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Some Properties of Higher Dimensional Homotopy Groups for Digital Images

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Abstract: By combining the algebraic topological concepts such as Euler characteristics, (co)homology groups, fundamental and homotopy groups with digital topology we can compare, classify or identify the digital images between each other. In this paper, we explore the digital relative homotopy relation between two continuous functions on a pointed digital image whose domains are n-cube and which map the boundary of an n-cube to a fix point. Then we introduce the n^{th} homotopy group of a pointed digital image and give a relation between the homotopy groups of two pointed digital images.

Keywords: digital relative homotopy groups, digital homotopy groups

1 Introduction

Digital Topology is to study and characterize the properties of digital images. Therefore it plays an important role in Computer Vision, Image Processing and Computer Graphics. Researchers (Rosenfeld, Kong, Kopperman, Kovalevsky, Malgouyres, Ayala, Boxer, Chen, Han, Karaca and others) in this area have studied to determine the properties of digital images with tools from Topology (especially Algebraic Topology).

Rosenfeld [25] introduced a notion of continuity for functions between digital images. This notion has an importance in the applications of image processing and particularly in the study of mappings between the digital images. Boxer extends the results of Rosenfeld's by examining the digital versions of several classical notions from topology including homeomorphism, retraction, and homotopy to compare the digital images in [5]. The digital fundamental group of a discrete object was first introduced by Kong [20]. Later Boxer has showed how classical methods of Algebraic Topology may be used in calculating the digital fundamental form group based on the notions of digitally continuous functions [25], and the digital homotopy [5]. Boxer [9] has also examined the digital fundamental groups of unbounded digital images. Ayala and al. [3] have observed that the digital fundamental object is naturally isomorphic to the fundamental group of its continuous analogue. Han [18] also introduces the digital covering space and proves the non-product property of the digital fundamental group of the digital product image. Boxer [8] discusses the errors of Han's results [18], and develops further the topic of the digital covering space by concerning the existence and properties of digital universal covering spaces. Boxer and Karaca [10] show that the digital covering spaces are classified by the conjugacy class corresponding to a digital covering space. Boxer and al. [13] compute the simplicial homology groups of certain minimal simple closed surfaces and the Euler characteristics of certain digital surfaces.

One of the achievements of Algebraic Topology is to turn the global topological problems into Homotopy Theory problems. In mathematics, homotopy groups are important invariants in Algebraic Topology and used in classifying the topological spaces and generalize the fundamental group which gives an information about loops and homotopy groups give information about holes of a space. The fundamental group was defined by Poincaré in *Analysis Situs* [24] and in the same paper he introduced Homology Theory and the relation between homology and homotopy. To describe higher dimensional connectivity by the homotopy concept, mathematicians need a generalization of the fundamental group to higher dimensions. One would hope for a sequence of groups

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which are amenable to computation and have the property that two spaces are homeomorphic if and only if their corresponding groups are isomorphic. The definitions of homotopy groups were given by Eduard Čech and Witold Hurewicz in 1932-1935. In recent years, one of the most famous problem of algebraic topology is to determine the homotopy groups of n-spheres. Homotopy is more easily defined and conceptionally simpler. It doesn't require elaborating the explanations of chains, boundaries, cycles or quotient groups. Homotopy applies immediately to general topological spaces and does not require the special polyhedral structure that are used in homology.

Poincaré [24] was greatly preoccupied with the classification problem. He hoped that the fundamental group would overcome the deficiencies of Homology Theory in the classification of 3-manifolds. However, J.W. Alexander [1] showed that there exists non-homeomorphic 3-manifolds having isomorphic homology groups and isomorphic fundamental groups. We think that the same problem may occur in Digital Topology; we might have two digital images that are not isomorphic (we use the term "isomorphic" instead of "homemorphic") to each other but their corresponding digital fundamental groups and digital homology groups are isomorphic. The aim of this paper is to explore the digital homotopy groups of a digital image and show that they are invariants of a digital image.

This paper is organised as follows. In the preliminary part, we give some basic definitions such as digital κ-adjacencies, a digital (κ_1,κ_2)-continuous function, a digital (κ_1,κ_2)-isomorphism, digitally (κ_1,κ_2)-homotopy, a pointed digital image, a continuous map between two pointed digital images, and a pointed contractible digital image. In Section 3, we introduce a 'relative homotopy map' between two continuous functions that map the n-cube to a pointed digital image and the boundary of an n-cube is mapped to a base point. Then we show that the relative homotopy relation is an equivalence relation and the equivalence classes together with the operation '$\star$' that we introduce is a digital n-th homotopy group of a pointed digital image. Also this construction yields a covariant functor from the category of pointed digital images and a pointed continuous functions to the category of groups and homomorphisms. As a result we get some conclusions.

