) operator for aggregating intuitionistic fuzzy information and studied its application in multiple attribute group decision making.

One more thing that may be mentioned is that the above aggregation operators for IFNs have assumed that the attributes and the decision makers are at same priority level. However, in the real life multiple attribute group decision making problems, attributes and decision makers have different priority levels. To imbue this issue, motivated by the idea of prioritized aggregation operators [42,43], Yu [45] developed some intuitionistic fuzzy prioritized aggregation operators, such as the intuitionistic fuzzy prioritized weighted average (IFPWA) operator, the intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator and proposed two approaches to solve multi-criteria group decision making problems under intuitionistic fuzzy environment. Further, Yu [46] introduced a new generalized intuitionistic fuzzy prioritized geometric aggregation operator based on Archimedean t-conorm and t-norm. Recently, Yu [47] also proposed two new generalized prioritized aggregation operators such as the generalized intuitionistic fuzzy prioritized weighted average (GIFPWA) operator, generalized intuitionistic fuzzy prioritized weighted geometric (GIFPWG) operator and discussed their applications in multi criteria decision making. However, it seems that there is no investigation on prioritized aggregation technique using Einstein operations with IFNs. Therefore, the focus of this paper is to develop some intuitionistic fuzzy prioritized weighted average operator based on Einstein operations. The paper is organized as follows: In Section 2 some basic concepts related to intuitionistic fuzzy sets, Einstein operations on intuitionistic fuzzy sets and prioritized average operator are briefly given. In Section 3 we introduce two new prioritized weighted aggregation operators such as the intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator and the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator and discuss their particular cases. Some properties of IFEPWA and IFEPWG operators are also studied here. In Section 4 we develop a method for multiple attribute group decision making based on the proposed operators under intuitionistic fuzzy environment in which the attributes and decision makers are in different priority levels. In Section 5 finally, a numerical example is presented to illustrate the proposed approach to multiple attribute group decision making with intuitionistic fuzzy information. Our conclusions are presented in Section 6.

2 Preliminaries

We briefly review some basic concepts related to intuitionistic fuzzy sets and prioritized weighted averaging operator, which will be needed in the following analysis.

Definition 1. ***Intuitionistic Fuzzy Set*** [1]: An intuitionistic fuzzy set A in a discrete universe of discourse $X = (x_1, x_2, \ldots, x_n)$ is given by

$$A = \{\langle \mu_A(x), \nu_A(x)\rangle \,|x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$. For each $x \in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ denote the degree of membership and degree of non-membership of x in A respectively.

Further, $\pi_A = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of hesitancy or the intuitionistic index of x in A.

For an element $x \in X$, Xu and Yager [32] and Xu [33] defined the pair $(\mu_A(x), \nu_A(x))$ as intuitionistic fuzzy number (IFN) and denoted it by $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$.

Definition 2. ***Einstein Operations on IFNs*** [19, 20]: Let $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle$ be two intuitionistic fuzzy numbers and $\lambda > 0$, then following Einstein operations on α_1 and α_2 are defined as

$$\text{(i)}\alpha_1 \oplus_\varepsilon \alpha_2 = \left\langle \frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{1+\mu_{\alpha_1}\mu_{\alpha_2}}, \frac{\nu_{\alpha_1}\nu_{\alpha_2}}{1+(1-\nu_{\alpha_1})(1-\nu_{\alpha_2})} \right\rangle,$$
$$\text{(ii)}\alpha_1 \otimes_\varepsilon \alpha_2 = \left\langle \frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1+(1-\mu_{\alpha_1})(1-\mu_{\alpha_2})}, \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{1+\nu_{\alpha_1}\nu_{\alpha_2}} \right\rangle,$$
$$\text{(iii)}\lambda \cdot_\varepsilon \alpha_1 = \left\langle \frac{(1+\mu_{\alpha_1})^\lambda-(1-\mu_{\alpha_1})^\lambda}{(1+\mu_{\alpha_1})^\lambda+(1-\mu_{\alpha_1})^\lambda}, \frac{2\nu_{\alpha_1}^\lambda}{(2-\nu_{\alpha_1})^\lambda+\nu_{\alpha_1}^\lambda} \right\rangle,$$
$$\text{(iv)}(\alpha_1)^{\wedge\varepsilon\lambda} = \left\langle \frac{2\mu_{\alpha_1}^\lambda}{(2-\mu_{\alpha_1})^\lambda+\mu_{\alpha_1}^\lambda}, \frac{(1+\nu_{\alpha_1})^\lambda-(1-\nu_{\alpha_1})^\lambda}{(1+\nu_{\alpha_1})^\lambda+(1-\mu_{\alpha_1})^\lambda} \right\rangle.$$

Definition 3. ***Score Function*** [4]: Let $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$ be an intuitionistic fuzzy number, the score function S of an IFN is defined as follows:

$$S(\alpha) = \mu_\alpha - \nu_\alpha, \quad S(\alpha) \in [-1,1]. \tag{2}$$

Definition 4. ***Accuracy Function*** [7]: Let $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$ be an intuitionistic fuzzy number, the score function H of an IFN is defined as follows:

$$H(\alpha) = \mu_\alpha + \nu_\alpha, \quad H(\alpha) \in [0,1]. \tag{3}$$

To rank any two $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$, $i = 1,2$, Xu and Yager [32] proposed the following method:

Definition 5. ***Ranking Method for IFNs*** [32]: Let α_1 and α_2 be two intuitionistic fuzzy numbers, $S(\alpha_1)$ and $S(\alpha_2)$ be the scores of α_1 and α_2 respectively and $H(\alpha_1)$ and $H(\alpha_2)$ be the accuracy values of α_1 and α_2, then

(i)If $S(\alpha_1) > S(\alpha_2)$, then α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$.
(ii)If $S(\alpha_1) = S(\alpha_2)$, then we check their accuracy values and decide as follows:
  (a)If $H(\alpha_1) = H(\alpha_2)$, then α_1 and α_2 represent the same information, denoted by$\alpha_1 = \alpha_2$.
  (b)However, if $H(\alpha_1) > H(\alpha_2)$, then α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$.

It is noted that above defined score function $S(\alpha)$ ranges from -1 to 1. Liu [14] introduced another score function of an IFN as follows

$$S^*(\alpha) = \frac{1+S(\alpha)}{2} = \frac{1+\mu_\alpha - \nu_\alpha}{2} \in [0,1]. \tag{4}$$

According to Liu [14], if we replace the score function $S(\alpha)$ by $S^*(\alpha)$, the order relation between two IFNs α_1 and α_2 introduced by Xu and Yager [32] is also valid.

The Prioritized Weighted Average (PWA) operator was originally introduced by Yager [42,43] as follows:

Definition 6. ***Prioritized Weighted Average (PWA) Operator*** [42,43]: Let $G = \{G_1, G_2, \ldots, G_n\}$ be a collection of attributes and let there be a prioritization between the attributes expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \cdots \succ G_n$, indicating that the attribute G_j has a higher priority than G_k, if $j < k$. Also let $G_j(x)$ be the performance value of any alternative x under attribute G_j, and satisfies $G_j(x) \in [0,1]$. If

$$PWA(G_1(x), G_2(x), \ldots, G_n(x)) = \sum_{j=1}^n \left(\frac{T_j}{\sum_{j=1}^n T_j} G_j(x) \right), \tag{5}$$

where $T_j = \prod_{k=1}^{j-1} G_k(x)$, $j = 2,3,\ldots,n$, $T_1 = 1$, then PWA is called the prioritized weighted average (PWA) operator.

In the next section, we investigate the prioritized weighted average operator under intuitionistic fuzzy environment based on Einstein operations and propose the intuitionistic fuzzz Einstein prioritized weighted average (IFEPWA) operator and the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator.

