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On Some Applications of Multi-Dimensional Laplace Transform

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Abstract: In this paper, the aim is to study the possible extension of the one-dimensional fractional differential equations to the multi-dimensional fractional differential equations with their applications. For this purpose, the multi-dimensional Laplace Transform method (L.T.M) is developed in order to solve multi-dimensional fractional differential equations with constant and variable coefficients. It is also observed that the proposed technique is possible and highly suitable for such problems. The results of the proposed scheme are encouraging and efficient.

Keywords: Fractional ordinary differential equations, Fractional Transform, Laplace Transform

1 Introduction

Integral transforms are widely applied to solve several different type of differential and integro-differential equations. In the literature there are several integral transforms and each of suitable for different type differential equations. Recently some new integral transforms were introduced, see [19, 20] and applied to solve some ODE as well as PDEs, see for example [21, 22]. A fractional differential equation is an equation which contains arbitrary order derivatives. Fractional differential equations used in many branches of sciences such as, mathematics, physics, chemistry and engineering. The fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering.
Since fractional derivatives provide an excellent instrument for description of long time memory. This is the main advantage of arbitrary order derivatives in comparison with classical integer-order derivatives. Thus many authors and researchers have been studying the fractional order differential equations, for example see [4] and [16]. Since a new fractional calculus, which allows us to perform local analysis of non-differentiable functions. This is a new notion, which can be seen as a local version of the classical Riemann–Liouville derivative and give many properties of ordinary derivatives, see [11, 12, 13] the authors show that by using Taylor's of fractional order, further the stochastic differential equation $dx = \sigma x db(t,a)$, where $b(t,a)$ is a fractional Brownian motion of order a, which can be converted into an equation involving fractional derivative, further the solution can be expressed in terms of the Mittag–Leffler function, see [9] and [26]. In [5], the authors studied fractional differential equations that such kind of equations appear in many problems. In particular, they have find a fractional differential equation related to the classical Schrodinger equation, by studying Nottales approach to quantum mechanics via a fractal space–time, see [18], Similarly, Laplace transform method was used to obtain the explicit solution of a certain kind of ordinary differential equations with fractional derivatives, the work in [10] was devoted the applications of the one-dimensional Laplace transform to construct the solutions of linear non-homogenous fractional order differential equations involving the Riemann–Liouville fractional derivatives with constant coefficients, see [2]. By using Laplace transform, one can also find the exact solution of time fractional partial differential equation and some fractional order integral and integro-differential equations, see [14]. Of course the procedure is similar to the ordinary Laplace transform, in order to solve a differential equations we transform of the unknown function, and later apply the inverse Laplace to obtain the

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desired solution. In [7], it was given the idea of fractional derivatives and fractional integrals with their basic properties. In order to solve fractional equations, there are several different methods available in the current literature , see [15] and [3], [25]. In order to solve finance related problems see, [8] where fractional stochastic differential equations with applications to finance were considered. In [27] the authors give a new nonlinear dynamic econometric model with fractional derivative. Later, the fractional derivative is defined in the Jumarie type and in [1], and it was used for the singular perturbations method for fractional differential equation and new development of the variational iteration method, and the homotopy decomposition method were also available in the litertaure.
In the next section we follow [6] and give the definition and some theorems of the multi-dimensional Laplace transform.

2 Fractional Differential Equations

The one-dimensional Laplace transform method is widely used in engineering mathematics, where it has numerous applications. Particularly useful in problems where the mechanical or electrical driving force has discontinuities, for instance, acts for a short time only, see, [23].

2.1 Basic Definitions

We start this section by the following definitions and properties of the fractional derivative, there exist different definitions of the classical and sequential derivatives are:

Definition (2.1.1)[Grunwald–Letnikov Fractional Derivative]: Let f be a function of t from the Cauchy formula is defined by:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau $$

where f(t) has m+1 continuous derivatives in the closed interval $[a,t]$, then we get the fractional integral of order α:

$$ {}_aD_t^{-\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} + \frac{1}{\Gamma(\alpha+m+1)} \int_a^t (t-\tau)^{\alpha-m} f^{(m+1)}(\tau)d\tau, $$

where $m < \alpha < m+1$.

From the above equation and replacing each α by $-\alpha$, we can get:

$$ {}_aD_t^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+m+1)} \int_a^t (t-\tau)^{-\alpha-m} f^{(m+1)}(\tau)d\tau, $$

is known as the fractional derivative of the Grunwald–Letnikov sense.

Example(1): Let the fractional derivative of the power function:

$$ f(t) = (t-a)^{v}, v > -1 $$

where v is a real number.

