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Common Fixed Point Theorems of Generalized Lipschitz Mappings in Cone Metric Spaces over Banach Algebras

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Abstract: In this paper, we prove some common fixed point theorems in cone metric spaces over Banach algebras. Furthermore, we give some examples to support our assertions and to show our results in the setting of cone metric spaces over Banach algebras are never equivalent to the counterparts in metric spaces.

Keywords: Generalized Lipschitz constant, Common fixed point, Weakly compatible, c-sequence

1 Introduction

Recently, it is not popular with the fixed point theorems in cone metric spaces any more. Investigating its reason, some scholars find that cone metric space is the generalization of the usual metric space but it is a fake generalization if one works with the assumption of normal cone. Constructing some equivalent metrics by using different approaches, they claim that the fixed point results in cone metric spaces just are repeated as the usual metric cases in metric spaces. Moreover, they make a conclusion that cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved (see [1,2,3,4,5,6]).

But the current situation changed, since, very recently, Liu and Xu [7] introduced the concept of cone metric space over Banach algebra, replacing Banach space by Banach algebra as the underlying space of cone metric space. In this way, they proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant k by means of spectral radius $\rho(k)$. Note that it is significant to introduce the concept of cone metric space over Banach algebra since one can prove that cone metric spaces over Banach algebras are not equivalent to metric spaces in terms of the existence of the fixed points of the generalized Lipschitz mappings (see [7]). In this paper, we obtain some common fixed point theorems of generalized Lipschitz mappings in cone metric spaces over Banach algebras without assumption of normality. Our main results greatly generalize the previous work in the literature. Otherwise, we give some examples to support our assertions and to show the non-equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in usual sense).

In order to start this paper, we need to briefly recall some basic terms and notions as follows.

Let $\mathscr{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathscr{A}$. A nonempty closed convex subset P of $\mathscr{A}$ is called a cone if $\{\theta,e\}\subset P$, $P^2=PP\subset P$, $P\cap(-P)=\{\theta\}$ and $\lambda P+\mu P\subset P$ for all $\lambda,\mu\geq 0$. On this basis, we define a partial ordering $\preceq$ with respect to P by $x\preceq y$ if and only if $y-x\in P$. We shall write $x\prec y$ to indicate that $x\preceq y$ but $x\neq y$, while $x\ll y$ will indicate that $y-x\in \mathrm{int}P$, where $\mathrm{int}P$ stands for the interior of P. If $\mathrm{int}P\neq\emptyset$, then P is called a solid cone. Write $\|\cdot\|$ as the norm on $\mathscr{A}$. A cone P is called normal if there is a number $M>0$ such that for all $x,y\in\mathscr{A}$, $\theta\preceq x\preceq y$ implies $\|x\|\leq M\|y\|$. The least positive number satisfying above is called the normal constant of P. It is well known that $M\geq 1$.

In what follows, we always suppose that $\mathscr{A}$ is a Banach algebra with a unit e, P is a solid cone, and $\preceq$ is a partial ordering with respect to P.

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Definition 1.1 ([7]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathscr{A}$ satisfies:

(i) $\theta \prec d(x,y)$ for all $x,y \in X$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over Banach algebra $\mathscr{A}$.

Definition 1.2 ([7]) Let (X,d) be a cone metric space over Banach algebra $\mathscr{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$.

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Example 1.3 Let $\mathscr{A} = C[a,b]$ be the Banach space of all real continuous functions on a closed interval $[a,b]$, with the supremum norm. Define multiplication in the usual way: $(xy)(t) = x(t)y(t)$. This makes $\mathscr{A}$ into a Banach algebra; the constant function 1 is the unit element. Let $P = \{x \in \mathscr{A} : x(t) \geq 0, t \in [a,b]\}$ and $X = \mathbb{R}$. Define a mapping $d : X \times X \to \mathscr{A}$ by $d(x,y) = |x-y|\varphi$ for all $x,y \in X$, where $\varphi : [a,b] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X,d) is a complete cone metric space over Banach algebra $\mathscr{A}$.

Definition 1.4 ([10]) Let $f,g : X \to X$ be mappings on a set X.

(1) If $y = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and y is called a point of coincidence of f and g;

(2) The pair (f,g) is called weakly compatible if f and g commute at all of their coincidence points, that is, $fgx = gfx$ for all $x \in C(f,g) = \{x \in X : fx = gx\}$.

Definition 1.5 ([11]) Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 1.6 Let P be a solid cone in a Banach space E and $\{u_n\} \subset P$ be a sequence with $\|u_n\| \to 0 (n \to \infty)$, then $\{u_n\}$ is a c-sequence.

Proof Let $c \gg \theta$ be given. It follows that there is $\delta > 0$ such that

$$U(c,\delta) = \{x \in E : \|x-c\| < \delta\} \subset P.$$

On account of $\|u_n\| \to 0 (n \to \infty)$, then there exists N such that $\|u_n\| < \delta$ for all $n > N$. Hence $\|(c-u_n)-c\| = \|u_n\| < \delta$, which implies that $c - u_n \in U(c,\delta) \subset P$, that is, $c - u_n \in \mathrm{int}P$, thus $u_n \ll c$ for all $n > N$.