3 Intuitionistic Fuzzy Einstein Prioritized Weighted Average Operators

On the basis of the definition of PWA operator, we give the definition of intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator as follows:

Definition 7. ***Intuitionistic Fuzzy Einstein Prioritized Weighted Average (IFEPWA) Operator***: Given a set of intuitionistic fuzzy numbers, $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$, the intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator is defined as follows

$$\begin{aligned} IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(\bigoplus_{j=1}^{n}{}_{\varepsilon} \frac{T_j}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_j \right) \\ &= \left(\frac{T_1}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_n}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_n \right), \end{aligned} \tag{6}$$

where $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ is the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$.

Next, based on the Einstein operational laws of IFNs, we have results in the following theorems:

Theorem 1. Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$, be a set of intuitionistic fuzzy numbers, then the aggregated value by using the IFEPWA operator is also an intuitionistic fuzzy number, and

$$\begin{aligned} &IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &= \left(\frac{T_1}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_n}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_n \right) \\ &= \left\langle \frac{\prod_{j=1}^n (1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}} - \prod_{j=1}^n (1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}}}{\prod_{j=1}^n (1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}} + \prod_{j=1}^n (1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}}}, \right. \\ &\qquad \left. \frac{2\prod_{j=1}^n \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^n T_j}}}{\prod_{j=1}^n (2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}} + \prod_{j=1}^n \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^n T_j}}} \right\rangle, \end{aligned} \tag{7}$$

where $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ is the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$.

Proof: The first result follows directly from Definition 2. We prove (7) by mathematical induction on n.

(i) First let $n=2$, then for α_1 and α_2, according to the Einstein operational laws of the IFNs, we have

$$\frac{T_1}{\sum_{j=1}^{2} T_j} \cdot_\varepsilon \alpha_1 = \left\langle \frac{(1+\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} - (1-\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}}}{(1+\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_i}} + (1-\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}}}, \frac{2\nu_{\alpha_1}^{\frac{T_1}{\sum_{j=1}^{2} T_j}}}{(2-\nu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} + \nu_{\alpha_1}^{\frac{T_1}{\sum_{j=1}^{2} T_j}}} \right\rangle, \tag{8}$$

and

$$\frac{T_2}{\sum_{j=1}^{2} T_i} \cdot_\varepsilon \alpha_2 = \left\langle \frac{(1+\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}} - (1-\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}{(1+\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}} + (1-\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}, \frac{2\nu_{\alpha_2}^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}{(2-\nu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}} + \nu_{\alpha_2}^{\frac{T_2}{\sum_{j=1}^{2} T_j}}} \right\rangle. \tag{9}$$

Now from Definition 7, we have

$$IFEPWA(\alpha_1,\alpha_2) = \left(\frac{T_1}{\sum_{j=1}^{2} T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^{2} T_j} \cdot_\varepsilon \alpha_2 \right)$$
$$= \left\langle \frac{(1+\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} (1+\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}} - (1-\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} (1-\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}{(1+\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} (1+\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_i}} + (1-\mu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} (1-\mu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}, \frac{2\nu_{\alpha_1}^{\frac{T_1}{\sum_{j=1}^{2} T_j}} \nu_{\alpha_2}^{\frac{T_2}{\sum_{j=1}^{2} T_j}}}{(2-\nu_{\alpha_1})^{\frac{T_1}{\sum_{j=1}^{2} T_j}} (2-\nu_{\alpha_2})^{\frac{T_2}{\sum_{j=1}^{2} T_j}} + \nu_{\alpha_1}^{\frac{T_1}{\sum_{j=1}^{2} T_j}} \nu_{\alpha_2}^{\frac{T_2}{\sum_{j=1}^{2} T_j}}} \right\rangle$$

$$= \left\langle \frac{\prod_{j=1}^{2}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{2} T_j}} - \prod_{j=1}^{2}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{2} T_j}}}{\prod_{j=1}^{2}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{2} T_j}} + \prod_{j=1}^{2}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{2} T_j}}}, \frac{2\prod_{j=1}^{2} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{2} T_j}}}{\prod_{j=1}^{2}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{2} T_j}} + \prod_{j=1}^{2} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{2} T_j}}} \right\rangle. \tag{10}$$

This shows that the result (7) holds for $n=2$.

Next, let (7) holds for $n=k$, that is

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_k) = \left(\frac{T_1}{\sum_{j=1}^{k} T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^{k} T_j} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_k}{\sum_{j=1}^{k} T_j} \cdot_\varepsilon (\alpha_k) \right)$$

$$= \left\langle \frac{\prod_{j=1}^{k}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k} T_j}} - \prod_{j=1}^{k}\left(1-\mu_{\alpha_j}\right)^{\frac{T_j}{\sum_{j=1}^{k} T_j}}}{\prod_{j=1}^{k}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k} T_j}} + \prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k} T_j}}}, \frac{2\prod_{j=1}^{k} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k} T_j}}}{\prod_{j=1}^{k}\left(2-\nu_{\alpha_j}\right)^{\frac{T_j}{\sum_{j=1}^{k} T_j}} + \prod_{j=1}^{k} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k} T_j}}} \right\rangle. \tag{11}$$

When $n=k+1$, by the Einstein operational laws of the IFNs, we have

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_{k+1}) = \left(\frac{T_1}{\sum_{j=1}^{k+1} T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^{k+1} T_j} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j} \cdot_\varepsilon (\alpha_k) \right) \oplus_\varepsilon \frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j} \cdot_\varepsilon (\alpha_{k+1})$$

$$= \left\langle \frac{\prod_{j=1}^{k}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} - \prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}{\prod_{j=1}^{k}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} + \prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}, \frac{2\prod_{j=1}^{k} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}{\prod_{j=1}^{k}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} + \prod_{j=1}^{k} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}} \right\rangle$$
$$\oplus \left\langle \frac{(1+\mu_{\alpha_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} - (1-\mu_{\alpha_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}}}{(1+\mu_{\alpha_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} + (1-\mu_{\alpha_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}}}, \frac{2\nu_{\alpha_{k+1}}^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}}}{(2-\nu_{\alpha_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}} + \nu_{\alpha_{k+1}}^{\frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}}} \right\rangle$$

$$= \left\langle \frac{\prod_{j=1}^{k+1}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} - \prod_{j=1}^{k+1}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}{\prod_{j=1}^{k+1}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} + \prod_{j=1}^{k+1}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}, \frac{2\prod_{j=1}^{k+1} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}}{\prod_{j=1}^{k+1}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}} + \prod_{j=1}^{k+1} \nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{k+1} T_j}}} \right\rangle, \tag{12}$$

i.e.,(7) holds for $n=k+1$. Therefore by mathematical induction (7) holds for all n.
This completes the proof of the Theorem 1.

The IFEPWA operator has the following properties:

Theorem 2. (Idempotency): Let $\alpha_j = \left\langle \mu_{\alpha_j}, \nu_{\alpha_j} \right\rangle$, $j=1,2,\ldots n$, be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k), j=2,3,\ldots,n$, with $T_1 = 1$. Also, let $S^*(\alpha_k)$ be the score of $\alpha_k = \left\langle \mu_{\alpha_k} \nu_{\alpha_k} \right\rangle$. If all the intuitionistic fuzzy numbers α_j, $j=1,2,\ldots,n$, are equal, i.e., $\alpha_j = \alpha = \langle \mu_\alpha, \nu_\alpha \rangle, \forall j$, then

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha. \tag{13}$$

Proof: From Definition 7, we have

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = IFEPWA\Big(\alpha,\alpha,\ldots,\alpha\Big)$$
$$= \left(\frac{T_1}{\sum_{j=1}^{n} T_j} \cdot_\varepsilon \alpha \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^{n} T_j} \cdot_\varepsilon \alpha \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_n}{\sum_{j=1}^{n} T_j} \cdot_\varepsilon \alpha \right)$$
$$= \left\langle \frac{\prod_{j=1}^{n}(1+\mu_\alpha)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} - \prod_{j=1}^{n}(1-\mu_\alpha)^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}{\prod_{j=1}^{n}(1+\mu_\alpha)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} + \prod_{j=1}^{n}(1-\mu_\alpha)^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}, \frac{2\prod_{j=1}^{n} \nu_\alpha^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}{\prod_{j=1}^{n}(2-\nu_\alpha)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} + \prod_{j=1}^{n} \nu_\alpha^{\frac{T_j}{\sum_{j=1}^{n} T_j}}} \right\rangle,$$

$$= \left\langle \frac{1+\mu_\alpha - 1 + \mu_\alpha}{1+\mu_\alpha + 1 - \mu_\alpha}, \frac{2\nu_\alpha}{2-\nu_\alpha+\nu_\alpha} \right\rangle = \langle \mu_\alpha, \nu_\alpha \rangle = \alpha. \quad (14)$$

This proves the theorem.