Solution: By using the Cauchy formula and replacing α by $-\alpha$, we can get:

$$ {}_aD_t^{\alpha} (t-a)^{v} = \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} (\tau-a)^{v} d\tau. $$

Letting $\tau = a+\xi(1-a)$ and by the definition of the beta function, we can get:

$$ \begin{aligned} {}_aD_t^{\alpha} (t-a)^{v} &= \frac{1}{\Gamma(-\alpha)} (t-a)^{v-\alpha} \int_0^1 \xi^{v} (t-\xi)^{-\alpha-1} d\xi \\ &= \frac{1}{\Gamma(-\alpha)} \beta(-\alpha, v+1)(t-a)^{v-\alpha} \\ &= \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (t-a)^{v-\alpha}, (\alpha < 0, v > 0). \end{aligned} $$

Grunwald–Letnikov Fractional Derivative

(i)If $p < 0$ and q is any real number, then:

$$ {}_aD_t^{q} ({}_aD_t^{p} f(t)) = {}_a D_t^{p+q} f(t), $$

for $(m < p < m+1, n < q < n+1)$.

(ii)If $p > 0$ and q is any real number when $f^{(k)}(a) = 0, (k = 0,1,...,m-1)$, then:

$$ {}_aD_t^{q} ({}_aD_t^{p} f(t)) = {}_a D_t^{p+q} f(t). $$

Next, the following definition gives the definition of the Riemann–Liouville fractional derivative.

Definition (2.1.2)[Riemann–Liouville Fractional Derivative]:

Consider this definition is the most widely known definition of the fractional derivative:

$$ {}_aD_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-\alpha} f(\tau)d\tau, $$

$(m \leq \alpha < m+1)$ and the above equation can be written as:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-\alpha-1} f(\tau)d\tau, $$

where $(k-1 \leq \alpha < k)$.

Riemann–Liouville Fractional Derivative:

(i)If p,q are two positive real number and $t > a$, then:

$$ {}_aD_t^{p} ({}_aD_t^{-q} f(t)) = {}_a D_t^{p-q} f(t). $$

(ii)If $0 \leq k-1 \leq q < k$, then:

$$ {}_aD_t^{-p}({}_aD_t^q f(t)) = {}_aD_t^{q-p} f(t) - \sum_{j=1}^{k} [{}_aD_t^{q-j} f(t)]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}. $$

(iii)If $f(t)$ is continuous for $t \geq a$, then:

$$ {}_aD_t^{-p}({}_aD_t^{-q} f(t)) = {}_aD_t^{-p-q} f(t). $$

Next, the following definition gives the definition of the Caputo's fractional derivative.

Definition (2.1.3)[The Caputo's Fractional Derivative]:

Let f be a function of t the Caputo's fractional derivative defined by:

$$ {}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}\tau d\tau}{(t-\tau)^{\alpha+1-n}}, $$

where $(n-1 < \alpha < n)$.

Caputo's Fractional Derivative:

i- If $\mu, \alpha \geq 0$ and $t > 0$, then:

$$ {}_a^cD_t^{-\mu}({}_a^cD_t^{\alpha} f(t)) = {}_a^cD_t^{\alpha-\mu} f(t) - \sum_{k=0}^{l-1} f^{(k)}(0^+) \frac{t^{k+\mu-\alpha}}{\Gamma(\mu-\alpha+k+1)}, $$

where $0 < \alpha < \mu$, $m-1 < \mu < m$, $l-1 < \alpha \leq l < m-1$, $(m,l) \in N$.

ii- From the above relation and taking $\alpha = \mu$, we get:

$$ {}_a^cD_t^{-\alpha}({}_a^cD_t^{\alpha} f(t)) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, (m-1 < \alpha \leq m). $$

Next, we define Sequential fractional derivative.

Definition (2.1.4)[The Sequential Fractional Derivative]:

Let f be a function of t, n-th order differentiation is simply a series of first-order differentiations and replacing each first-order derivative by fractional derivatives of orders. Then, we can get:

$$ D^{\alpha} f(t) = D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(t), $$

where $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is known as the sequential fractional derivative.

Some Properties of the Fractional Derivatives:

1- Linearity:-

Let the fractional differentiation is a linear operation:

$$ D^p(C_1 f_1(t), C_2 f_2(t), \ldots, C_n f_n(t)) = C_1 D^p f_1(t) + C_2 D^p f_2(t) + \ldots + C_n D^p f_n(t) $$

where D^p is any mutation of the above equation.