2 Preliminaries

Let $\mathbb{Z}$ be the set of integers. A (binary) digital image is a pair (X,κ), where $X\subset\mathbb{Z}^n$ for some positive integer n and κ indicates some adjacency relation for the members of X. A variety of adjacency relations are used in the study of digital images. The following terminology is used in [20]. Two points p and q in $\mathbb{Z}^2$ are 8-*adjacent* if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathbb{Z}^2$ are 4-*adjacent* if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are 26-*adjacent* if they are distinct and differ by at most 1 in each coordinate; they are 18-*adjacent* if they are 26-adjacent and differ in at most two coordinates; they are 6-*adjacent* if they are 18-adjacent and differ in exactly one coordinate. We generalize these adjacencies as follows [9]. Let l,n be positive integers, $1\le l\le n$ and two distinct points $p=(p_1,\dots,p_n)$, $q=(q_1,q_2,\dots,q_n)$ in $\mathbb{Z}^n$, p and q are κ_l-*adjacent* if there are at most l distinct coordinates j for which $|p_j-q_j|=1$, and for all other coordinates j, $p_j=q_j$. A κ_l-adjacency relation on $\mathbb{Z}^n$ may be denoted by the number of points that are κ_l adjacent to a point $p\in\mathbb{Z}^n$. Thus, the κ_1-adjacency on $\mathbb{Z}$ may be denoted by the number 2, and κ_1-adjacent points of $\mathbb{Z}$ are called 2-adjacent. Similarly, κ_1-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent; κ_2-adjacent points of $\mathbb{Z}^2$ are called 8-adjacent; and in $\mathbb{Z}^3$, κ_1-, κ_2-, and κ_3-adjacent points are called 6-adjacent, 18-adjacent, and 26-adjacent, respectively.

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A κ-*neighbor* of $p\in\mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p. A digital image $X\subset\mathbb{Z}^n$ is κ-*connected* [**?**] if and only if for every pair of different points $x,y\in X$, there is a set $\{x_0,x_1,\dots,x_r\}$ of points of X such that $x=x_0$, $y=x_r$ and x_i and x_{i+1} are κ-neighbors where $i\in\{0,1,\dots,r-1\}$. A κ-*component* of a digital image X is a maximal κ-connected subset of X.

Let $a,b\in\mathbb{Z}$ with $a\le b$. *A digital interval* [6] is a set of the form

$$[a,b]_{\mathbb{Z}}=\{z\in\mathbb{Z}:\ a\le z\le b\}$$

in which 2-adjacency is assumed.

Let $X\subset\mathbb{Z}^{n_1}$ and $Y\subset\mathbb{Z}^{n_2}$. Let κ_i be an adjacency relation defined on $\mathbb{Z}^{n_i}$, $i\in\{1,2\}$. We say that a function $f:X\to Y$ is (κ_1,κ_2)-*continuous* [8] if the image under f of every κ_0-connected subset of X is κ_1-connected subset of Y.

Let $X\subset\mathbb{Z}^{n_1}$ and $Y\subset\mathbb{Z}^{n_2}$ be digital images with κ_1-adjacency and κ_2-adjacency respectively. Then the function $f:X\to Y$ is (κ_1,κ_2)-continuous [25,6] if and only if for every pair of κ_1-adjacent points $\{x_0,x_1\}$ of X, either $f(x_0)=f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_2-adjacent in Y.

Composition preserves digital continuity [6], i.e., if $f:X\to Y$ and $g:Y\to Z$ are (κ_1,κ_2)-continuous and (κ_2,κ_3)-continuous respectively, then the composite function $(g\circ f):X\to Z$ is (κ_1,κ_3)-continuous.

Let $X\subset\mathbb{Z}^{n_1}$ and $Y\subset\mathbb{Z}^{n_2}$ be digital images with κ_1-adjacency and κ_2-adjacency respectively. A function $f:X\to Y$ is a (κ_1,κ_2)-*isomorphism* [5] if f is

(κ_1,κ_2)-continuous and bijective and further $f^{-1}: Y \to X$ is (κ_2,κ_1)-continuous.

Definition 2.1. ([6]; see also [19]) Let X and Y be digital images. Let $f,g : X \to Y$ be (κ_1,κ_2)-continuous functions. Suppose there is a positive integer m and a function

$$F : X \times [0,m]_{\mathbb{Z}} \to Y \text{ such that}$$

–for all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x)$;
–for all $x \in X$, the induced function $F_x : [0,m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,\kappa_2)$-continuous; and
–for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ for all $x \in X$ is (κ_1,κ_2)-continuous,

then F is a digital (κ_1,κ_2)-homotopy between f and g, and f and g are digitally (κ_1,κ_2)-homotopic in Y, and denoted by $f \simeq_{\kappa_1,\kappa_2} Y$.

Digital (κ_1,κ_2)-homotopy is an equivalence relation among digitally continuous functions $f : X \to Y$ (see [19, 6]).

If (X,κ) is a digital image and $p \in X$, the triple (X,p,κ) is a pointed digital image ([9]). *A pointed digitally continuous function* f [8,9] from a pointed digital image (X,p,κ_1) to a pointed digital image (Y,q,κ_2) such that $f(p) = q$.

Let f and g be pointed digitally continuous functions from (X,p) to (Y,q). A digital homotopy

$$H : X \times [0,m]_{\mathbb{Z}} \to Y$$

between f and g is called *a pointed digital homotopy* between f and g if for all $t \in [0,m]_{\mathbb{Z}}$, $H(p,t) = q$ (see [8]).