Corollary 1: If $\alpha_j = \left\langle \mu_{\alpha_j}, \nu_{\alpha_j} \right\rangle$, $j = 1,2,\ldots,n$, is a collection of the largest IFNs, i.e., $\alpha_j = \alpha^* = \langle 1,0 \rangle, \forall j$, then

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = IFEPWA(\alpha^*,\alpha^*,\ldots,\alpha^*) = \langle 1,0 \rangle \quad (15)$$

Proof: Corollary 1 follows directly from Theorem 2.

Corollary 2. (Non-compensatory): If $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ is the smallest IFN, i.e., $\alpha_1 = \alpha_* = \langle 0,1 \rangle$, then

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = IFEPWA(\alpha_*,\alpha_2,\ldots,\alpha_n) = \langle 0,1 \rangle. \quad (16)$$

Proof: Since $\alpha_1 = \alpha_* = \langle 0,1 \rangle$, then by definition of the score function, we have

$$S^*(\alpha_1) = 0. \quad (17)$$

Further,

$$T_j = \prod_{k=1}^{j-1} S^*(\alpha_k), \quad j = 2,3,\ldots,n, \quad \text{and} \quad T_1 = 1. \quad (18)$$

So that from Equations (17) and (18), we have

$$T_j = \prod_{k=1}^{j-1} S^*(\alpha_k) = S^*(\alpha_1) \times S^*(\alpha_2) \times \cdots \times S^*(\alpha_{j-1}) \quad (19)$$

$$= 0 \times S^*(\alpha_2) \times \cdots \times S^*(\alpha_{j-1}) = 0, \quad j = 2,3,\ldots,n, \quad (20)$$

and

$$\sum_{j=1}^{n} T_j = 1. \quad (21)$$

By Definition 7, we have

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{T_1}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{T_2}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \frac{T_n}{\sum_{j=1}^n T_j} \cdot_\varepsilon \alpha_n \right)$$
$$= \left(\frac{1}{1} \cdot_\varepsilon \alpha_1 \oplus_\varepsilon \frac{0}{1} \cdot_\varepsilon \alpha_2 \oplus_\varepsilon, \ldots \frac{0}{1} \cdot_\varepsilon \alpha_n \right)$$
$$= \alpha_1 = \alpha_* = \langle 0,1 \rangle. \quad (22)$$

This corollary carries a very significant conclusion. According to it highest priority attribute in decision making is not compromised. This amount to saying that if the highest priority attribute is not met than other attributes even if they are partially/fully met will not matter. That is, they will have no contribution to the final decision.

Theorem 3. (Monotonicity): Let $\alpha_j = \left\langle \mu_{\alpha_j}, \nu_{\alpha_j} \right\rangle$ and $\alpha'_j = \left\langle \mu_{\alpha'_j}, \nu_{\alpha'_j} \right\rangle$, $j = 1,2,\ldots,n$, be two sets of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $T'_j = \prod_{k=1}^{j-1} S^*(\alpha'_k)$, $j = 2,3,\ldots,n$, $T_1 = T'_1 = 1$. Also let $S^*(\alpha_k)$ and $S^*(\alpha'_k)$ be the scores of $\alpha_k = \left\langle \mu_{\alpha_k}, \nu_{\alpha_k} \right\rangle$ and $\alpha'_k = \left\langle \mu_{\alpha'_k}, \nu_{\alpha'_k} \right\rangle$ respectively. If $\alpha_j \le \alpha'_j$, i.e., $\mu_{\alpha_j} \le \mu'_{\alpha_j}$ and $\nu_{\alpha_j} \ge \nu'_{\alpha_j}$ for all j, then

$$IFEPWA(\alpha_1,\alpha_2,\ldots\alpha_n) \le IFEPWA(\alpha'_1,\alpha'_2,\ldots,\alpha'_n). \quad (23)$$

Proof: We know that $f(x) = \frac{1-x}{1+x}$, $x \in [0,1]$ is a decreasing function of x. If $\mu_{\alpha_j} \le \mu_{\alpha'_j}$, for all j, then $f(\mu_{\alpha'_j}) \le f(\mu_{\alpha_j})$ i.e., $\frac{1-\mu_{\alpha'_j}}{1+\mu_{\alpha'_j}} \le \frac{1-\mu_{\alpha_j}}{1+\mu_{\alpha_j}}$ for all j.
Now let

$$w = \left(\frac{T_1}{\sum_{j=1}^n T_j}, \frac{T_2}{\sum_{j=1}^n T_j}, \frac{T_3}{\sum_{j=1}^n T_j}, \cdots, \frac{T_n}{\sum_{j=1}^n T_j} \right)^T,$$

and

$$w' = \left(\frac{T'_1}{\sum_{j=1}^n T'_j}, \frac{T'_2}{\sum_{j=1}^n T'_j}, \frac{T'_3}{\sum_{j=1}^n T'_j}, \cdots, \frac{T'_n}{\sum_{j=1}^n T'_j} \right)^T,$$

be the prioritized weight vectors of $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ and $\alpha'_j = (\mu_{\alpha'_j}, \nu_{\alpha'_j})$, $j = 1,2,\ldots,n$, such that $\frac{T_j}{\sum_{j=1}^n T_j}, \frac{T'_j}{\sum_{j=1}^n T'_j} \in [0,1]$ with the condition $\sum_{j=1}^n \left(\frac{T_j}{\sum_{j=1}^n T_j} \right) = 1$,
$\sum_{j=1}^n \left(\frac{T'_j}{\sum_{j=1}^n T'_j} \right) = 1$.
From the result above, we have

$$\left(\frac{1-\mu_{\alpha'_j}}{1+\mu_{\alpha'_j}} \right)^{\frac{T'_j}{\sum_{j=1}^n T'_j}} \le \left(\frac{1-\mu_{\alpha_j}}{1+\mu_{\alpha_j}} \right)^{\frac{T_j}{\sum_{j=1}^n T_j}}, \ j = 1,2,\ldots,n. \quad (24)$$

Thus

$$\prod_{j=1}^n \left(\frac{1-\mu_{\alpha'_j}}{1+\mu_{\alpha'_j}} \right)^{\frac{T'_j}{\sum_{j=1}^n T'_j}} \le \prod_{j=1}^n \left(\frac{1-\mu_{\alpha_j}}{1+\mu_{\alpha_j}} \right)^{\frac{T_j}{\sum_{j=1}^n T_j}}$$

$$\Leftrightarrow \frac{2}{1+\prod_{j=1}^n \left(\frac{1-\mu_{\alpha_j}}{1+\mu_{\alpha_j}} \right)^{\frac{T_j}{\sum_{j=1}^n T_j}}} - 1 \le \frac{2}{1+\prod_{j=1}^n \left(\frac{1-\mu_{\alpha'_j}}{1+\mu_{\alpha'_j}} \right)^{\frac{T'_j}{\sum_{j=1}^n T'_j}}} - 1 \quad (25)$$

i.e.,

$$\frac{\prod_{j=1}^n (1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}} - \prod_{j=1}^n (1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}}}{\prod_{j=1}^n (1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}} + \prod_{j=1}^n (1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^n T_j}}}$$
$$\le \frac{\prod_{j=1}^n (1+\mu_{\alpha'_j})^{\frac{T'_j}{\sum_{j=1}^n T'_j}} - \prod_{j=1}^n (1-\mu_{\alpha'_j})^{\frac{T'_j}{\sum_{j=1}^n T'_j}}}{\prod_{j=1}^n (1+\mu_{\alpha'_j})^{\frac{T'_j}{\sum_{j=1}^n T'_j}} + \prod_{j=1}^n (1-\mu_{\alpha'_j})^{\frac{T'_j}{\sum_{j=1}^n T'_j}}}. \quad (26)$$