Lemma 1.7 ([19]) Let $\mathscr{A}$ be a Banach algebra with a unit e, $x \in \mathscr{A}$, then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathscr{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.8 ([19]) Let $\mathscr{A}$ be a Banach algebra with a unit e, $a,b \in \mathscr{A}$. If a commutes with b, then

$$\rho(a+b) \leq \rho(a) + \rho(b), \qquad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 1.9 ([16]) Let P be a solid cone in a Banach space E.

(1) If $a,b,c \in E$ and $a \preceq b \ll c$, then $a \ll c$.

(2) If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.

Lemma 1.10 ([8]) Let P be a solid cone in a Banach algebra $\mathscr{A}$ and $\{u_n\}$ be a c-sequence in P. If $k \in P$ is an arbitrarily given vector, then $\{ku_n\}$ is a c-sequence.

Proof Without loss of generality, put $\theta \prec k$. Let $c \gg \theta$. There exists $\delta > 0$ such that

$$U(c,\delta) = \{x \in \mathscr{A} : \|x-c\| < \delta\} \subset P.$$

Choose $c_0 \gg \theta$ with $\|c_0\| < \frac{\delta}{\|k\|}$. Note that

$$\|(c-kc_0)-c\| = \|kc_0\| \leq \|k\|\|c_0\| < \delta$$
$$\Rightarrow c - kc_0 \in U(c,\delta) \subset P,$$

which means that $c - kc_0 \in \mathrm{int}P$, that is, $kc_0 \ll c$. Since $\{u_n\}$ is a c-sequence, then there exists N such that $u_n \ll c_0$ for all $n > N$, so by (1) of Lemma 1.9, $ku_n \ll c$ $(n > N)$

2 Main results

In this section, we give some valuable lemmas in Banach algebras which will be used in the sequel. Moreover, we obtain several common fixed point theorems in cone metric spaces over Banach algebras instead of the theorems only in cone metric spaces over usual Banach spaces. All conclusions are new. Further, we illustrate our results by some examples. These examples indicate that our results in the setting of cone metric spaces over Banach algebras are never equivalent to the versions of usual metric spaces.

Lemma 2.1 Let $\mathscr{A}$ be a Banach algebra with a unit e and P be a solid cone in $\mathscr{A}$. Let $a,k,l \in P$ hold $l \preceq k$ and $a \preceq la$. If $\rho(k) < 1$, then $a = \theta$.

Proof Since $\rho(k) = \lim_{n\to\infty} \|k^n\|^{\frac{1}{n}} < 1$, then there exists $\alpha > 0$ such that $\lim_{n\to\infty} \|k^n\|^{\frac{1}{n}} < \alpha < 1$. Letting n be big enough, we obtain $\|k^n\|^{\frac{1}{n}} \leq \alpha$, so that $\|k^n\| \leq \alpha^n \to 0$ $(n \to \infty)$. Thus $\|k^n\| \to 0$ $(n \to \infty)$. Hence by Lemma 1.6, $\{k^n\}$ is a c-sequence, and by Lemma 1.10, $\{k^n a\}$ is also a c-sequence. As $l \preceq k$ leads to $a \preceq la \preceq l^2 a \preceq \cdots \preceq l^n a \preceq k^n a$, thus by Lemma 1.9 that $a = \theta$.

Lemma 2.2 Let $\mathscr{A}$ be a Banach algebra with a unit e. Let $k \in \mathscr{A}$. If $\rho(k) < 1$, then

$$\rho\left((e-k)^{-1}\right) \leq \frac{1}{1-\rho(k)}.$$

Proof Since $\rho(k) < 1$, it follows by Lemma 1.7 that $e-k$ is invertible and $(e-k)^{-1} = \sum_{i=0}^{\infty} k^i$. Set $s = \sum_{i=0}^{\infty} k^i$, $s_n = \sum_{i=0}^{n} k^i$, then $s_n \to s$ $(n \to \infty)$ and s_n commutes with s for all n. It follows immediately from Lemma 1.8 that

$$\begin{aligned}
&\rho(s_n) = \rho(s_n - s + s) \leq \rho(s_n - s) + \rho(s)\\
&\Rightarrow \rho(s_n) - \rho(s) \leq \rho(s_n - s),\\
&\rho(s) = \rho(s - s_n + s_n) \leq \rho(s - s_n) + \rho(s_n)\\
&\Rightarrow \rho(s) - \rho(s_n) \leq \rho(s - s_n),
\end{aligned}$$

which imply that

$$\begin{aligned}
&|\rho(s_n) - \rho(s)| \leq \rho(s_n - s) \leq \|s_n - s\|\\
&\Rightarrow \rho(s_n) \to \rho(s) (n \to \infty).
\end{aligned}$$

Thus again by Lemma 1.8,

$$\begin{aligned}
\rho\left((e-k)^{-1}\right) &= \rho\left(\sum_{i=0}^{\infty} k^i\right)\\
&= \rho(s) = \lim_{n\to\infty} \rho(s_n) = \lim_{n\to\infty} \rho\left(\sum_{i=0}^{n} k^i\right)\\
&\leq \lim_{n\to\infty} \sum_{i=0}^{n} [\rho(k)]^i = \sum_{i=0}^{\infty} [\rho(k)]^i = \frac{1}{1-\rho(k)}.
\end{aligned}$$