If (X,p,κ_1) and (Y,q,κ_2) are pointed digital images,

$$f : X \to Y$$

a (κ_1,κ_2)-continuous function such that $f(p) = q$,

$$g : Y \to X$$

a (κ_2,κ_1)-continuous function such that $g(q) = p$,

$$H : X \times [0,m_X]_{\mathbb{Z}} \to Y$$

a (κ_1,κ_2)-homotopy between $g \circ f$ and 1_X such that

$$H(p,t) = q$$

for all $t \in [0,m_X]_{\mathbb{Z}}$, and

$$K : Y \times [0,m_Y]_{\mathbb{Z}} \to Y$$

a (κ_2,κ_1)-homotopy between $f \circ g$ and 1_Y such that

$$K(q,t) = p$$

for all $t \in [0,m_Y]_{\mathbb{Z}}$, then (X,p,κ_1) and (Y,q,κ_2) have the same (κ_1,κ_2)-homotopy type and are called (κ_1,κ_2)-*pointed homotopy equivalent* [9]. These functions f and g are called *pointed homotopy equivalences*.

For $p \in Y$, let $\overline{p}$ denote the constant function for some $p \in X$ defined by

$$\overline{p}(x) = p$$

for all $x \in X$. A digital image (X,κ) is said to be κ-*contractible* [6,19], if its identity map is (κ,κ)-homotopic to the constant function $\overline{p}$ for some $p \in X$. If the construction of homotopy holds p fixed, we say (X,p,κ) is pointed κ-contractible.

3 Digital Homotopy Groups

It is well known that digital interval $[0,m]_{\mathbb{Z}}$ is 2-connected. Thus, n-ary cartesian power of $[0,m]_{\mathbb{Z}}$

$$[0,m]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}} \times ... \times [0,m]_{\mathbb{Z}} = [0,m]_{\mathbb{Z}}^n$$

is $2n$-connected.

The n-boundary of $[0,m]_{\mathbb{Z}}^n$, denoted by $\partial[0,m]_{\mathbb{Z}}$, is defined as follows:

$$\partial[0,m]_{\mathbb{Z}}^n = \{(t_1,...,t_n) : \; \exists i \in \{1,2,..,m\} \; t_i = 0 \text{ or } t_i = m\}.$$

Let (X,p,κ) be a pointed digital image. Let $f : ([0,m]_{\mathbb{Z}}^n, \partial[0,m]_{\mathbb{Z}}^n) \to (X,p)$ be a $(2n,\kappa)$-continuous map, that is, the restriction map of f

$$f : [0,m]_{\mathbb{Z}}^n \to X$$

is $(2n,\kappa)$-continuous and

$$f(\partial[0,m]_{\mathbb{Z}}^n) = p.$$

Definition 3.1. Define $S_n^{\kappa}(X,p)$ as the set of all $(2n,\kappa)$-continuous maps on the digital image X and a base point p of the form $f : ([0,m]_{\mathbb{Z}}^n, \partial[0,m]_{\mathbb{Z}}^n) \to (X,p)$.

A digital analogue of homotopy relation for higher homotopy groups in algebraic topology given in [16,26] is as follows:

Definition 3.2. Let (X,p,κ) be a pointed digital image and

$$f,g : ([0,m]_{\mathbb{Z}}^n, \partial[0,m]_{\mathbb{Z}}^n) \to (X,p)$$

be two $(2n,\kappa)$-continuous maps. If there is a positive integer m_1 and a map

$$F : [0,m]^n_{\mathbb{Z}} \times [0,m_1]_{\mathbb{Z}} \to X$$

such that the following conditions are satisfied, then we say that f and g are relative homotopic with respect to $\partial[0,m]^n_{\mathbb{Z}}$ and denote it by $f \simeq g \ \text{rel}\ \partial[0,m]^n_{\mathbb{Z}}$:

–For all $t = (t_1,t_2,...,t_n) \in [0,m]^n_{\mathbb{Z}}$, $F(t,0) = f(t)$, and $F(t,m_1) = g(t)$;
–for all $s \in [0,m_1]_{\mathbb{Z}}$, and for all $t \in \partial[0,m]^n_{\mathbb{Z}}$, $F(t,s) = p$;
–for all $t \in [0,m]^n_{\mathbb{Z}}$, the induced map $F_t : [0,m_1]_{\mathbb{Z}} \to X$ defined by $F_t(s) = F(t,s)$ for all $s \in [0,m_1]_{\mathbb{Z}}$ is $(2,\kappa)$-continuous;
–for all $s \in [0,m_1]_{\mathbb{Z}}$, the induced map $F_s : [0,m]^n_{\mathbb{Z}} \to X$ defined by $F_s(t) = F(t,s)$ for all $t \in [0,m]^n_{\mathbb{Z}}$ is $(2n,\kappa)$-continuous.

The following theorem is the analogous version for in [6, Proposition 2.8]:

Proposition 3.3. The relative homotopy relation is an equivalence relation on the set $S^{\kappa}_n(X,p)$. The set of all equivalent classes of $S^{\kappa}_n(X,p)$ under the relative homotopy relation will be denoted by $\Pi^{\kappa}_n(X,p)$ and the equivalence class of $f \in S^{\kappa}_n(X,p)$ will be denoted by $[f]$.