2- The Leibnitz rule for Fractional Derivatives:-

Let f be a continuous function of τ in the interval $[a,t]$ and $\varphi(t)$ has n+1 continuous derivatives in this interval. Then

$$ {}_aD_t^p(\varphi(t) f(t)) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^k(t) {}_aD_t^{p-k} f(t). $$

3- Fractional Derivative of a Composite Function:
Let an analytic composite function $\varphi(t) = F(h(t))$ and by using the Leibnitz rule, we can obtain:

$$ {}_aD_t^p F(h(t)) = \frac{(t-a)^{-p}}{\Gamma(1-p)} \varphi(t) + \sum_{k=0}^{\infty} \binom{p}{k} \frac{k!(t-a)^{k-p}}{\Gamma(k-p+1)} \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r} $$

where the sum extends over all combinations of non-negative integer values of $a_1, a_2, \ldots, a_k$ such that,

$$ \sum_{r=1}^{k} r a_r = k \text{ and } \sum_{r=1}^{k} a_r = m. $$

2.2 Some Basic Methods for Solving Fractional Differential Equations

Fractional Differential Equations appear in various research areas and engineering applications. An effective and easy to use method for solving such equations. Now, we give some of these methods for details see [24]:

Definition (2.2.1):

Let f be a function of x specified for $x > 0$ The Laplace transform of $f(x)$ denoted by $L\{f(x)\}$ or $F(s)$ is defined by:

$$ Lf(x) = F(s) = \int_0^{\infty} e^{-sx} f(x)\, dx $$

where s is a complex number. The Laplace transform of $f(x)$ exists if the integral that appeared in the above equation converges for some values of s.

Definition (2.2.2):

Let $L\{f(x)\} = F(s)$ then $f(x)$ is called an inverse Laplace transform of $F(s)$ and we write $f(x) = L^{-1}\{F(s)\}$ where L^{-1} is called the inverse Laplace transformation operator.

2.3 Solution of the Linear Ordinary Fractional Differential Equations Via the Laplace Transform

Consider Fractional order for one-dimensional Differential equation with constant coefficient:

$$D^v y(x) = D^n[D^{n-v} y(x)] = D^n[D^{-u} y(x)]$$

where n is the smallest integer greater than $v > 0$, $u = n - v$, we can write this equation as:

$$D^v y(x) = D^n[D^{-(n-v)} y(x)].$$

Now, by taking Laplace Transform to both sides of above equation, we have:

$$\begin{aligned}
Ł\{[D^v y(x)]\} &= Ł\left\{D^n[D^{-(n-v)} y(x)]\right\} \\
&= s^n Ł\left\{D^{-(n-v)} y(x)\right\} - \sum_{i=1}^{n} s^{n-i} D^{i-1-(n-v)} y(0) \\
&= s^n[s^{-(n-v)} Y(s)] - \sum_{i=1}^{n} s^{n-i} D^{i-1-(n-v)} y(0). \\
&= s^v Y(s) - \sum_{i=1}^{n} s^{n-i} D^{i-1-n+v} y(0).
\end{aligned}$$

In particular,

$$n = 1 \Rightarrow s^v Y(s) - D^{-(1-v)} y(0), \;\; 0 < v \le 1.$$
$$n = 2 \Rightarrow s^v Y(s) - sD^{-(2-v)} y(0) - D^{-(1-v)} y(0), \;\; 1 < v \le 2.$$

If $n = 3$ then

$$s^v Y(s) - s^2 D^{-(3-v)} y(0) - sD^{-(2-v)} y(0) - D^{-(1-v)} y(0),$$

for $2 < v \le 3$.

Example [2]: Let solve $D^{\frac{2}{3}} y(t) = ay(t)$, where a is a constant, $0 < v = 2/3 \le 1$.

Solution: Now, by taking Laplace Transform to both sides of above equation, we have:

$$L\left\{D^{\frac{2}{3}} y(t)\right\} = aL\{y(t)\},$$

implies

$$s^{2/3} Y(s) - D^{-(1-2/3)} y(0) = aY(s).$$

Assume that $D^{-(1-2/3)} y(0) = D^{-1/3} y(0)$ is the value of $D^{-1/3} y(t)$ at $t = 0$ exists, and call it c_1. Then:

$$\begin{aligned}
&s^{2/3} Y(s) - c_{-1} = aY(s), \\
&Y(s) = \frac{c_{-1}}{s^{2/3} - a}, \\
&y(t) = L^{-1}\left\{\frac{c_{-1}}{s^{2/3} - a}\right\} = c_{-1} t \frac{1}{3} E_{2/3,2/3}\left(at^{\frac{2}{3}}\right)
\end{aligned}$$

and

$$L\left\{D^{-1/3} y(t)\right\} = s^{-1/3} Y(s).$$

Since

$$Y(s) = \frac{c_{-1}}{s^{2/3} - a}.$$

Then,

$$L\left\{D^{-1/3} y(t)\right\} = \frac{c_{-1} s^{-1/3}}{s^{2/3} - a}.$$

So,

$$\begin{aligned}
D^{-1/3} y(t) &= L^{-1}\left\{\frac{c_{-1} s^{-1/3}}{s^{2/3} - a}\right\} \\
&= c_{-1} E_{2/3,1}\left(at^{\frac{2}{3}}\right).
\end{aligned}$$