Theorem 2.3 Let (X,d) be a cone metric space over Banach algebra $\mathscr{A}$ and P be a solid cone in $\mathscr{A}$. Suppose that f,g,S,T are four self-maps on X such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$ and suppose that at least one of these four subsets of X is complete. Let

$$\begin{aligned}
d(Sx,Ty) \preceq\ & k_1 d(fx,Sx) + k_2 d(gy,Sx) + k_3 d(fx,Ty)\\
&+ k_4 d(gy,Ty) + k_5 d(fx,gy), \qquad (2.1)
\end{aligned}$$

for all $x,y \in X$, where $k_i \in P$ are generalized Lipschitz constants with $k_i k_j = k_j k_i$ $(i,j = 1,\ldots,5)$. If $\rho(k_1 + k_2) + \rho(k_1 + k_3 + k_5) < 1$ and $\rho(k_3 + k_4) + \rho(k_2 + k_4 + k_5) < 1$, then the pairs (f,S) and (g,T) have a unique common point of coincidence. Moreover, f,g,S and T have a unique common fixed point provided that the pairs (f,S) and (g,T) are weakly compatible.

Proof Let x_0 be an arbitrary point in X and define a sequence $\{y_n\}$ in X as follows:

$$y_{2n} = Sx_{2n} = gx_{2n+1}, \qquad y_{2n+1} = Tx_{2n+1} = fx_{2n+2},$$

for all $n \geq 0$. Taking advantage of (2.1), we have

$$\begin{aligned}
&d(y_{2n},y_{2n+1}) = d(Sx_{2n},Tx_{2n+1})\\
&\preceq k_1 d(fx_{2n},Sx_{2n}) + k_2 d(gx_{2n+1},Sx_{2n})\\
&\quad + k_3 d(fx_{2n},Tx_{2n+1}) + k_4 d(gx_{2n+1},Tx_{2n+1})\\
&\quad + k_5 d(fx_{2n},gx_{2n+1})\\
&\preceq k_1 d(y_{2n-1},y_{2n}) + k_3[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})]\\
&\quad + k_4 d(y_{2n},y_{2n+1}) + k_5 d(y_{2n-1},y_{2n})\\
&= (k_1+k_3+k_5) d(y_{2n-1},y_{2n}) + (k_3+k_4) d(y_{2n},y_{2n+1}),
\end{aligned}$$

which implies that

$$(e-k_3-k_4) d(y_{2n},y_{2n+1}) \preceq (k_1+k_3+k_5) d(y_{2n-1},y_{2n}).$$

Since $\rho(k_3 + k_4) + \rho(k_2 + k_4 + k_5) < 1$ leads to $\rho(k_3 + k_4) < 1$, it concludes by Lemma 1.7 that $e-k_3-k_4$ is invertible, so

$$d(y_{2n},y_{2n+1}) \preceq (e-k_3-k_4)^{-1}(k_1+k_3+k_5) d(y_{2n-1},y_{2n}).$$

Put $\lambda = (e-k_3-k_4)^{-1}(k_1+k_3+k_5)$, it is evident that

$$d(y_{2n},y_{2n+1}) \preceq \lambda d(y_{2n-1},y_{2n}). \qquad (2.2)$$

Again by using (2.1),

$$\begin{aligned}
&d(y_{2n+1},y_{2n+2}) = d(Sx_{2n+2},Tx_{2n+1})\\
&\preceq k_1 d(fx_{2n+2},Sx_{2n+2}) + k_2 d(gx_{2n+1},Sx_{2n+2})\\
&\quad + k_3 d(fx_{2n+2},Tx_{2n+1}) + k_4 d(gx_{2n+1},Tx_{2n+1})\\
&\quad + k_5 d(fx_{2n+2},gx_{2n+1})\\
&\preceq k_1 d(y_{2n+1},y_{2n+2}) + k_2[d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2})]\\
&\quad + k_4 d(y_{2n},y_{2n+1}) + k_5 d(y_{2n+1},y_{2n})\\
&= (k_2+k_4+k_5) d(y_{2n},y_{2n+1}) + (k_1+k_2) d(y_{2n+1},y_{2n+2}),
\end{aligned}$$

which means that

$$(e-k_1-k_2) d(y_{2n+1},y_{2n+2}) \preceq (k_2+k_4+k_5) d(y_{2n},y_{2n+1}).$$

Because $\rho(k_1 + k_2) + \rho(k_1 + k_3 + k_5) < 1$ indicates $\rho(k_1+k_2) < 1$, it follows by Lemma 1.7 that $e-k_1-k_2$ is invertible, hence

$$d(y_{2n+1},y_{2n+2}) \preceq (e-k_1-k_2)^{-1}(k_2+k_4+k_5) d(y_{2n},y_{2n+1}).$$

Set $\mu = (e-k_1-k_2)^{-1}(k_2+k_4+k_5)$, it establishes that

$$d(y_{2n+1},y_{2n+2}) \preceq \mu d(y_{2n},y_{2n+1}). \qquad (2.3)$$

Combining (2.2) and (2.3), we obtain

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &\preceq \mu\lambda d(y_{2n-1},y_{2n}),\\
d(y_{2n},y_{2n+1}) &\preceq \lambda\mu d(y_{2n-2},y_{2n-1}).
\end{aligned}$$