Now we adapt the concept of the *'trivial extension map'* [7] in the case $n = 1$ for higher dimensions.

Definition 3.4. Let (X,p,κ) be a pointed digital image and

$$f : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]^n_{\mathbb{Z}}) \to (X,p)$$

be $(2n,\kappa)$-continuous map. If there is a positive integer integer $m_1 \geq m$, and a map

$$f' : ([0,m_1]^n_{\mathbb{Z}}, \partial[0,m_1]^n_{\mathbb{Z}}) \to (X,p) \quad \text{defined as}$$

$$f'(t_1,...,t_n) = \begin{cases} f(t), & 0 \leq \forall\, t_i \leq m, \ \ i = 1,2,...,n; \\ p, & \text{otherwise.} \end{cases}$$

for $t = (t_1,...,t_n) \in [0,m_1]^n_{\mathbb{Z}}$, then f' is called *the trivial extension* of f.

Digital $(2n,\kappa)$-continuous maps f and g in $S^{\kappa}_n(X,p)$ belong to the same equivalence class in $\Pi^{\kappa}_n(X,p)$ if there are trivial extensions f' and g' of f and g, respectively, whose domains have the same cardinality, and a relative digital homotopy between f' and g'.

Definition 3.5. Let (X,p,κ) be a pointed digital image. Let

$$f : ([0,m_1]^n_{\mathbb{Z}}, \partial[0,m_1]^n_{\mathbb{Z}}) \to (X,p)$$

and

$$g : ([0,m_2]^n_{\mathbb{Z}}, \partial[0,m_2]^n_{\mathbb{Z}}) \to (X,p)$$

be $(2n,\kappa)$-continuous maps. By the definition, these maps belong to $S^{\kappa}_n(X,p)$. The 'product' of f and g, written $f \star g$, is the generalization of definition of the product given in [19]. Formally;

$$(f \star g) : ([0,m_1+m_2]^n_{\mathbb{Z}}, \partial[0,m_1+m_2]^n_{\mathbb{Z}}) \to (X,p)$$

For a fixed index $i \in \{1,...,n\}$

$$(f\star g)(t) = \begin{cases} f(t_1,...,t_n), & t_1 \in [0,m_1]_{\mathbb{Z}} \\ & \text{and for } j \neq 1,\ t_j \leq m_1; \\ g(t_1,...,t_i - m_i,...,t_n), & t_i \in [m_1, m_1+m_2]_{\mathbb{Z}} \\ & \text{and for } j \neq 1,\ t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

From now on, without loss of generality we fix the index $i \in \{1,...,n\}$ as 1, i.e., we assume that:

$$(f\star g)(t) = \begin{cases} f(t_1,...,t_n), & t_1 \in [0,m_1]_{\mathbb{Z}} \\ & \text{and for } j \neq 1,\ t_j \leq m_1; \\ g(t_1 - m_1, t_2,...,t_n), & t_1 \in [m_1, m_1+m_2]_{\mathbb{Z}} \\ & \text{and for } j \neq 1,\ t_j \leq m_2; \\ x_0, & \text{otherwise.} \end{cases}$$

Lemma 3.6 The operation is well-defined on the set of $\Pi^{\kappa}_n(X,p)$, i.e, if $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$, then

$$[f_1 \star g_1] = [f_2 \star g_2].$$

Proof Let

$$f_1, f_2 : ([0,m_1]^n_{\mathbb{Z}}, \partial[0,m_1]^n_{\mathbb{Z}}) \to (X,p) \quad \text{and}$$

$$g_1, g_2 : ([0,m_2]^n_{\mathbb{Z}}, \partial[0,m_2]^n_{\mathbb{Z}} \to (X,p)$$

be $(2n,\kappa)$-continuous maps.

Since $f_1 \simeq f_2 \ \text{rel}\ \partial[0,m_1]^n_{\mathbb{Z}}$, there is a positive integer n_1 and a map

$$G : [0,m_1]^n_{\mathbb{Z}} \times [0,n_1]_{\mathbb{Z}} \quad \text{such that}$$

–for all $t = (t_1,...,t_n) \in [0,m_1]^n_{\mathbb{Z}}$, $G(t,0) = f_1(t)$ and $G(t,n_1) = f_2(t)$;
–for all $s \in [0,n_1]_{\mathbb{Z}}$, and for all $t \in \partial[0,m_1]^n_{\mathbb{Z}}$, $G(t,s) = p$;
–for all $t \in [0,m_1]^n_{\mathbb{Z}}$, the induced map $G_t : [0,n_1]_{\mathbb{Z}} \to X$ defined by $G_t(s) = G(t,s)$ for all $s \in [0,n_1]_{\mathbb{Z}}$ is $(2,\kappa)$-continuous;
–for all $s \in [0,n_1]_{\mathbb{Z}}$, the induced map $G_s : [0,m_1]^n_{\mathbb{Z}} \to X$ defined by $G_s(t) = G(t,s)$ for all $t \in [0,m_1]^n_{\mathbb{Z}}$ is $(2n,\kappa)$-continuous.