At $t = 0$,

$$D^{-1/3} y(t) = E_{2/3,1}(0) = c_{-1}.$$

Example [3]: Let solve $D^{\frac{4}{3}} y(t) = 0$, $1 < v = 4/3 \le 2$.

Solution: By taking the Laplace transform of both sides of the equation, we have:

$$L\left\{D^{\frac{4}{3}} y(t)\right\} = 0,$$

implies that,

$$s^{\frac{4}{3}} Y(s) - sD^{-(2-4/3)} y(0) - D^{-(1-4/3)} y(0) = 0.$$

We will assume that constants $D^{-(2-4/3)} y(0) = c_1$ and $D^{-(1-4/3)} y(0) = c_2$ exist. Then:

$$s^{\frac{4}{3}} Y(s) - c_1 s - c_2 = 0.$$

Solving for $Y(s)$, we obtain:

$$Y(s) = \frac{c_1 s}{s^{4/3}} + \frac{c_2}{s^{4/3}}.$$

Finally, we find the inverse Laplace of $Y(s)$:

$$\begin{aligned}
y(t) &= L^{-1}\left\{\frac{c_1 s}{s^{4/3}}\right\} + L^{-1}\left\{\frac{c_2}{s^{4/3}}\right\} \\
y(t) &= \frac{c_1}{\Gamma(\frac{1}{3})} t^{-2/3} + \frac{c_2}{\Gamma(\frac{4}{3})} t^{1/3}.
\end{aligned}$$

2.4 The Multi- Dimensional Laplace Transform

As seen before, the one-dimensional Laplace transform is defined for functions of only one independent variable. In 1999, Dahiya and Nadjafi defined the multi-dimensional Laplace transform for functions of more than one independent variable. Also, they study some properties that concerned with this definition.

We start this section, with the following definition.

Definition (3.1), [6]:

Let u be a function of $x_1, x_2, \dots, x_n$ specified for $x_i > 0$ for each i=1,2,...,n. Then the multi-dimensional Laplace transform of $u(x_1, x_2, \dots, x_n)$ denoted by $L_n\{u(x_1, x_2, \dots, x_n)\}$ or $U(s_1, s_2, \dots, s_n)$ is defined by:

$$L_n\{u(x_1, x_2, \dots, x_n)\} = U(s_1, s_2, \dots, s_n)$$
$$= \int_0^\infty \int_0^\infty \dots \int_0^\infty e^{-\sum_{i=1}^n s_i x_i} u(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

where s_i is a complex number for each $i = 1, 2, \dots, n$. The multi- dimensional Laplace transform of $u(x_1, x_2, \dots, x_n)$ exists if the integral that appeared in the above equation converges for some values of $s_1, s_2, \dots, s_n$.

3 Initial and Boundary Value Problems and Linear Fractional Partial Differential Equations

Consider the initial and boundary value problem which consists of the v-th order linear partial fractional differential equation with constant coefficients:

$$\frac{\partial^{\sum_{i=1}^n v_i} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n \partial x_i^{v_i}}$$
$$= \frac{\partial^{\sum_{i=1}^n m_i} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n \partial x_i^{m_i}} \left\{ \frac{\partial^{\sum_{i=1}^n -(m_i - v_i)} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n \partial x_i^{-(m_i - v_i)}} \right\}$$

together with appropriate initial and boundary conditions. The general form for two-variable with derivative of one variable is:

$$\left(\frac{\partial}{\partial x}\right)^\alpha u(x,y) = \left(\frac{\partial}{\partial x}\right)^n \left\{ \left(\frac{\partial}{\partial x}\right)^{-n+\alpha} u(x,y) \right\}.$$

By taking L_2 to both sides of above equation, then implies that:

$$s_1^\alpha L_2 \left\{ \left(\frac{\partial}{\partial x}\right)^{-(n+\alpha)} u(x,y) \right\}$$
$$- \sum_{i=1}^n s_1^{n-i} L_1 \left\{ \left(\frac{\partial}{\partial x}\right)^{i-1-n+\alpha} u(0,y) \right\}.$$

