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Hardy's Type Integral Inequalities on Time Scales

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Abstract: In this paper, we prove some new integral inequalities of Hardy's type on time scales. The Hardy inequalities have many applications especially in proving the boundedness Cesàro operators. The main results will be proved by making use of some algebraic inequalities, the Hölder inequality and a simple consequence of Keller's chain rule on time scales. The discrete inequalities that we will derive from our results in the discrete time scales are essentially new.

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1 Introduction

The classical Hardy inequality states that for $f \geq 0$ and integrable over any finite interval $(0,x)$ and f^p is integrable and convergent over $(0,\infty)$ and $p>1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \quad (1)$$

The constant $(p/(p-1))^p$ is the best possible. The discrete version of the inequality (1) due to Hardy is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a(i)\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a^p(n), \quad p>1. \quad (2)$$

Since the discovery of these inequalities various papers which deal with new proofs, generalizations and extensions have appeared in the literature. We refer the reader to the books [14,15,20] and the papers [2,3,6,11, 12,13,16,17,18,19,22,23,27]. One of the applications of the dicrete inequality is its proof of the boundedness of the Cesàro operator $T:\ell^p \to \ell^p$, for $1<p<\infty$, which is defined by

$$(Ta)_k = \frac{1}{k}\sum_{j=1}^{k} a_j, \quad k \in \mathbb{N}, \text{ where } a=(a_k)_{k=1}^{\infty}.$$

Kaijser et al. [13] established the more general (Hardy-Knopp type) inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \leq \int_0^\infty \Phi\left(f(x)\right)\frac{dx}{x}, \quad (3)$$

where Φ is a convex function on $(0,\infty)$. The Hardy inequalities have applications in the theory of differential equations (ordinary or partial) and led to many interesting questions and connections between different areas of mathematical analysis. For example, Hardy inequalities are closely related to the quasiadditivity properties of capacities [1] and have recently been used to find the gaps between zeros of differential equations which appear in the bending of beams [25].

Hardy's inequality (1) has been generalized by Hardy himself in [8]. There he showed that, for any $m>1$, $p>1$, and any integrable function $f(x)>0$ on $(0,\infty)$, then

$$\int_0^\infty \frac{1}{x^m}\left(\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{m-1}\right)^p \int_0^\infty \frac{1}{x^m}(xf(x))^p dx, \quad (4)$$

where the constant here also is the best possible. It is easy to see that (4) cannot be applied in the case $m=1$. This problem has been treated in [6] by splitting $[0,\infty)$, the interval of integration, into $[0,1]$ and $[1,\infty)$ and he proved the following inequalities: if $1<p<\infty$ and $f(x)>0$ is a

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integrable on $(1,\infty)$, then

$$\int_1^\infty \frac{1}{x}\left(\int_x^\infty f(t)dt\right)^p dx \leq p^p \int_1^\infty \frac{1}{x}(x\log x)^p f^p(x)dx, \tag{5}$$

$$\int_0^1 \frac{1}{x}\left(\int_0^x f(t)dt\right)^p dx \leq p^p \int_0^1 \frac{1}{x}(x(-\log x))^p f^p(x)dx. \tag{6}$$

A number of dynamic inequalities of Hardy type were established in [21,24,26,28]. In particular the authors establish dynamic inequalities where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. We assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_\mathbb{T}$ by $[t_0,\infty)_\mathbb{T} := [t_0,\infty)\cap\mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [4], [5] which summarize and organize much of the time scale calculus.

In [24] the author established a time scale version of the Hardy inequality (1) and proved that if $p>1$ and g is a nonnegative and such that the delta integral $\int_a^\infty (g(t))^p\Delta t$ exits as a finite number, then

$$\int_a^\infty \frac{1}{(\sigma(x)-a)^p}\left(\int_a^{\sigma(x)} g(t)\Delta t\right)^p \Delta x \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty g^p(x)\Delta x. \tag{7}$$

If in addition $\mu(t)/t\to 0$ as $t\to\infty$, then the constant is the best possible. However it is an open problem whether the constant in inequality (7) is the best possible also on time scales than do not satisfy $\lim_{t\to\infty}(\mu(t)/t)=0$. Also, it is easy to see that (7) cannot be applied when the term $(\sigma(x)-a)^p$ is replaced by $(\sigma(x)-a)$.