Since $g_1 \simeq g_2$ rel $\partial[0,m_2]^n_{\mathbb{Z}}$, there is a positive integer n_2 and a map

$$H : [0,m_2]^n_{\mathbb{Z}} \times [0,n_2]_{\mathbb{Z}} \quad \text{such that}$$

–for all $t = (t_1,...,t_n) \in [0,m_2]^n_{\mathbb{Z}}$, $H(t,0) = g_1(t)$ and $H(t,n_2) = g_2(t)$;
–for all $s \in [0,n_2]_{\mathbb{Z}}$, and for all $t \in \partial[0,m_2]^n_{\mathbb{Z}}$, $H(t,s) = p$;
–for all $t \in [0,m_2]^n_{\mathbb{Z}}$, the induced map $H_t : [0,n_2]_{\mathbb{Z}} \to X$ defined by $G_t(s) = G(t,s)$ for all $s \in [0,n_2]_{\mathbb{Z}}$ is $(2,\kappa)$-continuous;
–for all $s \in [0,n_2]_{\mathbb{Z}}$, the induced map $H_s : [0,m_2]^n_{\mathbb{Z}} \to X$ defined by $H_s(t) = H(t,s)$ for all $t \in [0,m_2]^n_{\mathbb{Z}}$ is $(2n,\kappa)$-continuous.

Let $m_3 = m_1 + m_2$ and $n_3 = n_1 + n_2$

$$(f_1 \star g_1) : ([0,m_3]^n_{\mathbb{Z}}, \partial[0,m_3]^n_{\mathbb{Z}}) \to X$$

$$(f_1 \star g_1)(t) = \begin{cases} f_1(t), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_1; \\ g_1(t^*), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

where $t = (t_1,...,t_n)$ and $t^* = (t_1 - m_1,...,t_n)$.

$$(f_2 \star g_2) : ([0,m_3]^n_{\mathbb{Z}}, \partial[0,m_3]^n_{\mathbb{Z}}) \to X$$

$$(f_1 \star g_1)(t) = \begin{cases} f_2(t), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_1; \\ g_2(t^*), & t_{\in}[m_1,m_3]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

where $t = (t_1,...,t_n)$ and $t^* = (t_1 - m_1,...,t_n)$.
Now let's define a relative homotopy map between $(f_1 \star g_1)$ and $(f_2 \star g_2)$.

$$F : [0,m_3]^n_{\mathbb{Z}} \times [0,n_3]^n_{\mathbb{Z}} \to X$$

$$F(t,s) = \begin{cases} G(t,\min\{s,n_1\}), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1,\ t_j \leq m_1; \\ H(t^*,\max\{s-n_1,0\}), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1,\ t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

where $t = (t_1,...,t_n)$ and $t^* = (t_1 - m_1,...,t_n)$.

Then the map F is a relative homotopy between $(f_1 \star g_1)$ and $(f_2 \star g_2)$ and;

$$F(t,0) = \begin{cases} G(t,\min\{0,n_1\}), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1,\ t_j \leq m_1; \\ H(t^*,\max\{0-n_1,0\}), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1,\ t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} G(t,0), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1, t_j \leq m_1; \\ H(t^*,0), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for} \\ & j \neq 1, t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

$$= (f_1 \star g_1)(t).$$

$$F(t,n_3) = \begin{cases} G(t,\min\{n_3,n_1\}), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_1; \\ H(t^*,\max\{n_3-n_1,0\}), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for} j \neq 1, \\ & t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} G(t,n_1), & t_1 \in [0,m_1]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_1; \\ H(t^*,n_2), & t_1 \in [m_1,m_3]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m_2; \\ p, & \text{otherwise.} \end{cases}$$

$$= (f_2 \star g_2)(t)$$

where $t = (t_1,...,t_n)$ and $t^* = (t_1 - m_1,...,t_n)$.

□

By Lemma 3.6, the operation

$$[f] \star [g] = [f \star g]$$

is well-defined on $\Pi^{\kappa}_n(X,p)$. We will show that the set $\Pi^{\kappa}_n(X,p)$ together with the '$\star$' operation is a group.

Lemma 3.7. The operation '$\star$' is associative on the set $\Pi^{\kappa}_n(X,p)$. That is, if $[f],[g],[h] \in \Pi^{\kappa}_n(X,p)$, then

$$([f \star g] \star h) = ([f] \star [g \star h]).$$

Proof The proof follows from the analogous result in [6, Lemma 4.5].