Next using $g(y) = \frac{2-y}{y}$, $y \in [0,1]$, a decreasing function of y, if $\nu_{\alpha_j} \ge \nu_{\alpha'_j}$ for all j, then $g(\mu_{\alpha'_j}) \ge g(\mu_{\alpha_j})$ i.e., $\frac{2-\nu_{\alpha'_j}}{\nu_{\alpha'_j}} \ge \frac{2-\nu_{\alpha_j}}{\nu_{\alpha_j}}$ for all j.
Then

$$\left(\frac{2-\nu_{\alpha'_j}}{\nu_{\alpha'_j}} \right)^{\frac{T'_j}{\sum_{j=1}^n T'_j}} \ge \left(\frac{2-\nu_{\alpha_j}}{\nu_{\alpha_j}} \right)^{\frac{T_j}{\sum_{j=1}^n T_j}}, \ j = 1,2,\ldots,n. \quad (27)$$

Thus

$$\prod_{j=1}^n \left(\frac{2-\nu_{\alpha'_j}}{\nu_{\alpha'_j}} \right)^{\frac{T'_j}{\sum_{j=1}^n T'_j}} \ge \prod_{j=1}^n \left(\frac{2-\nu_{\alpha_j}}{\nu_{\alpha_j}} \right)^{\frac{T_j}{\sum_{j=1}^n T_j}} \quad (28)$$

$$\Leftrightarrow \frac{2}{\prod_{j=1}^{n}\left(\frac{2-\nu_{\alpha_j}}{\nu_{\alpha_j}}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+1} \geq \frac{2}{\prod_{j=1}^{n}\left(\frac{2-\nu_{\alpha'_j}}{\nu_{\alpha'_j}}\right)^{\frac{T'_j}{\sum_{j=1}^{n}T'_j}}+1} \tag{29}$$

i.e.,

$$\frac{2\prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \geq \frac{2\prod_{j=1}^{n}\nu_{\alpha'_j}^{\frac{T'_j}{\sum_{j=1}^{n}T'_j}}}{\prod_{j=1}^{n}\left(2-\nu_{\alpha'_j}\right)^{\frac{T'_j}{\sum_{j=1}^{n}T'_j}}+\prod_{j=1}^{n}\nu_{\alpha'_j}^{\frac{T'_j}{\sum_{j=1}^{n}T'_j}}}. \tag{30}$$

Note that (30) also holds even if $\nu_{\alpha_j} = \nu_{\alpha'_j} = 0$ for all j. Then, according to Definition 5, we obtain that

$$IFGEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq IFEPWA(\alpha'_1, \alpha'_2, \ldots, \alpha'_n). \tag{31}$$

This proves the theorem.

Theorem 4. (Boundedness): Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1, 2, \ldots, n$ be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $k = 2, 3, \ldots, n$, $T_1 = 1$ and $S^*(\alpha_k)$ be the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$. Also, let

$$\alpha^- = \left\langle \min_j \mu_{\alpha_j}, \max_j \nu_{\alpha_j} \right\rangle \text{ and } \alpha^+ = \left\langle \max_j \mu_{\alpha_j}, \min_j \nu_{\alpha_j} \right\rangle. \tag{32}$$

Then

$$\alpha^- \leq IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+. \tag{33}$$

Proof: It directly follows from Theorem 3.

Relationship between intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator and intuitionistic fuzzy prioritized weighted average (IFPWA) operator [45]:

In the next theorem, we prove a relation between the IFEPWA operator and IFPWA operator proposed by Yu [45] as follows

$$\begin{aligned} IFPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(\frac{T_1}{\sum_{j=1}^{n}T_j}\alpha_1 \oplus \frac{T_2}{\sum_{j=1}^{n}T_j}\alpha_2 \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n}T_i}\alpha_n\right) \\ &= \left\langle 1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}, \prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \right\rangle. \end{aligned} \tag{34}$$

Theorem 5. Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1, 2, \ldots, n$, be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $j = 2, 3, \ldots, n$, $T_1 = 1$ and $S^*(\alpha_k)$ be the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$. Then

$$IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq IFPWA(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{35}$$

with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n$.

Proof: Using weighted AM-GM inequality [15,41], we have

$$\begin{aligned} &\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} + \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \\ &\leq \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(1+\mu_{\alpha_j})\right) + \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(1-\mu_{\alpha_j})\right) = 2, \end{aligned} \tag{36}$$

then

$$\begin{aligned} &\frac{\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} + \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \\ &= 1 - \frac{2\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} + \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \\ &\leq 1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}, \end{aligned} \tag{37}$$

where the equality holds if and only if $\mu_{\alpha_1} = \mu_{\alpha_2} = \mu_{\alpha_3} = \cdots = \mu_{\alpha_n}$.

In addition, since

$$\begin{aligned} &\prod_{j=1}^{n}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} + \prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \\ &\leq \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(2-\nu_{\alpha_j})\right) + \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{i=1}^{n}T_j}\nu_{\alpha_j}\right) = 2, \end{aligned} \tag{38}$$

we have

$$\frac{2\prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(2-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} + \prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \geq \prod_{j=1}^{n}\nu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}, \tag{39}$$

where the equality holds if and only if $\nu_{\alpha_1} = \nu_{\alpha_2} = \nu_{\alpha_3} = \cdots = \nu_{\alpha_n}$.

Then, according to Definition 5, we obtain that

$$IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq IFPWA(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{40}$$

with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n$.

This proves the theorem.

Special cases of IFEPWA operator:

(i) If the priority levels of the aggregated arguments reduced to the same level, then the IFEPWA operator reduces to the intuitionistic fuzzy Einstein weighted average (IFEWA) operator [19]:

$$\begin{aligned} IFEPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) &= (w_1 \cdot_\varepsilon \alpha_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \alpha_2 \oplus_\varepsilon \cdots \oplus_\varepsilon w_n \cdot_\varepsilon \alpha_n), \\ &= \left\langle \frac{\prod_{j=1}^{n}\left(1+\mu_{\alpha_j}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\alpha_j}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}\right)^{w_j}}, \right. \\ &\qquad \left. \frac{2\prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}}{\prod_{j=1}^{n}\left(2-\nu_{\alpha_j}\right)^{w_j} + \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}} \right\rangle. \end{aligned} \tag{41}$$

(ii) If $\nu_{\alpha_j} = 1-\mu_{\alpha_j}$ for all $j = 1,2,3,\ldots,n$, and the priority levels of the aggregated arguments reduced to the same level, then the IFEPWA operator gives

$$IFEPWA(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left\langle \frac{\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{w_j} - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}}{\prod_{j=1}^{n}(1+\mu_{\alpha_j})^{w_j} + \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}} \right\rangle, \quad (42)$$

which we call the fuzzy Einstein weighted average (IFEA) operator.

From the geometric perspective, here we define the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator.