In [21] the authors established a new inequality with weighted functions, which can be considered as the time scale version of the inequality (3). In particular, they proved that if $u\in C_{rd}([a,b],\mathbb{R})$ is a nonnegative function such that the delta integral $\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)}\Delta s$ exists as a finite number and the function v is defined by

$$v(t)=(t-a)\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)}\Delta s,\ \ t\in[a,b],$$

and $\Phi:(c,d)\to\mathbb{R}$, is continuous and convex, where $c,d\in\mathbb{R}$, then the inequality

$$\int_a^b u(t)\Phi\left(\frac{1}{(\sigma(t)-a)}\int_a^{\sigma(t)} g(s)\Delta s\right)\frac{\Delta t}{t-a} \leq \int_a^b v(t)\Phi(g(t))\frac{\Delta t}{t-a}, \tag{8}$$

holds for all delta integrable functions $g\in C_{rd}([a,b],\mathbb{R})$ such that $g(t)\in(c,d)$.

In this paper, we will prove some new inequalities of Hardy's type on time scales where power p will be replaced by p/q where p and q are positive real numbers. The technique in this paper depends on the application of the chain rule, Hölder's inequality and some algebraic inequalities. The results in this paper contain some continuous and discrete inequalities as special cases and can be considered as time scale versions of the inequalities (5) and (6). These inequalities can be considered as extensions and generalizations of some Hardy type inequalities proved in [6].

2 Main Results

In this section, we will prove the main results. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t):=\inf\{s\in\mathbb{T}:\ s>t\},\quad \rho(t):=\sup\{s\in\mathbb{T}:\ s<t\},$$

where $\sup\emptyset=\inf\mathbb{T}$. A point $t\in\mathbb{T}$, is said to be left–dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, is right–dense if $\sigma(t)=t$, is left–scattered if $\rho(t)<t$ and right–scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The space $\mathscr{R}$ of regressive functions ([4, page 58]) defined by

$$\mathscr{R}:=\{x:\mathbb{T}\to\mathbb{R}:\ \ x \text{ is rd-continuous on } \sum \text{ and } \ 1+\mu(t)x(t)\neq 0\}.$$

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We will assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_\mathbb{T}$ by $[a,b]_\mathbb{T}:=[a,b]\cap\mathbb{T}$. Fix $t\in\mathbb{T}$ and let $x:\mathbb{T}\to\mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given any $\varepsilon>0$ there is a neighborhood U of t with

$$|[x(\sigma(t))-x(s)]-x^\Delta(t)[\sigma(t)-s]|\leq\varepsilon|\sigma(t)-s|,\quad \text{for all } s\in U.$$

In this case, we say $x^\Delta(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t.

We will frequently use the following results which are due to Hilger [10]. Assume that $g:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}$.
(i) If g is differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^\Delta(t)=\frac{g(\sigma(t))-g(t)}{\mu(t)}$.
(iii) If g is differentiable and t is right-dense, then

$$g^\Delta(t)=\lim_{s\to t}\frac{g(t)-g(s)}{t-s}.$$

(iv) If g is differentiable at t, then $g(\sigma(t))=g(t)+\mu(t)g^\Delta(t)$.

Note that if $\mathbb{T}=\mathbb{R}$ then

$$\sigma(t)=t,\ \mu(t)=0,\ f^{\Delta}(t)=f'(t),\ \int_a^b f(t)\Delta t=\int_a^b f(t)dt$$

if $\mathbb{T}=\mathbb{Z}$, then

$\sigma(t)=t+1,\ \mu(t)=1,\ f^{\Delta}(t)=\Delta f(t),\ \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1} f(t),$

if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, and

$y^{\Delta}(t)=\Delta_h y(t):=\frac{y(t+h)-y(t)}{h},\ \int_a^b f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h,$

and if $\mathbb{T}=\{t:t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, then $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$x^{\Delta}(t)=\Delta_q x(t)=\frac{(x(qt)-x(t))}{(q-1)t},\quad \int_{t_0}^{\infty} f(t)\Delta t=\sum_{k=n_0}^{\infty} f(q^k)\mu(q^k),$

where $t_0=q^{n_0}$, and if $\mathbb{T}=\mathbb{N}_0^2:=\{n^2:n\in\mathbb{N}_0\}$, then $\sigma(t)=(\sqrt{t}+1)^2$,

$$\mu(t)=1+2\sqrt{t},\ \Delta_N y(t)=\frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}}.$$