As $k_i k_j = k_j k_i$ $(i,j = 1,\ldots,5)$, one has $\mu\lambda = \lambda\mu$. Denoting $h = \lambda\mu$, we claim that

$$d(y_{2k+1},y_{2k+2}) \preceq h d(y_{2k-1},y_{2k}) \preceq \cdots \preceq h^k d(y_1,y_2) \qquad (2.4)$$

and

$$d(y_{2k},y_{2k+1}) \preceq hd(y_{2k-2},y_{2k-1}) \preceq \cdots \preceq h^k d(y_0,y_1) \tag{2.5}$$

for any k. So by (2.4) and (2.5) it concludes that

$$d(y_n,y_{n+1}) \preceq h^{\frac{n-1}{2}} d(y_1,y_2) \quad (n=2k+1) \tag{2.6}$$

and

$$d(y_n,y_{n+1}) \preceq h^{\frac{n}{2}} d(y_0,y_1) \qquad (n=2k). \tag{2.7}$$

Note by Lemma 1.8 and Lemma 2.2 that

$$\begin{aligned}\rho(h) &= \rho(\lambda\mu)\\ &= \rho\big[(e-k_3-k_4)^{-1}(k_1+k_3+k_5)\\ &\quad (e-k_1-k_2)^{-1}(k_2+k_4+k_5)\big]\\ &\le \rho\big((e-k_3-k_4)^{-1}\big)\rho(k_1+k_3+k_5)\\ &\quad \rho\big((e-k_1-k_2)^{-1}\big)\rho(k_2+k_4+k_5)\\ &\le \frac{\rho(k_1+k_3+k_5)}{1-\rho(k_3+k_4)}\cdot\frac{\rho(k_2+k_4+k_5)}{1-\rho(k_1+k_2)}\\ &= \frac{\rho(k_1+k_3+k_5)}{1-\rho(k_1+k_2)}\cdot\frac{\rho(k_2+k_4+k_5)}{1-\rho(k_3+k_4)}\\ &< 1,\end{aligned}$$

which means that $(e-h)^{-1} = \sum_{i=0}^{\infty} h^i$ and $\|h^n\| \to 0$ as $n \to \infty$. For each $m > n$, without loss of generality, let n be odd and m be even. Thus by (2.6) and (2.7) it follows that

$$\begin{aligned}d(y_n,y_m) &\preceq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots + d(y_{m-1},y_m)\\ &\preceq (h^{\frac{n-1}{2}} + h^{\frac{n+1}{2}} + \cdots + h^{\frac{m-2}{2}})d(y_1,y_2)\\ &\quad + (h^{\frac{n+1}{2}} + h^{\frac{n+3}{2}} + \cdots + h^{\frac{m-2}{2}})d(y_0,y_1)\\ &\preceq h^{\frac{n-1}{2}}(e+h+h^2+\cdots)d(y_1,y_2)\\ &\quad + h^{\frac{n+1}{2}}(e+h+h^2+\cdots)d(y_0,y_1)\\ &= (e-h)^{-1}[h^{\frac{n-1}{2}}d(y_1,y_2) + h^{\frac{n+1}{2}}d(y_0,y_1)].\end{aligned}$$

In view of

$$\begin{aligned}&\|h^{\frac{n-1}{2}}d(y_1,y_2) + h^{\frac{n+1}{2}}d(y_0,y_1)\|\\ &\le \|h^{\frac{n-1}{2}}\|\|d(y_1,y_2)\| + \|h^{\frac{n+1}{2}}\|\|d(y_0,y_1)\| \to 0 (n\to\infty),\end{aligned}$$

by Lemma 1.6, we have $\{h^{\frac{n-1}{2}}d(y_1,y_2) + h^{\frac{n+1}{2}}d(y_0,y_1)\}$ is a c-sequence. Next by using Lemma 1.9 and Lemma 1.10, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Suppose, for example, that $f(X)$ is a complete subset of X. Then there exists some point $y \in X$ such that $y_n \to y = fu$ $(n \to \infty)$ for some $u \in X$. Of course, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ also converge to y. Let us prove that $y = Su$. Indeed, by using (2.1), it is clear that

$$\begin{aligned}d(Su,y) &\preceq d(Su,Tx_{2n+1}) + d(Tx_{2n+1},y)\\ &\preceq k_1 d(fu,Su) + k_2 d(gx_{2n+1},Su)\\ &\quad + k_3 d(fu,Tx_{2n+1}) + k_4 d(gx_{2n+1},Tx_{2n+1})\\ &\quad + k_5 d(fu,gx_{2n+1}) + d(Tx_{2n+1},y)\\ &\preceq k_1 d(y,Su) + k_2 d(gx_{2n+1},y) + k_2 d(Su,y)\\ &\quad + k_3 d(y,Tx_{2n+1}) + k_4 d(gx_{2n+1},y)\\ &\quad + k_4 d(y,Tx_{2n+1}) + k_5 d(y,gx_{2n+1})\\ &\quad + d(Tx_{2n+1},y).\end{aligned} \tag{2.8}$$