The general form for two-variable with derivative of two variable is:

$$\frac{\partial^{\sum_{i=1}^2 v_i} u(x_1, x_2)}{\prod_{i=1}^2 \partial x_i^{v_i}} = \frac{\partial^{\sum_{i=1}^2 m_i}}{\prod_{i=1}^2 \partial x_i^{m_i}} \left\{ \frac{\partial^{\sum_{i=1}^2 -(m_i - v_i)} u(x_1, x_2)}{\prod_{i=1}^2 \partial x_i^{-(m_i - v_i)}} \right\}$$

Now, by taking L_2 to both sides of above equation, then follows that

$$s_1^{m_1} s_2^{m_2} L_2 \left\{ \frac{\partial^{\sum_{i=1}^2 -(m_i - v_i)} u(x_1, x_2)}{\prod_{i=1}^2 \partial x_i^{-(m_i - v_i)}} \right\} -$$
$$s_1^{m_1} \sum_{l_2=1}^{m_2} s_2^{l_2-1} L_1 \left\{ \frac{\partial^{-(m_2 - l_2 + v_2)} u(x_1, 0)}{\partial x_2^{-(m_2 - l_2 + v_2)}} \right\} -$$
$$s_2^{m_2} \sum_{l_1=1}^{m_1} s_1^{l_1-1} L_1 \left\{ \frac{\partial^{-(m_1 - l_1 + v_1)} u(0, x_2)}{\partial x_1^{-(m_1 - l_1 + v_1)}} \right\} +$$
$$\sum_{l_1=1}^{m_1} s_1^{l_1-1} \sum_{l_2=1}^{m_2} s_2^{l_2-1} \left\{ \frac{\partial^{-\sum_{i=1}^2 (m_i - l_i + v_i)} u(0,0)}{\prod_{i=1}^2 \partial x_i^{-(m_i - l_i + v_i)}} \right\}$$

and

$$s_1^{m_1} s_2^{m_2} L_2 \left\{ \frac{\partial^{\sum_{i=1}^2 -(m_i - v_i)} u(x_1, x_2)}{\prod_{i=1}^2 \partial x_i^{-(m_i - v_i)}} \right\} = s_1^{v_1} s_2^{v_2} U(s_1, s_2).$$

The general form for three-variable with derivative of three variable is:

$$\frac{\partial^{\sum_{i=1}^3 v_i} u(x_1, x_2, x_3)}{\prod_{i=1}^3 \partial x_i^{v_i}} = \frac{\partial^{\sum_{i=1}^3 m_i}}{\prod_{i=1}^3 \partial x_i^{m_i}} \left\{ \frac{\partial^{\sum_{i=1}^3 -(m_i - v_i)} u(x_1, x_2, x_3)}{\prod_{i=1}^3 \partial x_i^{-(m_i - v_i)}} \right\}$$

Now, by taking L_3 to both sides of above equation, then:

$$\Rightarrow s_1^{m_1} s_2^{m_2} s_3^{m_3} L_3 \left\{ \frac{\partial^{\sum_{i=1}^3 -(m_i - v_i)} u(x_1, x_2, x_3)}{\prod_{i=1}^3 \partial x_i^{-(m_i - v_i)}} \right\} -$$
$$s_1^{m_1} \sum_{l_2=1}^{m_2} s_2^{l_2-1} \sum_{l_3=1}^{m_3} s_3^{l_3-1} L_1 \left\{ \frac{\partial^{-\sum_{i=2}^3 (m_i - l_i + v_i)} u(x_1, 0, 0)}{\prod_{i=2}^3 \partial x_i^{-(m_i - l_i + v_i)}} \right\} -$$
$$s_2^{m_2} \sum_{l_1=1}^{m_1} s_1^{l_1-1} \sum_{l_3=1}^{m_3} s_3^{l_3-1} L_1 \left\{ \frac{\partial^{-\sum_{i=1, i\neq 2}^3 (m_i - l_i + v_i)} u(0, x_2, 0)}{\prod_{i=1, i\neq 2}^3 \partial x_i^{-(m_i - l_i + v_i)}} \right\} -$$
$$s_3^{m_3} \sum_{l_1=1}^{m_1} s_1^{l_1-1} \sum_{l_2=1}^{m_2} s_2^{l_2-1} L_1 \left\{ \frac{\partial^{\sum_{i=1}^2 -(m_i - v_i)} u(0, 0, x_3)}{\prod_{i=1}^2 \partial x_i^{-(m_i - v_i)}} \right\} -$$
$$s_1^{m_1} s_2^{m_2} \sum_{l_3=1}^{m_3} s_3^{l_3-1} L_2 \left\{ \frac{\partial^{-(m_3 - l_3 + v_3)} u(x_1, x_2, 0)}{\partial x_3^{-(m_3 - l_3 + v_3)}} \right\} -$$
$$s_1^{m_1} s_3^{m_3} \sum_{l_2=1}^{m_2} s_2^{l_2-1} L_2 \left\{ \frac{\partial^{-(m_2 - l_2 + v_2)} u(x_1, 0, x_3)}{\partial x_2^{-(m_2 - l_2 + v_2)}} \right\} -$$
$$s_2^{m_2} s_3^{m_3} \sum_{l_1=1}^{m_1} s_1^{l_1-1} L_2 \left\{ \frac{\partial^{-(m_1 - l_1 + v_1)} u(0, x_2, x_3)}{\partial x_1^{-(m_1 - l_1 + v_1)}} \right\} -$$
$$\sum_{l_1=1}^{m_1} s_1^{l_1-1} \sum_{l_2=1}^{m_2} s_2^{l_2-1} \sum_{l_3=1}^{m_3} s_1^{l_3-1} \left\{ \frac{\partial^{-\sum_{i=1}^3 (m_i - l_i + v_i)} u(0,0,0)}{\prod_{i=1}^3 \partial x_i^{-(m_i - l_i + v_i)}} \right\}$$