In this paper we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t)=g(t)$, then the Cauchy (delta) integral of g is defined by

$$\int_a^t g(s)\Delta s:=G(t)-G(a).$$

It can be shown (see [4]) that if $g\in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t):=\int_{t_0}^t g(s)\Delta s$ exists, $t_0\in\mathbb{T}$, and satisfies $G^{\Delta}(t)=g(t)$, $t\in\mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t.$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$, here $g^{\sigma}=g\circ\sigma$) of two differentiable function f and g

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma},$$
$$\text{and}\ \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{9}$$

We say that a function $p:\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$, $t\in\mathbb{T}$. The chain rule formula (see [4, Theorem 1.90]) that we will use in this paper is

$$(f(g(t))^{\Delta}=\int_0^1 f'\left[hg^{\sigma}+(1-h)g\right]dhg^{\Delta}(t), \tag{10}$$

Using the fact that $g(\sigma(t))=g(t)+\mu(t)g^{\Delta}(t)$, we obtain

$$(f(g(t))^{\Delta}=\int_0^1 f'\left[g+h\mu(t)g^{\Delta}(t)\right]dhg^{\Delta}(t). \tag{11}$$

The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t=[u(t)v(t)]_a^b-\int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{12}$$

To prove the main results, we will use the following Hölder inequality [4, Theorem 6.13]. Let $a,\ b\in\mathbb{T}$. For $u,v\in C_{rd}(\mathbb{T},\mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\Delta t\leq\left[\int_a^b |u(t)|^q\Delta t\right]^{\frac{1}{q}}\left[\int_a^b |v(t)|^p\Delta t\right]^{\frac{1}{p}}, \tag{13}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Throughout the paper, we will assume that the functions are nonnegative rd-continuous functions, $\Delta-$differentiable, locally delta integrable and the left hand side of the inequalities exists if the right hand side exists. We also assume that all the powers in the integrals are positive real numbers.

Before we state and prove the main results we need to find the integral of $1/t$ on time scales. From the chain rule (11), we see that

$$(\log t)^{\Delta}=\int_0^1\frac{1}{[t+h\mu(t)]}dh=\frac{1}{\mu(t)}\log\left(\frac{t+\mu(t)}{t}\right),\ \text{when}\ \mu(t)\neq 0.$$

This allows us to define the new function $z(t)$ on a time scale $\mathbb{T}$ by

$$(\log t)^{\Delta}=z(t):=\begin{cases}\frac{1}{\mu(t)}\log\left(\frac{t+\mu(t)}{t}\right), & \text{when}\ \mu(t)\neq 0,\\ \frac{1}{t}, & \text{when}\ \mu(t)=0.\end{cases} \tag{14}$$

Thus on a time scale $\mathbb{T}$, we have that (here $t_0\in\mathbb{T}$)

$$\int_{t_0}^t z(s)\Delta s=\log\left(\frac{t}{t_0}\right),\quad \text{for}\ t\in\mathbb{T}. \tag{15}$$

As a generalization of (14) we have (here g is a nonnegative function)

$$(\log g(t))^{\Delta}=Z(t):=\begin{cases}\frac{1}{\mu(t)}\log\left(1+\mu(t)\frac{g^{\Delta}(t)}{g(t)}\right), & \text{when}\ \mu(t)\neq 0,\\ \frac{g^{\Delta}(t)}{g(t)}, & \text{when}\ \mu(t)=0,\end{cases} \tag{16}$$

provided that $\frac{g^{\Delta}(t)}{g(t)}\in\mathscr{R}$. Thus on a time scale $\mathbb{T}$, we have that

$$\int_{t_0}^t Z(s)\Delta s=\log\left(\frac{g(t)}{g(t_0)}\right),\quad \text{for}\ t\in\mathbb{T}. \tag{17}$$