Because $e-k_1-k_2$ is invertible, then by (2.8), it follows that

$$\begin{aligned}d(Su,y) \preceq\ & (e-k_1-k_2)^{-1}[(k_2+k_4+k_5)d(gx_{2n+1},y)\\ & + (k_3+k_4+e)d(Tx_{2n+1},y)].\end{aligned}$$

Since $\{d(gx_{2n+1},y)\}$ and $\{d(Tx_{2n+1},y)\}$ are c-sequences, then by utilizing Lemma 1.9 and Lemma 1.10, it establishes that $y = Su$. So $y = Su = fu$. By virtue of $y = Su \in S(X) \subseteq g(X)$, there exists $v \in X$ such that $y = gv$. Let us prove that $y = Tv$. Actually, by (2.1), we gain that

$$\begin{aligned}d(y,Tv) &\preceq d(y,Sx_{2n}) + d(Sx_{2n},Tv)\\ &\preceq d(y,Sx_{2n}) + k_1 d(fx_{2n},Sx_{2n})\\ &\quad + k_2 d(gv,Sx_{2n}) + k_3 d(fx_{2n},Tv)\\ &\quad + k_4 d(gv,Tv) + k_5 d(fx_{2n},gv)\\ &\preceq d(y,Sx_{2n}) + k_1 d(fx_{2n},y) + k_1 d(y,Sx_{2n})\\ &\quad + k_2 d(y,Sx_{2n}) + k_3 d(fx_{2n},y) + k_3 d(y,Tv)\\ &\quad + k_4 d(y,Tv) + k_5 d(fx_{2n},y).\end{aligned} \tag{2.9}$$

Note that $e-k_3-k_4$ is invertible, then by (2.9), it follows that

$$\begin{aligned}d(y,Tv) \preceq\ & (e-k_3-k_4)^{-1}[(e+k_1+k_2)d(y,Sx_{2n})\\ & + (k_1+k_3+k_5)d(fx_{2n},y)].\end{aligned}$$

Now that $\{d(y,Sx_{2n})\}$ and $\{d(fx_{2n},y)\}$ are c-sequences, then by using Lemma 1.9 and Lemma 1.10, it concludes that $y = Tv$. Hence $y = gv = Tv$. We have proved that y is a common point of coincidence for pairs (f,S) and (g,T). Next we shall show that the common point of coincidence is unique. In fact, if there exists another common point of coincidence y' such that $y' = Su' = Tv' = fu' = gv'$ (say), then by (2.1), we have

$$\begin{aligned}d(y',y) &= d(Su',Tv)\\ &\preceq k_1 d(fu',Su') + k_2 d(gv,Su') + k_3 d(fu',Tv)\\ &\quad + k_4 d(gv,Tv) + k_5 d(fu',gv)\\ &= (k_2+k_3+k_5)d(y',y).\end{aligned}$$

By utilizing Lemma 1.8, it is not hard to verify that

$$\begin{aligned}
&\rho(2k_1+2k_2+2k_3+2k_4+2k_5)\\
=&\rho[(k_1+k_2)+(k_1+k_3+k_5)\\
&+(k_3+k_4)+(k_2+k_4+k_5)]\\
\preceq&\rho(k_1+k_2)+\rho(k_1+k_3+k_5)\\
&+\rho(k_3+k_4)+\rho(k_2+k_4+k_5)\\
<&2,
\end{aligned}$$

which establishes that $\rho(k_1+k_2+k_3+k_4+k_5)<1$. As a result of $k_2+k_3+k_5\preceq k_1+k_2+k_3+k_4+k_5$, then by Lemma 2.1 it yields that $y'=y$.

Now assuming that the pairs (f,S) and (g,T) are weakly compatible, we shall prove y is the common fixed point of f,g,S and T. Since (f,S) and (g,T) are weakly compatible, it establishes that $Sfu=fSu$ and $Tgv=gTv$. In other words, we demonstrate that $Sy=fy$ and $Ty=gy$.

On account of (2.1), we deduce that

$$\begin{aligned}
d(Sy,y)&=d(Sy,Tv)\\
&\preceq k_1d(fy,Sy)+k_2d(gv,Sy)+k_3d(fy,Tv)\\
&\quad+k_4d(gv,Tv)+k_5d(fy,gv)\\
&=(k_2+k_3+k_5)d(Sy,y)
\end{aligned}$$

and

$$\begin{aligned}
d(Ty,y)&=d(y,Ty)=d(Su,Ty)\\
&\preceq k_1d(fu,Su)+k_2d(gy,Su)+k_3d(fu,Ty)\\
&\quad+k_4d(gy,Ty)+k_5d(fu,gy)\\
&=(k_2+k_3+k_5)d(Ty,y).
\end{aligned}$$

Note the facts that $\rho(k_1+k_2+k_3+k_4+k_5)<1$ and $k_2+k_3+k_5\preceq k_1+k_2+k_3+k_4+k_5$, then by Lemma 2.1, we speculate $Sy=y,\ Ty=y$. Therefore, $fy=gy=Sy=Ty=y$. That is, y is a common fixed point of f,g,S and T.