Definition 8. ***Intuitionistic Fuzzy Einstein Prioritized Weighted Geometric (IFEPWG) Operator***: Given a set of intuitionistic fuzzy numbers, $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$, the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator is defined by

$$\begin{aligned} IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) &= \bigotimes_{\varepsilon\, j=1}^{n} (\alpha_j)^{\wedge\varepsilon \frac{T_j}{\sum_{j=1}^{n} T_j}} \\ &= \left((\alpha_1)^{\wedge\varepsilon \frac{T_1}{\sum_{j=1}^{n} T_j}} \otimes_\varepsilon (\alpha_2)^{\wedge\varepsilon \frac{T_2}{\sum_{j=1}^{n} T_j}} \otimes_\varepsilon \cdots \otimes_\varepsilon (\alpha_n)^{\wedge\varepsilon \frac{T_n}{\sum_{j=1}^{n} T_j}} \right), \end{aligned} \quad (43)$$

where $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k), j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ is the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$.

Next, based on the Einstein operational laws of IFNs, we have the following theorem:

Theorem 6. Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$ be a set of intuitionistic fuzzy numbers, then using the IFEPWG operator the aggregated value is also an intuitionistic fuzzy number, and

$$\begin{aligned} &IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ &= \left((\alpha_1)^{\wedge\varepsilon \frac{T_1}{\sum_{j=1}^{n} T_j}} \otimes_\varepsilon (\alpha_2)^{\wedge\varepsilon \frac{T_2}{\sum_{j=1}^{n} T_j}} \otimes_\varepsilon \cdots \otimes_\varepsilon (\alpha_n)^{\wedge\varepsilon \frac{T_n}{\sum_{j=1}^{n} T_j}} \right) \\ &= \left\langle \frac{2\prod_{j=1}^{n} \mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}} + \prod_{j=1}^{n} \mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}, \right. \\ &\qquad \left. \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}} - \prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}} + \prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}}} \right\rangle, \end{aligned} \quad (44)$$

where $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k), j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ is the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$.

Proof: It can be proved on lines similar to that of Theorem 1.

Some other properties of the IFEPWG operator are proved in the following theorems:

Theorem 7. (Idempotency): Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots n$, be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k), j = 2,3,\ldots,n$, with $T_1 = 1$. Also, let $S^*(\alpha_k)$ be the score of $\alpha_k = \langle \mu_{\alpha_k} \nu_{\alpha_k} \rangle$. If all the intuitionistic fuzzy numbers α_j, $j = 1,2,\ldots,n$, are equal, i.e., $\alpha_j = \alpha = \langle \mu_\alpha, \nu_\alpha \rangle, \forall i$, then

$$IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha. \quad (45)$$

Proof: The proof of Theorem 7 is similar to that of Theorem 2.

Theorem 8. (Monotonicity): Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$ and $\alpha'_j = \langle \mu_{\alpha'_j}, \nu_{\alpha'_j} \rangle$, $j = 1,2,\ldots,n$, be two sets of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $T'_j = \prod_{k=1}^{j-1} S^*(\alpha'_k)$, $j = 2,3,\ldots,n$, $T_1 = T'_1 = 1$. Also let $S^*(\alpha_k)$ and $S^*(\alpha'_k)$ be the scores of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$ and $\alpha'_k = \langle \mu_{\alpha'_k}, \nu_{\alpha'_k} \rangle$ respectively. If $\alpha_j \le \alpha'_j$, i.e., $\mu_{\alpha_j} \le \mu'_{\alpha_j}$ and $\nu_{\alpha_j} \ge \nu'_{\alpha_j}$ for all j, then

$$IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \le IFEPWG(\alpha'_1,\alpha'_2,\ldots,\alpha'_n). \quad (46)$$

Proof: The proof of Theorem 8 is similar to Theorem 3.

Theorem 9. (Boundedness): Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$, be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ be the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$. Also, let

$$\alpha^- = \left\langle \min_j \mu_{\alpha_j}, \max_j \nu_{\alpha_j} \right\rangle \text{ and } \alpha^+ = \left\langle \max_j \mu_{\alpha_j}, \min_j \nu_{\alpha_j} \right\rangle. \quad (47)$$

Then

$$\alpha^- \le IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \le \alpha^+. \quad (48)$$

Proof: It directly follows from Theorem 8.

Relation between intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator and intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator [45].

Based on the algebraic laws on IFNs, Yu [45] defined the IFPWG operator as follows

$$\begin{aligned} IFPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) &= \left(\alpha_1^{\frac{T_1}{\sum_{j=1}^{n} T_j}} \oplus \alpha_2^{\frac{T_2}{\sum_{j=1}^{n} T_j}} \oplus \cdots \oplus \alpha_n^{\frac{T_n}{\sum_{j=1}^{n} T_j}} \right) \\ &= \left\langle \prod_{j=1}^{n} \mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n} T_j}}, 1 - \prod_{j=1}^{n} (1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\rangle. \end{aligned} \quad (49)$$

We have the following additional theorem:

Theorem 10. Let $\alpha_j = \langle \mu_{\alpha_j}, \nu_{\alpha_j} \rangle$, $j = 1,2,\ldots,n$, be a set of intuitionistic fuzzy numbers, $T_j = \prod_{k=1}^{j-1} S^*(\alpha_k)$, $j = 2,3,\ldots,n$, $T_1 = 1$ and $S^*(\alpha_k)$ be the score of $\alpha_k = \langle \mu_{\alpha_k}, \nu_{\alpha_k} \rangle$. Then

$$IFPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \le IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n), \quad (50)$$

with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n$.

Proof: Using weighted AM-GM inequality [14,41], we have

$$\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \leq \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(2-\mu_{\alpha_j})\right)+\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\mu_{\alpha_j}\right)=2,$$

or

$$\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \geq \prod_{j=1}^{n}\mu_{\alpha_j}^{\frac{T_j}{\sum_{j=1}^{n}T_j}}, \tag{51}$$

where the inequality holds if and only if $\mu_{\alpha_1}=\mu_{\alpha_2}=\mu_{\alpha_3}=\cdots=\mu_{\alpha_n}$.

Additionally, since

$$\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \leq \sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(1+\nu_{\alpha_j})\right)+\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(1-\nu_{\alpha_j})\right)=2,$$

or

$$\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} = 1-\frac{2\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\frac{T_j}{\sum_{j=1}^{n}T_j}}} \leq 1-\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \tag{52}$$

where the inequality holds if and only if $\nu_{\alpha_1}=\nu_{\alpha_2}=\nu_{\alpha_3}=\cdots=\nu_{\alpha_n}$.

Then, according to Definition 5, we get

$$IFPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \leq IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n) \tag{53}$$

with equality if and only if $\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_n$.

This proves the theorem.

Special cases of IFEPWG operator:

(i) If the priority levels of the aggregated arguments are reduced to the same level, then the IFEPWG operator (43) reduces to the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator [20]:

$$IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n)=((\alpha_1)^{\wedge\varepsilon w_i}\otimes_\varepsilon(\alpha_2)^{\wedge\varepsilon w_i}\otimes_\varepsilon\cdots\otimes_\varepsilon(\alpha_n)^{\wedge\varepsilon w_i}) = \left\langle\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}}{\prod_{j=1}^{n}\left(2-\mu_{\alpha_j}\right)^{w_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+\nu_{\alpha_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}\right)^{w_i}}{\prod_{j=1}^{n}\left(1+\nu_{\alpha_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}\right)^{w_j}}\right\rangle. \tag{54}$$

(ii) If $\nu_{\alpha_j}=1-\mu_{\alpha_j}\ \forall j=1,2,\ldots,n$, and the priority levels of the aggregated arguments are reduced to the same level, then the IFEPWG operator gives

$$IFEPWG(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{w_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}}\right\rangle, \tag{55}$$

which we call the fuzzy Einstein weighted geometric (FEWG) operator.

In the following section, we suggest an application of the proposed aggregation operators to multiple attribute group decision making problems with intuitionistic fuzzy information and give an illustrative example.