□

Lemma 3.8. Let (X,p,κ) be a pointed digital image and

$$e_p : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]^n_{\mathbb{Z}}) \to (X,p)$$

be a map defined by $e_p(t) = p$ for all $t = (t_1, ..., t_n) \in [0,m]^n_{\mathbb{Z}}$. Then $[e_p]$ is an identity element for $\Pi^\kappa_n(X,p)$. That is, if $[f] \in \Pi^\kappa_n(X,p)$, then

$$[f] \star [e_p] = [e_p] \star [f] = [f].$$

Proof Let

$$f : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]^n_{\mathbb{Z}}) \to (X,p)$$

be a $(2n,\kappa)$-continuous map and

$$e_p : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]^n_{\mathbb{Z}}) \to (X,p),$$

$e_p(t) = p \ \forall t = (t_1,t_2,...,t_n) \in [0,m]^n_{\mathbb{Z}}$ be a constant map. We'd like to show that

$$f \star e_p \simeq f \text{ rel } \partial[0,m]^n_{\mathbb{Z}}.$$

$$(f \star e_p) : ([0,2m]^n_{\mathbb{Z}}, \partial[0,2m]^n_{\mathbb{Z}}) \to (X,p)$$

$$(f \star e_p)(t) = \begin{cases} f(t), & t_1 \in [0,m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ e_p(t^*), & t_1 \in [m,2m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ p, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(t), & t_1 \in [0,m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ p, & \text{otherwise.} \end{cases}$$

Then we see that $(f \star e_p)$ are trivial extensions of f, hence they are relative homotopic to each other: $f \simeq (f \star e_p)$ rel $\partial[0,m]^n_{\mathbb{Z}}$.

Similarly, it can be seen that $(e_p \star f) \simeq f$ rel $\partial[0,m]^n_{\mathbb{Z}}$.

□

Lemma 3.9. Let (X,p,κ) be a pointed digital image and $f : [0,m]^n_{\mathbb{Z}} \to X$ be a $(2n,\kappa)$-continuous map. Then the map

$$g : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]^n_{\mathbb{Z}}) \to (X,p)$$

defined by

$$g(t) = f(m - t_1, ...t_n)$$

for $t = (t_1,...,t_n) \in [0,m]^n_{\mathbb{Z}}$ is an element of $[f^{-1}]$ in $\Pi^\kappa_n(X,p)$.

Proof We'd like to show that there is a relative homotopy between $(f \star g)$ and e_p.

$$(f \star g) : ([0,2m]^n_{\mathbb{Z}}, \partial[0,2m]^n_{\mathbb{Z}}) \to (X,p)$$

$$(f \star g)(t) = \begin{cases} f(t), & t_1 \in [0,m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ g(t^*), & t_1 \in [m,2m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ p, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(t), & t_1 \in [0,m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ f(t'), & t_1 \in [m,2m]_{\mathbb{Z}}, \text{ and for } j \neq 1, \\ & t_j \leq m; \\ p, & \text{otherwise.} \end{cases}$$

where $t = (t_1,...,t_n)$ and $t' = (2m - t_1,...,t_n)$.

Define;

$$H : [0,2m]^n_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}} \to X$$

$$H(t,s) = \begin{cases} (f \star g)(t), & t_1 \in [0,m-s]_{\mathbb{Z}} \text{ or for } j \neq 1, \\ & m+s \leq t_1 \leq 2m; \\ (f \star g)(t''), & \text{otherwise.} \end{cases}$$

where $t = (t_1,...,t_n)$ and $t'') = (m-s,...,t_n)$

Then the digital map H is a relative homotopy between $(f \star g)$ and e_p.

With a similar proof, it can be seen that $(g \star f) \simeq e_p$ rel $\partial[0,m]^n_{\mathbb{Z}}$.

□

Definition 3.10. The set $\Pi^\kappa_n(X,p)$ together with the operation $\star$ has a group structure as the previous three lemmas hold. This group is called a digital *n-th homotopy group of a pointed digital image* (X,p).

Lemma 3.11. Let (X,p,κ_1) and (Y,q,κ_2) be two digital images, and let the digital map

$$\varphi : (X,p) \to (Y,q), \quad \varphi(p) = q$$

be a (κ_1,κ_2)-continuous. If $f : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]_{\mathbb{Z}}) \to (X,p)$ is $(2n,\kappa_1)$-continuous map, then

$$(\varphi \circ f) : ([0,m]^n_{\mathbb{Z}}, \partial[0,m]_{\mathbb{Z}}) \to (Y,q)$$

is $(2n,\kappa_2)$-continuous map.

Lemma 3.12. Let (X,p,κ_1) and (Y,q,κ_2) be two digital images, and

$$\varphi : (X,p) \to (Y,q), \quad \varphi(p) = q$$

be a (κ_1,κ_2)-continuous map. For

$$f,g:([0,m]^n_{\mathbb{Z}},\partial[0,m]_{\mathbb{Z}})\to(X,p)$$

$(2n,\kappa_1)$-continuous maps, if $f\simeq g$ rel $\partial[0,m]^n_{\mathbb{Z}}$, then $\varphi\circ f\simeq\varphi\circ g$ rel $\partial[0,m]^n_{\mathbb{Z}}$.