Finally, we shall show the common fixed point is unique. If there is another common fixed point z, then by (2.1), we arrive at

$$\begin{aligned}
d(y,z)&=d(Sy,Tz)\\
&\preceq k_1d(fy,Sy)+k_2d(gz,Sy)+k_3d(fy,Tz)\\
&\quad+k_4d(gz,Tz)+k_5d(fy,gz)\\
&=(k_2+k_3+k_5)d(y,z).
\end{aligned}$$

Again by Lemma 2.1, we acquire $y=z$. The proofs for cases in which $g(X)$, $S(X)$ and $T(X)$ are completely are similar and are therefore omitted. We complete the proof.

Corollary 2.4 Let (X,d) be a cone metric space over Banach algebra $\mathscr{A}$ and P be a solid cone in $\mathscr{A}$. Suppose that f,S,T are three self-maps on X such that $S(X)\bigcup T(X)\subseteq f(X)$ and suppose that at least one of these three subsets of X is complete. Let

$$\begin{aligned}
d(Sx,Ty)\preceq\ &k_1d(fx,Sx)+k_2d(fy,Sx)+k_3d(fx,Ty)\\
&+k_4d(fy,Ty)+k_5d(fx,fy),
\end{aligned}$$

for all $x,y\in X$, where $k_i\in P$ are generalized Lipschitz constants with $k_ik_j=k_jk_i$ $(i,j=1,\ldots,5)$. If $\rho(k_1+k_2)+\rho(k_1+k_3+k_5)<1$ and $\rho(k_3+k_4)+\rho(k_2+k_4+k_5)<1$, then the pairs (f,S) and (f,T) have a unique common point of coincidence. Moreover, f,S and T have a unique common fixed point provided that the pairs (f,S) and (f,T) are weakly compatible.

Proof By taking $f=g$ in Theorem 2.3, we get the proof.

Corollary 2.5 Let (X,d) be a cone metric space over Banach algebra $\mathscr{A}$ and P be a solid cone in $\mathscr{A}$. Suppose that f,S are two self-maps on X such that $S(X)\subseteq f(X)$ and suppose that at least one of these two subsets of X is complete. Let

$$\begin{aligned}
d(Sx,Sy)\preceq\ &k_1d(fx,Sx)+k_2d(fy,Sx)+k_3d(fx,Sy)\\
&+k_4d(fy,Sy)+k_5d(fx,fy),
\end{aligned}$$

for all $x,y\in X$, where $k_i\in P$ are generalized Lipschitz constants with $k_ik_j=k_jk_i$ $(i,j=1,\ldots,5)$. If $\rho(k_1+k_2)+\rho(k_1+k_3+k_5)<1$ and $\rho(k_3+k_4)+\rho(k_2+k_4+k_5)<1$, then the pair (f,S) has a unique point of coincidence. Moreover, f and S have a unique common fixed point provided that the pair (f,S) is weakly compatible.

Proof By taking $f=g$ and $S=T$ in Theorem 2.3, we obtain the proof.

Corollary 2.6 Let (X,d) be a cone metric space and P be a solid cone. Suppose that f,g,S,T are four self-maps on X such that $T(X)\subseteq f(X)$ and $S(X)\subseteq g(X)$ and suppose that at least one of these four subsets of X is complete. Let

$$\begin{aligned}
d(Sx,Ty)\preceq\ &k_1d(fx,Sx)+k_2d(gy,Sx)+k_3d(fx,Ty)\\
&+k_4d(gy,Ty)+k_5d(fx,gy),
\end{aligned}$$

for all $x,y\in X$, where $k_i\geq 0$ $(i=1,\ldots,5)$ are constants with $2k_1+k_2+k_3+k_5<1$ and $k_2+k_3+2k_4+k_5<1$. Then the pairs (f,S) and (g,T) have a unique common point of coincidence. Moreover, f,g,S and T have a unique common fixed point provided that the pairs (f,S) and (g,T) are weakly compatible.

Proof Since $\rho(k)=k$ for all $k\in\mathbb{R}^+$, then by Theorem 2.3, the proof is clear.

Remark 2.7 Theorem 2.3 greatly generalizes the main results of [7,8] and [17]. Corollary 2.6 is the version of usual cone metric spaces and greatly expands the main results of [13]. Indeed, let $f=g=i_X$ (identity mapping) and $S=T$ may arrive. Otherwise, Corollary 2.6 also generalizes [12, Theorem 2.1] and [15, Theorem 2.1].

Remark 2.8 Our main results do not need the assumption of normality. Actually, there exist lots of non-normal cones (see [16]). Based on these facts, our main results are very meaningful and may offer us an useful tool for the existence of fixed points.

Remark 2.9 Throughout the conclusions above, we consider common fixed point theorems in cone metric spaces over Banach algebras instead of the theorems only in cone metric spaces. All the coefficients are vectors and the multiplications such as $k_1d(fx,Sx)$ are vector multiplications instead of usual scalar ones, which may bring us more convenience in applications.

Remark 2.10 It is a valuable increase in introducing the concept of cone metric space over Banach algebra, since it establishes the non-equivalence of fixed point results between metric spaces and cone metric spaces over Banach algebras. The following examples illustrate our conclusion.