and similarly,

$$s_1^{m_1} s_2^{m_2} s_3^{m_3} L_3 \left\{ \frac{\partial^{\sum_{i=1}^3 -(m_i - v_i)} u(x_1, x_2, x_3)}{\prod_{i=1}^3 \partial x_i^{-(m_i - v_i)}} \right\} = s_1^{v_1} s_2^{v_2} s_3^{v_3} U(s_1, s_2, s_3).$$

The general form for multi-variable with derivative of multi (all) variable is:

$$\frac{\partial^{\sum_{i=1}^n v_i} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n \partial x_i^{v_i}} = \frac{\partial^{\sum_{i=1}^n m_i}}{\prod_{i=1}^n \partial x_i^{m_i}} \left\{ \frac{\partial^{\sum_{i=1}^n -(m_i - v_i)} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n \partial x_i^{-(m_i - v_i)}} \right\}.$$

Now, by taking L_n to both sides of above equation, then one can reach related formula.

To illustrate this method, consider the following examples.

1-When different derivative $\sum_{i=1}^2 v_i = \frac{5}{4}$ and $1 < \sum_{i=1}^2 v_i \le 2$. In particular, if $m_1 + m_2 = 2$, then:

$$\frac{\partial^{\frac{5}{4}} u(x_1, x_2)}{\partial x_1^{\frac{1}{2}} \partial x_2^{\frac{3}{4}}} = \frac{\partial^2}{\partial x_1 \partial x_2} \left\{ \frac{\partial^{-(2-\frac{5}{4})} u(x_1, x_2)}{\partial^{-(1-\frac{1}{2})} x_1 \partial^{-(1-\frac{3}{4})} x_2} \right\}$$
$$= s_1 s_2 L_2 \left\{ \frac{\partial^{-\frac{3}{4}} u(x_1, x_2)}{\partial x_1^{-\frac{1}{2}} \partial x_2^{-\frac{1}{4}}} \right\} - s_1 L_1 \left\{ \frac{\partial^{\frac{-3}{4}} u(x_1, 0)}{\partial x_2^{\frac{-3}{4}}} \right\}$$
$$- s_2 L_1 \left\{ \frac{\partial^{\frac{-1}{2}} u(0, x_2)}{\partial x_1^{\frac{-1}{2}}} \right\} + \left\{ \frac{\partial^{\frac{-5}{4}} u(0,0)}{\partial x_1^{\frac{-1}{2}} \partial x_2^{\frac{-3}{4}}} \right\}$$

and

$$s_1 s_2 \left\{ \frac{\partial^{-\frac{3}{4}} u(x_1,x_2)}{\partial x_1^{-\frac{1}{2}} \partial x_2^{-\frac{1}{4}}} \right\} = s_1^{\frac{1}{2}} s_2^{\frac{3}{4}} U(s_1,s_2).$$

2- When $n=1$ and $v=\frac{1}{2}$, we have:

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\frac{1}{2}} u(x,y) &= \frac{\partial}{\partial x}\left\{\left(\frac{\partial}{\partial x}\right)^{-1+\frac{1}{2}} u(x,y)\right\} \\ &= s_1 L_2 \left\{\left(\frac{\partial}{\partial x}\right)^{-\frac{1}{2}} u(x,y)\right\} - \left(\frac{\partial}{\partial x}\right)^{-\frac{1}{2}} u(0,y) \\ &= s_1\left[s_1^{-\frac{1}{2}} L_2\{u(x,y)\}\right] - \left(\frac{\partial}{\partial x}\right)^{-\frac{1}{2}} u(0,y) \\ &= s_1^{\frac{1}{2}} L_2\{u(x,y)\} - \left(\frac{\partial}{\partial x}\right)^{-\frac{1}{2}} u(0,y).\end{aligned}$$