Proof Let $f\simeq g$ rel $\partial[0,m]^n_{\mathbb{Z}}$. Then there exists a digital interval $[0,m_1]_{\mathbb{Z}}$, and a digital map

$$F:[0,m]^n_{\mathbb{Z}}\times[0,m_1]_{\mathbb{Z}}\to X \text{ such that}$$

–for all $t=(t_1,...,t_n)\in[0,m]^n_{\mathbb{Z}}$, $F(t,0)=f(t)$ and $F(t,m_1)=g(t)$;
–for all $s\in[0,m_1]_{\mathbb{Z}}$, and for all $t\in\partial[0,m]^n_{\mathbb{Z}}$, $F(t,s)=p$;
–for all $t\in[0,m]^n_{\mathbb{Z}}$, the induced map $F_t:[0,m_1]_{\mathbb{Z}}\to X$ defined by $F_t(s)=F(t,s)$ for all $s\in[0,m_1]_{\mathbb{Z}}$ is $(2,\kappa_1)$-continuous;
–for all $s\in[0,m_1]_{\mathbb{Z}}$, the induced map $F_s:[0,m]^n_{\mathbb{Z}}\to X$ defined by $F_s(t)=F(t,s)$ for all $t\in[0,m]^n_{\mathbb{Z}}$ is $(2n,\kappa_1)$-continuous.

We'd like to construct a relative homotopy map between

$$(\varphi\circ f),(\varphi\circ g):([0,m]^n_{\mathbb{Z}},\partial[0,m]_{\mathbb{Z}})\to(Y,q).$$

Define,

$$H:[0,m]^n_{\mathbb{Z}}\times[0,m_1]_{\mathbb{Z}}\to X$$

$$H(t,s)=\varphi\circ F(t,s)\quad\text{for } t=(t_1,...,t_n)\in[0,m]^n_{\mathbb{Z}}.$$

Then;

–for all $t=(t_1,...,t_n)\in[0,m]^n_{\mathbb{Z}}$, $H(t,0)=\varphi\circ f(t)$ and $H(t,m_1)=\varphi\circ g(t)$;
–for all $s\in[0,m_1]_{\mathbb{Z}}$ and for all $t\in\partial[0,m]^n_{\mathbb{Z}}$, $H(t,s)=q$;
–for all $t\in[0,m]^n_{\mathbb{Z}}$, the induced map $H_t:[0,m_1]_{\mathbb{Z}}\to Y$, $H_t(s)=\varphi\circ F(t,s)$ for all $s\in[0,m_1]_{\mathbb{Z}}$ is $(2,\kappa_2)$-continuous;
–for all $s\in[0,m_1]_{\mathbb{Z}}$, the induced map $H_s:[0,m]^n_{\mathbb{Z}}\to X$, $H_s(t)=\varphi\circ F(t,s)$ for all $t\in[0,m]^n_{\mathbb{Z}}$ is $(2n,\kappa_2)$-continuous.

Hence, $(\varphi\circ f)\simeq(\varphi\circ g)$ rel $\partial[0,m]^n_{\mathbb{Z}}$.

□

Theorem 3.13. The digital homotopy group construction induces a covariant functor from the category of pointed digital images and pointed digitally continuous functions to the category of groups and homomorphisms.

Proof The proof follows from the analogous version for [6, Proposition 2.8].

□

Theorem 3.14. If (X,p,κ_1) and (Y,q,κ_2) are (κ_1,κ_2) isomorphic and the point $p\in X$ is mapped to the point $q\in Y$ under the isomorphism map φ, then $\Pi^{\kappa_1}_n(X,p)$ and $\Pi^{\kappa_1}_n(Y,q)$ are isomorphic groups for all positive integer n.

Proof Let $\varphi:X\to Y$ be (κ_1,κ_2)-isomorphism map and $\varphi(p)=q$, φ has a (κ_2,κ_1)-continuous inverse, say,

$$\phi:Y\to X \text{ such that } \phi(q)=p.$$

Then,

$$\varphi\circ\phi=Id_Y \text{ and } \phi\circ\varphi=Id_X.$$

From Theorem 3.13, the homotopy functor Π_n is covariant functor and preserves the composition of maps and identity. Hence;

$$\Pi_n(\varphi\circ\phi)=\Pi_n(Id_Y)\ \to\ \Pi_n(\varphi)\circ\Pi_n(\phi)=Id_{\Pi_n(Y,q)}$$

so that the homomorphism $\Pi_n(\varphi)$ is surjective and;

$$\Pi_n(\phi\circ\varphi)=\Pi_n(Id_X)\ \to\ \Pi_n(\phi)\circ\Pi_n(\varphi)=Id_{\Pi_n(X,p)}$$

so that the homomorphism $\Pi_n(\varphi)$ is injective. Therefore, $\Pi_n(\varphi)$ is a group isomorphism between $\Pi^{\kappa_1}_n(X,p)$ and $\Pi^{\kappa_2}_n(Y,q)$.

□

Corollary 3.15. If the pointed digital images (X,p,κ_1) and (Y,q,κ_2) have the same (κ_1,κ_2)-homotopy type, then their n-th homotopy groups are isomorphic.

Corollary 3.16. Let (X,p,κ) be a pointed digital image that is contractible to a point p. Then $\Pi^{\kappa}_n(X,p)$ is trivial for all positive integer n.