4 An Approach to Multiple Attribute Group Decision Making under Intuitionistic Fuzzy Environment

Let us consider a multiple attribute group decision making problem involving a set of options $X=\{X_1,X_2,...,X_m\}$ to be considered under a set of attributes $G=\{G_1,G_2,...,G_n\}$ and let there be a prioritization between the attributes expressed by the linear ordering $G_1\succ G_2\succ\cdots\succ G_n$ (indicating attribute G_j has a higher priority than G_s, if $j<s$), and let $D=\{D_1,D_2,...,D_q\}$ be the set of decision makers and let there be a prioritization between the decision makers expressed by the linear ordering $D_1\succ D_2\succ\cdots\succ D_q$ (indicating decision maker D_η has a higher priority than D_ς, if $\eta<\varsigma$). Let $A^{(k)}=\left(\alpha_{ij}^{(k)}\right)_{m\times n}=\left(\left\langle\mu_{ij}^{(k)},\nu_{ij}^{(k)}\right\rangle\right)_{m\times n}$ be an intuitionistic fuzzy decision matrix, and $\alpha_{ij}^{(k)}=\left\langle\mu_{ij}^{(k)},\nu_{ij}^{(k)}\right\rangle$ be an attribute value provided by the decision maker $D_k\in D$, which is expressed in an IFN, where $\mu_{ij}^{(k)}$ indicates the degree that the option $X_i\in X$ satisfies the attribute $G_j\in G$ expressed by the decision maker D_k, and $\nu_{ij}^{(k)}$ indicates the degree that the option $X_i\in X$ does not satisfies the attribute $G_j\in G$ expressed by the decision maker D_k, such that

$$\mu_{ij}^{(k)}\in[0,1],\nu_{ij}^{(k)}\in[0,1],\mu_{ij}^{(k)}+\nu_{ij}^{(k)}\leq 1, \quad i=1,2,...,m;\ j=1,2,...,n. \tag{56}$$

To harmonize the data, first step is to look at the attributes. These in general can be of different types. If all the attributes $G=\{G_1,G_2,\ldots,G_n\}$ are of the same type, then the attribute values do not need harmonization. However if these involve different scales and/or units, there is need to convert them all to the same scale and/or unit. Just to make this point clear, let us consider two types of attributes, namely, (i) cost type and the (ii) benefit type. Considering their natures, a benefit attribute (the bigger the values better is it) and cost attribute (the smaller the values the better is it) are of rather opposite type. In such cases, we need to first transform the attribute values of cost type into the attribute values of benefit type. So, transform the intuitionistic fuzzy decision matrix $A^{(k)}=\left(\alpha_{ij}^{(k)}\right)_{m\times n}$ into the normalized intuitionistic fuzzy decision matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n}$ using the method given by Xu and Hu [40], where $r_{ij}^{(k)} = \left\langle \mu_{ij}^{(k)}, \nu_{ij}^{(k)} \right\rangle$ and

$$r_{ij}^{(k)} = \begin{cases} \alpha_{ij}^{(k)}, & \text{for benefit attribute } G_j \\ \left(\alpha_{ij}^{(k)}\right)^C, & \text{for cost attribute } G_j \end{cases}, \quad i=1,2,\ldots,m\,; \;\; j=1,2,\ldots,n, \tag{57}$$

where $\left(\alpha_{ij}^{(k)}\right)^C$ is the complement of $\alpha_{ij}^{(k)}$, such that $\left(\alpha_{ij}^{(k)}\right)^C = \left\langle \nu_{ij}^{(k)}, \mu_{ij}^{(k)} \right\rangle$.

With attributes harmonized and using the IFEPWA / IFEPWG operator, we now formulate an algorithm to solve multiple attribute group decision making problems with intuitionistic fuzzy information:

Step 1. Calculate the values of $T_{ij}^{(k)}$, $k = 1,2,\ldots,q$, as follows

$$T_{ij}^{(k)} = \prod_{\gamma=1}^{k-1} S^*(\tilde{r}_{ij}^{\gamma}), \quad k = 2,3,\ldots,q, \tag{58}$$

$$T_{ij}^{(1)} = 1. \tag{59}$$

Step 2. Utilize IFEPWA operator:

$$\begin{aligned} r_{ij} &= \langle \mu_{ij}, \nu_{ij} \rangle \\ &= IFEPWA(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(q)}) \\ &= \left(\frac{T_{ij}^{(1)}}{\sum_{k=1}^{q} T_{ij}^{(k)}} \cdot_{\varepsilon} \left(r_{ij}^{(1)}\right) \oplus_{\varepsilon} \frac{T_{ij}^{(2)}}{\sum_{k=1}^{q} T_{ij}^{(k)}} \cdot_{\varepsilon} \left(r_{ij}^{(2)}\right) \oplus_{\varepsilon} \cdots \oplus_{\varepsilon} \frac{T_{ij}^{(q)}}{\sum_{k=1}^{q} T_{ij}^{(k)}} \cdot_{\varepsilon} \left(r_{ij}^{(q)}\right) \right) \\ &= \left\langle \frac{\prod_{k=1}^{q} \left(1+\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} - \prod_{k=1}^{q} \left(1-\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}{\prod_{k=1}^{q} \left(1+\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} + \prod_{k=1}^{q} \left(1-\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}, \right. \\ &\qquad \left. \frac{2\prod_{k=1}^{q} \left(\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}{\prod_{k=1}^{q} \left(2-\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} + \prod_{k=1}^{q} \left(\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}} \right\rangle, \end{aligned} \tag{60}$$

or the IFEPWG operator:

$$\begin{aligned} r_{ij} &= \langle \mu_{ij}, \nu_{ij} \rangle \\ &= IFEPWG(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(q)}) \\ &= (r_{ij}^{(1)\wedge\varepsilon \frac{T_{ij}^{(1)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} \otimes_{\varepsilon} (r_{ij}^{(2)})^{\wedge\varepsilon \frac{T_{ij}(2)}{\sum_{k=1}^{q} T_{ij}^{(k)}}} \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} (r_{ij}^{(q)})^{\wedge\varepsilon \frac{T_{ij}^{(q)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}) \\ &= \left\langle \frac{2\prod_{k=1}^{q} \left(\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}{\prod_{k=1}^{q} \left(2-\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} + \prod_{k=1}^{q} \left(\mu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}, \right. \\ &\qquad \left. \frac{\prod_{k=1}^{q} \left(1+\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} - \prod_{k=1}^{q} \left(1-\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}}{\prod_{k=1}^{q} \left(1+\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} + \prod_{k=1}^{q} \left(1-\nu_{ij}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}} \right\rangle, \end{aligned} \tag{61}$$

to aggregate all the individual intuitionistic fuzzy decision matrices $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n}$, $k = 1,2,\ldots,q$, into a collective intuitionistic fuzzy decision matrix $R = (r_{ij})_{m\times n}$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

Step 3. Calculate the values T_{ij}, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, as follows

$$T_{ij} = \prod_{v=1}^{j-1} S^*(r_{iv}), \quad i = 1,2,\ldots,m; \;\; j = 2,3,\ldots,n, \tag{62}$$

$$T_{i1} = 1, \qquad i = 1,2,\ldots,m. \tag{63}$$

Step 4. Aggregate all intuitionistic fuzzy preference values r_{ij}, $j = 1,2,\ldots,n$, by the IFEPWA / IFEPWG operator:

$$\begin{aligned} r_i &= \langle \mu_i, \nu_i \rangle \\ &= IFEPWA(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}} \cdot_{\varepsilon} (r_{i1}) \oplus_{\varepsilon} \frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}} \cdot_{\varepsilon} (r_{i2}) \oplus_{\varepsilon} \cdots \oplus_{\varepsilon} \frac{T_{in}}{\sum_{j=1}^{n} T_{ij}} \cdot_{\varepsilon} (r_{in}) \right) \\ &= \left\langle \frac{\prod_{j=1}^{n} \left(1+\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n} \left(1-\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1+\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(1-\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right. \\ &\qquad \left. \frac{2\prod_{j=1}^{n} \left(\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(2-\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}} \right\rangle, \\ &\qquad i = 1,2,\ldots,m. \end{aligned} \tag{64}$$

or

$$\begin{aligned} r_i &= \langle \mu_i, \nu_i \rangle \\ &= IFEPWG(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left((r_{i1})^{\wedge\varepsilon \frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}}} \otimes_{\varepsilon} (r_{i2})^{\wedge\varepsilon \frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}}} \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} (r_{in})^{\wedge\varepsilon \frac{T_{in}}{\sum_{j=1}^{n} T_{ij}}} \right) \\ &= \left\langle \frac{2\prod_{j=1}^{n} \left(\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(2-\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(\mu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right. \\ &\qquad \left. \frac{\prod_{j=1}^{n} \left(1+\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n} \left(1-\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1+\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(1-\nu_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}} \right\rangle, \\ &\qquad i = 1, 2, \ldots, m, \end{aligned} \tag{65}$$

to derive the overall intuitionistic fuzzy preference values r_i, $i = 1,2,\ldots,m$, of the options X_i, $i = 1,2,\ldots,n$.