3- When $n=2$ and $v=\frac{3}{2}, 1<v\leq 2$ we have:

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\frac{3}{2}} u(x,y) &= \frac{\partial^2}{\partial x^2}\left\{\left(\frac{\partial}{\partial x}\right)^{-2+\frac{3}{2}} u(x,y)\right\} \\ &= \frac{\partial^2}{\partial x^2}\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}} u(x,y)\right\} \\ &= s_1^{\frac{3}{2}} L_2\{u(x,y)\} - s_1 L_2\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}} u(0,y)\right\} \\ &\quad - L_2\left\{\left(\frac{\partial}{\partial x}\right)^{\frac{-1}{2}} u(0,y)\right\}.\end{aligned}$$

4- When $n=3$ and $\sum_{i=1}^{3} v_i = \frac{8}{3}$, $2<\sum_{i=1}^{3} v_i \leq 3$. we have:

$$\frac{\partial^{\frac{8}{3}} u(x_1,x_2,x_3)}{\partial x_1^{\frac{1}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{5}{3}}} = \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\left\{\frac{\partial^{-\frac{1}{3}} u(x_1,x_2,x_3)}{\partial x_1^{-\frac{2}{3}} \partial x_2^{-\frac{1}{3}} \partial x_3^{\frac{2}{3}}}\right\}$$

$\Rightarrow$

$$\begin{aligned}& s_1 s_2 s_3 L_3\left\{\frac{\partial^{\frac{-1}{3}} u(x_1,x_2,x_3)}{\partial x_1^{\frac{-2}{3}} \partial x_2^{\frac{-1}{3}} \partial x_3^{\frac{2}{3}}}\right\} - s_1 L_1\left\{\frac{\partial^{\frac{-7}{3}} u(x_1,0,0)}{\partial x_2^{\frac{-2}{3}} \partial x_3^{\frac{-5}{3}}}\right\} \\ & - s_2 L_1\left\{\frac{\partial^{-2} u(0,x_2,0)}{\partial x_1^{\frac{-1}{3}} \partial x_3^{\frac{-5}{3}}}\right\} - s_3 L_1\left\{\frac{\partial^{-1} u(0,0,x_3)}{\partial x_1^{\frac{-1}{3}} \partial x_2^{\frac{-2}{3}}}\right\} \\ & - s_1 s_2 L_2\left\{\frac{\partial^{\frac{-5}{3}} u(x_1,x_2,0)}{\partial x_3^{\frac{-5}{3}}}\right\} - s_1 s_3 L_2\left\{\frac{\partial^{\frac{-2}{3}} u(x_1,0,x_3)}{\partial x_2^{\frac{-2}{3}}}\right\} \\ & - s_2 s_3 L_2\left\{\frac{\partial^{\frac{-1}{3}} u(0,x_2,x_3)}{\partial x_1^{\frac{-1}{3}}}\right\} - \left\{\frac{\partial^{\frac{-8}{3}} u(0,0,0)}{\partial x_1^{\frac{1}{3}} \partial x_2^{\frac{2}{3}} \partial x_3^{\frac{5}{3}}}\right\}.\end{aligned}$$

To illustrate this approach, consider the following examples.

5- Consider the initial and boundary value problem which consists of the fractional order linear partial differential equation with constant coefficients:

$$\frac{\partial^{\frac{3}{2}}(x,y)}{\partial x^{\frac{3}{4}} \partial y^{\frac{3}{4}}} = xy, \quad 1 < v_1 + v_2 = 3/2 \leq 2.$$

together with the initial and boundary conditions:

$$\frac{\partial^{\frac{-3}{4}}(x,0)}{\partial y^{\frac{-3}{4}}} = \frac{\partial^{\frac{-3}{4}}(0,y)}{\partial x^{\frac{-3}{4}}} = \frac{\partial^{\frac{-3}{2}} u(0,0)}{\partial x^{\frac{-3}{4}} \partial y^{\frac{-3}{4}}} = 0.$$