Example 2.11 Let $\mathscr{A}=\mathbb{R}^2$ and the norm be $\|(x_1,x_2)\|=|x_1|+|x_2|$. Define the multiplication by

$$xy=(x_1,x_2)(y_1,y_2)=(x_1y_1+x_1y_2+x_2y_1,x_2y_2),$$

where $x=(x_1,x_2),y=(y_1,y_2)\in\mathscr{A}$. Then $\mathscr{A}$ is a Banach algebra with a unit $e=(0,1)$. Taking $X=[0,1]$, $P=\{(x_1,x_2)\in\mathscr{A}:x_1,x_2\geq 0\}$ and

$d(x,y)=(|x-y|,|x-y|)$ for all $x,y\in X$,

we claim that (X,d) is a cone metric space over $\mathscr{A}$ and P is a normal solid cone. Take $k_1=(\frac{1}{25},\frac{1}{25})$, $k_2=(\frac{1}{8},\frac{1}{8})$, $k_3=(\frac{1}{8},\frac{1}{8})$, $k_4=(\frac{1}{32},\frac{1}{32})$ and $k_5=(\frac{1}{6},\frac{1}{6})$, it is clear that $k_ik_j=k_jk_i$ $(i,j=1,\ldots,5)$. Denote $t=(t_1,t_1)$ $(t_1>0)$, then

$$\rho(t)=\lim_{n\to\infty}\|t^n\|^{\frac{1}{n}}=\lim_{n\to\infty}\|((2^n-1)t_1^n,t_1^n)\|^{\frac{1}{n}}=2t_1.$$

Hence

$$\begin{aligned}&\rho(k_1+k_2)+\rho(k_1+k_3+k_5)\\&=2\Big(\frac{1}{25}+\frac{1}{8}\Big)+2\Big(\frac{1}{25}+\frac{1}{8}+\frac{1}{6}\Big)=\frac{149}{150}<1,\\&\rho(k_3+k_4)+\rho(k_2+k_4+k_5)\\&=2\Big(\frac{1}{8}+\frac{1}{32}\Big)+2\Big(\frac{1}{8}+\frac{1}{32}+\frac{1}{6}\Big)=\frac{23}{24}<1.\end{aligned}$$

Defining two mappings $S:X\to X$ and $f:X\to X$ by $Sx=\frac{1}{4}x^2+\frac{1}{2}x$ and $fx=x$, we obtain that $S(X)\subseteq f(X)$, $f(X)$ is complete and the pair (f,S) is weakly compatible. It is easy to see that

$$\begin{aligned}d(Sx,Sy)\preceq{}& k_1d(fx,Sx)+k_2d(fy,Sx)+k_3d(fx,Sy)\\&+k_4d(fy,Sy)+k_5d(fx,fy)\end{aligned}$$

for all $x,y\in X$. Therefore, all conditions of Theorem 2.3 or Corollary 2.5 are satisfied. Thus by Theorem 2.3 or Corollary 2.5, f and S have a unique common fixed point in X. This common fixed point is $x=0$.

Example 2.12 Let $\mathscr{A}=\left\{\begin{pmatrix}\alpha&\beta\\0&\alpha\end{pmatrix}\Big|\alpha,\beta\in\mathbb{R}\right\}$, $\left\|\begin{pmatrix}\alpha&\beta\\0&\alpha\end{pmatrix}\right\|=|\alpha|+|\beta|$. The multiplication is usual matrix multiplication. Then $\mathscr{A}$ is a Banach algebra with a usual unit. Choose $X=\mathbb{R}$, $P=\left\{\begin{pmatrix}\alpha&\beta\\0&\alpha\end{pmatrix}\Big|\alpha,\beta\geq 0\right\}$. Letting

$$d(x,y)=\begin{pmatrix}|x-y|&2|x-y|\\0&|x-y|\end{pmatrix},\quad x,y\in X,$$

we deduce that (X,d) is a cone metric space over $\mathscr{A}$ and P is a solid cone. Suppose the mappings $S,f:X\to X$ as

$$Sx=\begin{cases}\frac{\alpha}{\beta+1}x, & x\neq 0,\\ \gamma, & x=0.\end{cases}\qquad fx=\begin{cases}\alpha x, & x\neq 0,\\ \gamma, & x=0.\end{cases}$$

Where $\alpha>0,\beta\geq 1$ and $\gamma\neq 0$. It is easy to see that

$$\begin{aligned}d(Sx,Sy)\preceq{}& k_1d(fx,Sx)+k_2d(fy,Sx)+k_3d(fx,Sy)\\&+k_4d(fy,Sy)+k_5d(fx,fy)\end{aligned}$$

for all $x,y\in X$, where

$$k_1=\begin{pmatrix}\frac{1}{2}&\frac{1}{3}\\0&\frac{1}{2}\end{pmatrix},\quad k_2=\begin{pmatrix}\frac{1}{3}&\frac{1}{4}\\0&\frac{1}{3}\end{pmatrix},\quad k_3=\begin{pmatrix}\frac{1}{5}&\frac{1}{4}\\0&\frac{1}{5}\end{pmatrix},$$

$$k_4=\begin{pmatrix}\frac{1}{4}&\frac{1}{6}\\0&\frac{1}{4}\end{pmatrix},\quad k_5=\begin{pmatrix}\frac{1}{\beta}&\alpha\\0&\frac{1}{\beta}\end{pmatrix}.$$

Note that f and S cannot commute at the coincidence point $x=0$ of them, that is to say, the pair (f,S) is not weakly compatible, thus although most of conditions in Corollary 2.5 are satisfied, f and S have not any common fixed point in X. Thus, this example demonstrates the crucial role of weak compatibility in our results.