Step 5. Calculate the score values as follows:

$$S^*(r_i) = \frac{1+\mu_{r_i} - \nu_{r_i}}{2}, \quad i = 1,2,\ldots,m. \tag{66}$$

Step 6. Rank all the options X_i, $i = 1,2,\ldots,m$, according to the score values $S^*(r_i)$, $i = 1,2,\ldots,m$, in descending order. The leading X_i, with the highest value of $S^*(r_i)$, is the best option.

In order to demonstrate the applicability of the proposed method to multiple attribute group decision making, we consider below a university faculty recruitment group decision making problem.

Example: The department of mathematics in a university wants to appoint outstanding mathematics teachers. The appointment is done by a committee of three decision makers, President (D_1), Dean of Academics (D_2) and Human Resource Officer (D_3). After preliminary screening, five teachers X_i , $i = 1,2,3,4,5$, remain for further evaluation. Panel of decision makers made strict evaluation for five teachers X_i, according to the following four attributes: G_1, the past experience; G_2, the research capability; G_3, subject knowledge; G_4, the teaching skill. During the evaluation process, the university President (D_1) has the absolute priority for decision making, Dean of Academics comes next. The prioritization relationship for the attributes is as follows $G_1 \succ G_2 \succ G_3 \succ G_4$. The three decision makers evaluated the candidates X_i, $i = 1,2,3,4,5$ with respect to the attributes G_j, $j = 1,2,3,4$, and provided their evaluation values in terms of intuitionistic fuzzy numbers and constructed the following three intuitionistic fuzzy decision matrices $A^{(k)} = \left(a_{ij}^{(k)}\right)_{5\times4}$, $k = 1,2,3$, (see Tables 1-3)

Table 1: Intuitionistic fuzzy decision matrix $A^{(1)}$

	G_1	G_2	G_3	G_4
X_1	$\langle 0.5,0.5\rangle$	$\langle 0.7,0.3\rangle$	$\langle 0.8,0.1\rangle$	$\langle 0.9,0.1\rangle$
X_2	$\langle 0.8,0.1\rangle$	$\langle 0.9,0.1\rangle$	$\langle 0.7,0.3\rangle$	$\langle 0.7,0.2\rangle$
X_3	$\langle 0.9,0.0\rangle$	$\langle 0.8,0.0\rangle$	$\langle 0.8,0.2\rangle$	$\langle 0.6,0.3\rangle$
X_4	$\langle 0.7,0.1\rangle$	$\langle 0.9,0.0\rangle$	$\langle 0.8,0.1\rangle$	$\langle 0.4,0.6\rangle$
X_5	$\langle 0.9,0.0\rangle$	$\langle 0.4,0.5\rangle$	$\langle 0.8,0.2\rangle$	$\langle 0.7,0.3\rangle$

Table 2: Intuitionistic fuzzy decision matrix $A^{(2)}$

	G_1	G_2	G_3	G_4
X_1	$\langle 0.7,0.2\rangle$	$\langle 0.9,0.0\rangle$	$\langle 0.7,0.1\rangle$	$\langle 0.3,0.6\rangle$
X_2	$\langle 0.6,0.3\rangle$	$\langle 0.7,0.2\rangle$	$\langle 0.9,0.0\rangle$	$\langle 0.5,0.5\rangle$
X_3	$\langle 0.8,0.1\rangle$	$\langle 0.6,0.3\rangle$	$\langle 0.7,0.2\rangle$	$\langle 0.4,0.6\rangle$
X_4	$\langle 0.8,0.1\rangle$	$\langle 0.7,0.1\rangle$	$\langle 0.7,0.2\rangle$	$\langle 0.7,0.1\rangle$
X_5	$\langle 0.7,0.1\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.9,0.0\rangle$	$\langle 0.7,0.1\rangle$

Table 3: Intuitionistic fuzzy decision matrix $A^{(3)}$

	G_1	G_2	G_3	G_4
X_1	$\langle 0.8,0.1\rangle$	$\langle 0.8,0.1\rangle$	$\langle 0.5,0.3\rangle$	$\langle 0.8,0.1\rangle$
X_2	$\langle 0.7,0.0\rangle$	$\langle 0.6,0.2\rangle$	$\langle 0.8,0.2\rangle$	$\langle 0.5,0.3\rangle$
X_3	$\langle 0.6,0.1\rangle$	$\langle 0.7,0.2\rangle$	$\langle 0.7,0.0\rangle$	$\langle 0.5,0.5\rangle$
X_4	$\langle 0.8,0.2\rangle$	$\langle 0.7,0.1\rangle$	$\langle 0.6,0.4\rangle$	$\langle 0.6,0.4\rangle$
X_5	$\langle 0.7,0.3\rangle$	$\langle 0.9,0.1\rangle$	$\langle 0.6,0.3\rangle$	$\langle 0.5,0.2\rangle$

Based on the IFEPWA operator, the steps are as follows:

Step 1: Since all the attributes G_j, $j = 1,2,3,4$, are of the benefit type, then the attributes values do not need harmonization, therefore

$$R^{(k)} = A^{(k)} = \left(\alpha_{ij}^{(k)}\right)_{5\times4} = \left(r_{ij}^{(k)}\right)_{5\times4}.$$

Step 2: Using Equations in (58) and (59) to calculate the$T_{ij}^{(1)}$, $T_{ij}^{(2)}$ and $T_{ij}^{(3)}$, we get

$$\left[T_{ij}^{(1)}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \left[T_{ij}^{(2)}\right] = \begin{bmatrix} 0.5000 & 0.7000 & 0.8500 & 0.9000 \\ 0.8500 & 0.9000 & 0.7000 & 0.7500 \\ 0.9500 & 0.9000 & 0.8000 & 0.6500 \\ 0.8000 & 0.9500 & 0.8500 & 0.4000 \\ 0.9500 & 0.4500 & 0.8000 & 0.7000 \end{bmatrix},$$

$$\left[T_{ij}^{(3)}\right] = \begin{bmatrix} 0.3750 & 0.6650 & 0.6800 & 0.3150 \\ 0.5525 & 0.6750 & 0.6650 & 0.3750 \\ 0.8075 & 0.5850 & 0.6000 & 0.2600 \\ 0.6800 & 0.7600 & 0.6375 & 0.3200 \\ 0.7600 & 0.2250 & 0.7600 & 0.5600 \end{bmatrix}.$$

Step 3: Using the IFEPWA operator (Equation (60)) to aggregate all the individual decision matrices $R^{(k)}$, ($k = 1,2,3$) into the collective decision matrix $R = \left(r_{ij}\right)_{5\times4} = \left(\left\langle \mu_{r_{ij}}, \nu_{r_{ij}} \right\rangle\right)_{5\times4}$, we get the following table:

Table 4: Intuitionistic fuzzy collective decision matrix R

	G_1	G_2	G_3	G_4
X_1	$\langle 0.6315,0.2934\rangle$	$\langle 0.8046,0.0000\rangle$	$\langle 0.7030,0.1358\rangle$	$\langle 0.7383,0.2197\rangle$
X_2	$\langle 0.7173,0.0000\rangle$	$\langle 0.7843,0.1535\rangle$	$\langle 0.8046,0.0000\rangle$	$\langle 0.6037,0.3017\rangle$
X_3	$\langle 0.8060,0.0000\rangle$	$\langle 0.7150,0.0000\rangle$	$\langle 0.7459,0.0000\rangle$	$\langle 0.5240,0.4130\rangle$
X_4	$\langle 0.7638,0.1214\rangle$	$\langle 0.7971,0.0000\rangle$	$\langle 0.7238,0.1839\rangle$	$\langle 0.5205,0.3846\rangle$
X_5	$\langle 0.7971,0.0000\rangle$	$\langle 0.5358,0.4129\rangle$	$\langle 0.7987,0.0000\rangle$	$\langle 0.6576,0.1952\rangle$

Step 4: Following Equations in (62) and (63) to calculate the T_{ij}, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, we get

$$[T_{ij}] = \begin{bmatrix} 1 & 0.6690 & 0.6037 & 0.4730 \\ 1 & 0.8587 & 0.7001 & 0.6317 \\ 1 & 0.9030 & 0.7743 & 0.6759 \\ 1 & 0.8819 & 0.7924 & 0.6101 \\ 1 & 0.8985 & 0.5045 & 0.4537 \end{bmatrix}.$$

Step 5: Aggregating all intuitionistic fuzzy numbers $\tilde{r}_{ij}$, $j = 1,2,\ldots,n$, by the IFEPWA operator (Equation (64)) to derive the overall intuitionistic fuzzy preference values $\tilde{r}_i$, $i = 1,2,\ldots,m$ of the teachers X_i, we get

$$\tilde{r}_1 = \langle 0.7148, 0.0000\rangle, \tilde{r}_2 = \langle 0.7395, 0.0000\rangle, \tilde{r}_3 = \langle 0.7231, 0.0000\rangle, \\ \tilde{r}_4 = \langle 0.7292, 0.0000\rangle, \tilde{r}_5 = \langle 0.7106, 0.0000\rangle. \quad (67)$$

Step 6: Calculating the score values $S^*(\tilde{r}_i)$ of the teachers X_i, $i = 1,2,\ldots,m$, we have

$$S^*(\tilde{r}_1) = 0.8574, S^*(\tilde{r}_2) = 0.8698, S^*(\tilde{r}_3) = 0.8616, \\ S^*(\tilde{r}_4) = 0.8646, S^*(\tilde{r}_5) = 0.8553. \quad (68)$$

Step 7: Ranking the teachers X_i, $i = 1,2,3,4,5$, in accordance with the score values $S^*(\tilde{r}_i)$, $i = 1,2,3,4,5$, in descending order, we have

$$X_2 \succ X_4 \succ X_3 \succ X_1 \succ X_5.$$

Thus X_2 is the best teacher for this appointment.

Based on the IFEPWG operator, the main steps are as follows:

Step 1': Same as Step 1.

Step 2': Same as Step 2.

Step 3': Using IFEPWG operator (Equation (61)) to aggregate all the individual decision matrices $R^{(k)}$, $k = 1,2,3$, into a collective decision matrix $R = (r_{ij})_{5\times4} = \left(\left\langle \mu_{r_{ij}}, \nu_{r_{ij}} \right\rangle\right)_{5\times4}$, we get the following table

Table 5: Intuitionistic fuzzy collective decision matrix R

	G_1	G_2	G_3	G_4
X_1	$\langle 0.6064, 0.3514\rangle$	$\langle 0.7858, 0.1578\rangle$	$\langle 0.6799, 0.1553\rangle$	$\langle 0.5976, 0.3286\rangle$
X_2	$\langle 0.7073, 0.1501\rangle$	$\langle 0.7474, 0.1615\rangle$	$\langle 0.7858, 0.1857\rangle$	$\langle 0.5893, 0.3310\rangle$
X_3	$\langle 0.7737, 0.0639\rangle$	$\langle 0.7015, 0.1585\rangle$	$\langle 0.7410, 0.1509\rangle$	$\langle 0.5127, 0.4404\rangle$
X_4	$\langle 0.7590, 0.1277\rangle$	$\langle 0.7718, 0.0632\rangle$	$\langle 0.7125, 0.2148\rangle$	$\langle 0.4966, 0.4662\rangle$
X_5	$\langle 0.7718, 0.1214\rangle$	$\langle 0.4811, 0.4534\rangle$	$\langle 0.7680, 0.1694\rangle$	$\langle 0.6469, 0.2149\rangle$

Step 4': Using Equations in (62) and (63) to calculate the T_{ij}, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$. We get

$$[T_{ij}] = \begin{bmatrix} 1 & 0.6275 & 0.5108 & 0.3894 \\ 1 & 0.7768 & 0.6160 & 0.4928 \\ 1 & 0.8549 & 0.6596 & 0.5244 \\ 1 & 0.8156 & 0.6968 & 0.5218 \\ 1 & 0.8252 & 0.4240 & 0.3389 \end{bmatrix}.$$

Step 5': Aggregate all intuitionistic fuzzy numbers $\tilde{r}_{ij}$, $j = 1,2,\ldots,n$ by the IFEPWG operator (Equation (65)) to derive the overall intuitionistic fuzzy preference values $\tilde{r}_i$, $i = 1,2,\ldots,m$ of the teacher X_i, we have

$$\tilde{r}_1 = \langle 0.6624, 0.2626\rangle, \tilde{r}_2 = \langle 0.7120, 0.1927\rangle, \tilde{r}_3 = \langle 0.6984, 0.1784\rangle, \\ \tilde{r}_4 = \langle 0.7000, 0.1938\rangle, \tilde{r}_5 = \langle 0.6547, 0.2540\rangle. \quad (69)$$

Step 6': Calculating the score values $S^*(\tilde{r}_i)$ of the teachers $X_i, i = 1,2, ..., m$, we have

$$S^*(\tilde{r}_1) = 0.6999, S^*(\tilde{r}_2) = 0.7596, S^*(\tilde{r}_3) = 0.7600, \\ S^*(\tilde{r}_4) = 0.7531, S^*(\tilde{r}_5) = 0.7004. \quad (70)$$

Step 7': Ranking the teachers X_i, $i = 1,2,3,4,5$, in accordance with the score values $S^*(\tilde{r}_i)$, $i = 1,2,3,4,5$, in descending order, we have

$$X_3 \succ X_2 \succ X_4 \succ X_5 \succ X_1.$$

Thus, the best option is X_3.

This result is different from the result obtained by the IFEPWA operator because the IFEPWA operator focuses on the impact of overall data, while the IFEPWG operator highlights the role of the individual data.

5 Conclusion

In this paper, we investigate the intuitionistic fuzzy multiple attribute group decision-making problem in which the attributes and decision makers are at different priority levels. Based on Einstein operation, some new prioritized weighted aggregation operators called the intuitionistic fuzzy Einstein prioritized weighted average (IFEPWA) operator and the intuitionistic fuzzy Einstein prioritized weighted geometric (IFEPWG) operator for aggregating intuitionistic fuzzy numbers has been introduced. The prominent characteristic of these proposed operators is that they take into account prioritization among the attributes and decision makers. Some of their properties are investigated in detail. Based on proposed operators, an intuitionistic fuzzy multiple attribute group decision making approach is developed to solve multiple attribute group decision making problems under intuitionistic fuzzy environment. Finally, a numerical example is presented to illustrate the given approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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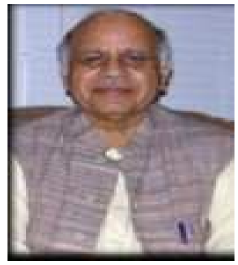

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