Then by taking the two-dimensional Laplace transform of both sides of the above fractional partial differential equation, one can have:

$$\begin{aligned}& L_2\left\{\frac{\partial^{\frac{3}{2}} u(x,y)}{\partial x^{\frac{3}{4}} \partial y^{\frac{3}{4}}}\right\} = L_2\{xy\} \\ & s_1^{\frac{3}{4}} s_2^{\frac{3}{4}} U(s_1,s_2) - s_1 L_1\left\{\frac{\partial^{\frac{-3}{4}}(x,0)}{\partial y^{\frac{-3}{4}}}\right\} - s_2 L_1\left\{\frac{\partial^{\frac{-3}{4}}(0,y)}{\partial x^{\frac{-3}{4}}}\right\} \\ & + \left\{\frac{\partial^{\frac{-3}{2}} u(0,0)}{\partial x^{\frac{-3}{4}} \partial y^{\frac{-3}{4}}}\right\} = \frac{1}{s_1^2 s_2^2}.\end{aligned}$$

Therefore

$$\begin{aligned}U(s_1,s_2) &= \frac{1}{s_1^{\frac{3}{4}} s_2^{\frac{3}{4}}} \frac{1}{s_1^2 s_2^2} \\ &= \frac{1}{s_1^{\frac{11}{4}} s_2^{\frac{11}{4}}}.\end{aligned}$$

By taking the inverse two-dimensional Laplace transform of both sides of the above equation, one can have:

$$L_2^{-1}\{U(s_1,s_2)\} = L_2^{-1}\left\{\frac{1}{s_1^{\frac{11}{4}} s_2^{\frac{11}{4}}}\right\}.$$

Hence

$$u(x,y) = \frac{x^{\frac{7}{4}} y^{\frac{7}{4}}}{\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{11}{4}\right)}$$

is the solution of the above initial and boundary value problem.

6- Consider the initial and boundary value problem which consists of the fractional order linear partial differential equation with constant coefficients:

$$\left\{\frac{\partial^{\frac{5}{2}} u(x,y,z)}{\partial x^{\frac{5}{2}}}\right\} = e^{ax} + e^{ay} - e^{by}, \quad 2 < \sum_{i=1}^{3} v_i \leq 3$$

together with the initial and boundary conditions:

$$\frac{\partial^{\frac{-1}{2}} u(0,y,z)}{\partial x^{\frac{-1}{2}}} = \frac{\partial^{\frac{1}{2}} u(0,y,z)}{\partial x^{\frac{1}{2}}} = \frac{\partial^{\frac{3}{2}} u(0,y,z)}{\partial x^{\frac{3}{2}}} = 0.$$

Then by taking the three-dimensional Laplace transform of both sides of the above fractional differential equation, one can have:

$$L_3\left\{\frac{\partial^{\frac{5}{2}}u(x,y,z)}{\partial x^{\frac{5}{2}}}\right\}=L_3\left\{e^{ax}+e^{ay}-e^{by}\right\}.$$

Therefore

$$s_1^{\frac{5}{2}}U(s_1,s_2,s_3)-L_2\left\{\frac{\partial^{\frac{-1}{2}}u(0,y,z)}{\partial x^{\frac{-1}{2}}}\right\}$$
$$-s_1L_2\left\{\frac{\partial^{\frac{1}{2}}u(0,y,z)}{\partial x^{\frac{1}{2}}}\right\}-s_1^2L_2\left\{\frac{\partial^{\frac{3}{2}}u(0,y,z)}{\partial x^{\frac{3}{2}}}\right\}$$
$$=\frac{1}{s_1-a}+\frac{1}{s_2-a}-\frac{1}{s_2-b}$$
$$\Rightarrow U(s_1,s_2,s_3)=\frac{1}{s_1^{\frac{5}{2}}(s_1-a)}+\frac{a-b}{s_1^{\frac{5}{2}}(s_2-a)(s_2-b)}$$

By taking the inverse three-dimensional Laplace transform of both sides of the above equation, one can have:

$$L_3^{-1}\{U(s_1,s_2,s_3)\}=L_3^{-1}\left\{\frac{1}{s_1^{\frac{5}{2}}(s_1-a)}+\frac{a-b}{s_1^{\frac{5}{2}}(s_2-a)(s_2-b)}\right\}.$$

Hence

$$\Rightarrow u(x,y,z)=x^{\frac{5}{2}}E_{1,\frac{7}{2}}(ax)+\frac{x^{\frac{3}{2}}(e^{ay}-e^{by})}{\Gamma(\frac{5}{2})}$$

is the solution of the above initial and boundary value problem.

4 Conclusion

In this work, we proposed multi-dimensional Laplace transforms method(M.D.L.T.M) for solving multi-dimensional fractional differential equations with constant and variable coefficients. It is illustrated that the method is effective and reliable tool for the solution of fractional linear partial differential equations. Furthermore, it accelerates the rate of convergence. The M.D.L.T.M has been successfully applied to find an exact solution of fractional partial differential equations with constant and variable coefficients.

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