Proof Since X is contractible to a point p, there exists a homotopy map between the identitiy map of X and a constant map p such that

$$H:X\times[0,m_1]_{\mathbb{Z}}\to X$$

–$H(x,0)=x,\ \forall x\in X$;
–$H(x,m_1)=p,\ \forall p\in X$ and,
–$H(p,s)=p,\ \forall s\in[0,m_1]_{\mathbb{Z}}$

Take any $(2n,\kappa)$-continuous map,

$$f:([0,m]^n_{\mathbb{Z}},\partial[0,m]^n_{\mathbb{Z}})\to(X,p).$$

Define the relative homotopy map

$$G:[0,m]^n_{\mathbb{Z}}\times[0,m_1]_{\mathbb{Z}}\to X$$

$$(t,s)\longmapsto G(t,s)=H(f(t),s)$$

for $t=(t_1,...,t_n)\in[0,m]^n_{\mathbb{Z}}$. Then

–For all $t=(t_1,...,t_n)\in[0,m]^n_{\mathbb{Z}}$,

$$G(t,0)=H(f(t_1,...,t_n),0)=f(t_1,...,t_n)$$

and

$$G(t,m_1)=H(f(t),m_1)=p;$$

–for all $s \in [0,m_1]_{\mathbb{Z}}$, and for all $t \in \partial[0,m]^n_{\mathbb{Z}}$,

$$G(t,s) = H(p,s) = p;$$

–for all $t \in [0,m]^n_{\mathbb{Z}}$, the induced map $G_t : [0,m_1]_{\mathbb{Z}} \to X$,

$$G_t(s) = H(f(t),s)$$

for all $s \in [0,m_1]_{\mathbb{Z}}$ is $(2,\kappa)$-continuous;
–for all $s \in [0,m_1]_{\mathbb{Z}}$, the induced map $G_s : [0,m]^n_{\mathbb{Z}} \to X$,

$$G_s(t) = H(f(t),s)$$

for all $t \in [0,m]^n_{\mathbb{Z}}$ is $(2n,\kappa)$-continuous.

So, $f \simeq e_p$ rel $\partial[0,m]^n_{\mathbb{Z}}$, and hence the group $\Pi^{\kappa}_n(X,p)$ is trivial.

□

4 Application

Digital image processing algorithms are the tools for correcting the quality of images such as satellite images, telescope images, videos, photos or photoshops. Any distinction that can be measured, idenitified, compared or visualized are extremely important. By combining the algebraic topological concepts such as Euler characteristics, (co)homology groups, fundamental and homotopy groups with digital topology we can distinguish or compare the digital images between each other. As an example computer scientists use the fundamental group to count the objects in the images. Also the contractibility is a kind of a thinning operation and it can be used to compare or classify the digital objects.

Since regions in 2 or 3 dimensional Euclidean spaces are digitized as the discrete grid of pixels or voxels which have integer coordinates respectively, it is of interest to consider topological invariants that are preserved by the digitization process for computer tomography. The digital homotopy groups are such a tool for the desired interest. As an example, by computing the digital homotopy groups of MRI and CAT scan o a brain, it may detect the abnormalities like a brain tumor and diagnose the size or the stage of it by comparing with the digital homotopy groups of fine brain scan.

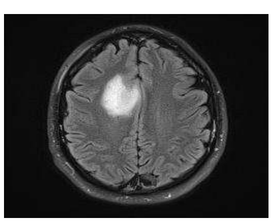

Fig. 1: MRI scan of a tumoral brain.

As an another example, the image of the seismic section is given below. The blue line is a fault and the red lines are erosional surfaces. The computation of the digital homotopy groups of the seismic section might be able to detect the breaking point of the fault. The digital

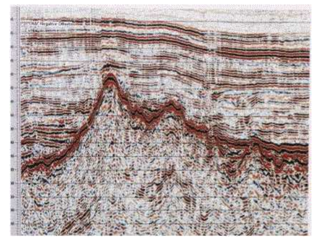

Fig. 2: The image of a fault and an erosional surfaces of a seismic section.

homotopy can be used to reduced the computer data of the image for saving of time. This is a tool for thinning operation for the digital image to reduce the data without loosing its own topological properties. Below the digital image

$$MSC'_8 = \{(1,0),(0,1),(-1,0)(0,-1)\}$$

is an 8-deformation retract to $\mathbb{Z}^2 - \{(0,0)\}$. In this example the infinite digital image $\mathbb{Z}^2 - \{(0,0)\}$ is reduced to bounded image without loosing its connectivity and homotopy type [17].

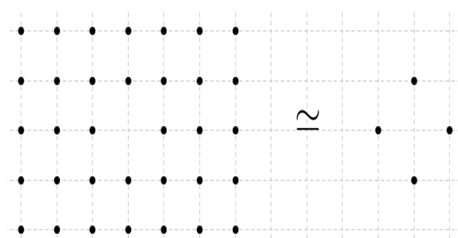

Fig. 3: $MSC8'$ is an 8-deformation retract to $\mathbb{Z}^2 - \{(0,0)\}$.

5 Conclusion

In this paper we introduce the construction of the digital homotopy groups and show that they are invariants for digital images. In the future, we will investigate the relationship between the digital higher homotopy groups and digital covering space map, and the relationship between the digital homology groups and the homotopy groups of a digital image. The homotopy theory in algebraic topology satisfies all the homology axioms except the 'excision'. We'll explore which homology axioms are satisfied without any certain conditions. Also we'll study the digital homotopy groups of a pair and we hope that this will yield us to compute the homotopy groups of some part of the digital image if we know the homotopy groups of an entire image.

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