Example 2.13 Let $\mathscr{A}=\mathbb{R}^2$ and the norm be $\|(x_1,x_2)\|=|x_1|+|x_2|$. The multiplication is defined by

$$xy=(x_1,x_2)(y_1,y_2)=(x_1y_1,x_1y_2+x_2y_1),$$

where $x=(x_1,x_2),y=(y_1,y_2)\in\mathscr{A}$. It is not hard to verify that $\mathscr{A}$ is a Banach algebra with a unit $e=(1,0)$. Let $X=[0,1]\times(-\infty,+\infty)$, $P=\{(x_1,x_2)\in\mathscr{A}:x_1,x_2\geq 0\}$ and

$$d(x,y)=(|x_1-y_1|,|x_2-y_2|)$$

for all $x=(x_1,x_2),y=(y_1,y_2)\in X$. Then (X,d) is a cone metric space over $\mathscr{A}$ and P is a normal solid cone.

Now define a mapping $S:X\to X$ by

$$\begin{aligned}Sx&=S(x_1,x_2)\\&=\Big(\frac{1}{2}\big(\sin\frac{x_1}{2}-|x_1-\frac{1}{2}|\big),\arctan(1+|x_2|)+\ln(x_1+2)\Big).\end{aligned}$$

By using mean value theorem of differentials, we have that

$$\begin{aligned}&d(Sx,Sy)=d(S(x_1,x_2),S(y_1,y_2))\\&=\Big(\big|\frac{1}{2}\big(\sin\frac{x_1}{2}-\sin\frac{y_1}{2}-|x_1-\frac{1}{2}|+|y_1-\frac{1}{2}|\big)\big|,\\&\quad|\arctan(1+|x_2|)-\arctan(1+|y_2|)\\&\quad+\ln(x_1+2)-\ln(y_1+2)|\Big)\\&\preceq\Big(\frac{|x_1-y_1|}{4}+\frac{1}{2}|x_1-y_1|,\frac{1}{2}|x_2-y_2|+\frac{1}{2}|x_1-y_1|\Big)\\&\preceq\Big(\frac{3}{4},1\Big)(|x_1-y_1|,|x_2-y_2|)\\&=\Big(\frac{3}{4},1\Big)d(x,y)\end{aligned}$$

for all $x,y \in X$. Denote $k_1 = k_2 = k_3 = k_4 = \theta, k_5 = (\frac{3}{4},1)$. It is easy to see that all conditions of Corollary 2.5 are fulfilled provided that $fx = x$ $(x \in X)$. As a result, f and S have a unique common fixed point in X.

The following statement indicates our fixed point results in cone metric spaces over Banach algebra $\mathscr{A}$ are not equivalent to those in metric spaces. Indeed, put

$$d_1(x,y) = \inf_{\{u \in P: d(x,y) \preceq u\}} \|u\|,$$

$$d_2(x,y) = \inf\{r \in \mathbb{R} : d(x,y) \preceq re\},$$

where $x,y \in X$ and $e = (e_1,e_2) \in \mathrm{int}P$. Then by Theorem 2.2 of [4], d_1 and d_2 are both equivalent metrics. Hence we need to consider only one of them. Let us refer to the metric d_2. We shall prove our conclusions are not equivalent to the well-known Banach contraction principle, which means Theorem 2.4 of [2] does not hold in the setting of cone metric spaces over Banach algebras. Actually, taking $x' = (\frac{1}{2},0)$, $y' = (0,0)$, $e = (1,\frac{1}{4})$, we arrive at

$$\begin{aligned} d_2(Sx',Sy') &= \inf\Big\{r \in \mathbb{R} : \Big(\Big|\frac{1}{2}\sin\frac{1}{4} + \frac{1}{4}\Big|, \Big|\ln\frac{5}{2} - \ln 2\Big|\Big) \\ &\qquad \preceq r\Big(1,\frac{1}{4}\Big)\Big\} \\ &= \max\Big\{\frac{1}{2}\sin\frac{1}{4} + \frac{1}{4}, 4\ln\frac{5}{4}\Big\} \\ &= 4\ln\frac{5}{4} \geq \frac{1}{2} = d_2(x',y'), \end{aligned}$$

which means that there is not $\lambda \in [0,1)$ satisfying

$$d_2(Sx,Sy) \leq \lambda d_2(x,y)$$

for all $x,y \in X$. Thus it does not hold the famous Banach contraction principle. In other words, Theorem 2.4 of [2] is unsuitable for cone metric spaces over Banach algebras.

Remark 2.14 Example 2.11 is used to support our main results are reasonable. Example 2.12 implies a fact that weak compatibility plays an important role for the existence of common fixed points of the pairs (f,S) and (g,T).

Remark 2.15 Example 2.13 shows that our fixed point theorems in cone metric spaces over Banach algebras are not the counterparts of metric spaces even with the hypothesis that the cones are normal cones. In other words, our results are never merely copies of the classical results in metric spaces. Based on these statements, cone metric spaces over Banach algebras offer a more generalized framework than usual metric spaces.

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