As a special case of (14), we see that if $\mathbb{T}=\mathbb{R}$, then $(\log t)^{\Delta}=(\log t)'=1/t$ and if $\mathbb{T}=\mathbb{N}$,

$$(\log t)^{\Delta}=\Delta\log t=\log(\frac{t+1}{t})=\log(t+1)-\log t,$$

where $\mu(t)=1$ in $\mathbb{N}$. Now, we are ready to state and prove the main results in this paper.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $b \in [1,\infty)_{\mathbb{T}}$ and $p,q>0$ such that $p/q>1$. Define*

$$\Lambda(t) := \int_t^b f(s)\Delta s, \quad \text{for any } t \in [1,\infty)_{\mathbb{T}}. \tag{18}$$

Then for any $b>1$

$$\int_1^b z(t)\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t \le \frac{p}{q}\left[\int_1^b z(t)\left(\frac{\log t}{z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}} \left[\int_1^b z(t)\left(\Lambda(t)\right)^{p/q}\Delta t\right]^{\frac{p-q}{p}} \tag{19}$$

Proof. Integrating the left hand side of (19) using the parts formula (12) with

$$u^\Delta(t) = z(t) \quad \text{and } v^\sigma(t) = (\Lambda^\sigma(t))^{p/q},$$

we have

$$\int_1^b z(t)\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t = \left(\Lambda(t)\right)^{p/q}\log t\Big|_1^b + \int_1^b (\log t)\left(-\left(\Lambda^{p/q}(t)\right)\right)^\Delta \Delta t. \tag{20}$$

Using the chain rule (11), we see that

$$-\left(\Lambda^{p/q}(t)\right)^\Delta = -\frac{p}{q}\int_0^1\left[\Lambda+\mu h\Lambda^\Delta\right]^{\frac{p}{q}-1} dh\Lambda^\Delta(t)$$
$$= \frac{p}{q}f(t)\int_0^1\left[\Lambda+\mu h\Lambda^\Delta\right]^{\frac{p}{q}-1} dh > 0. \tag{21}$$

Using the fact that $\Lambda(b)=0$, and substituting (21) into (20), we have

$$\int_1^b z(t)\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t = \frac{p}{q}\int_1^b f(t) \, logt\int_0^1\left[\Lambda(t)+h\mu(t)\Lambda^\Delta(t)\right]^{\frac{p}{q}-1} dh\Delta t. \tag{22}$$

Using the fact that $\Lambda^\Delta(t)<0$, we see that

$$(p/q)\int_0^1\left[\Lambda+h\mu\Lambda^\Delta\right]^{\frac{p}{q}-1}dh \le (\tfrac{p}{q})\Lambda^{\frac{p}{q}-1}(t), \;\; p/q>1. \tag{23}$$

Substituting (23) into (22), we have

$$\int_1^b z(t)\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t \le \frac{p}{q}\int_1^b \left(\Lambda(t)\right)^{p/q-1}(\log t)\, f(t)\Delta t.$$

This implies that

$$\int_1^b z(t)(\Lambda^\sigma(t))^{p/q}\Delta t \le \frac{p}{q}\int_1^b\left[(z(t))^{-(p-q)/p}(\log t f(t))\right] \left[(z(t))^{(p-q)/p}(\Lambda(t))^{(p-q)/q}\right]\Delta t. \tag{24}$$

Applying the Hölder inequality (20) on the term

$$\int_1^b\left[(z(t))^{-(p-q)/p}(\log t)f(t))\right]\left[(z(t))^{(p-q)/p}\Lambda^{(p-q)/q}\right]\Delta t,$$

with indices p/q and $p/(p-q)$, we see that

$$\int_1^b\left[(z(t))^{-(p-q)/p}(\log t f(t))\right]\left[(z(t))^{(p-q)/p}(\Lambda(t))^{(p-q)/q}\right]\Delta t$$
$$\le \left[\int_1^b\left[(z(t))^{-(p-q)/p}(\log t f(t))\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_1^b z(t)\left(\Lambda(t)\right)^{p/q}\Delta t\right]^{\frac{p-q}{p}}$$
$$= \left[\int_1^b (z(t))\left(\frac{\log t}{z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_1^b z(t)\left(\Lambda(t)\right)^{p/q}\Delta t\right]^{\frac{p-q}{p}}. \tag{25}$$

Substituting (25) into (24), we have

$$\int_1^b z(t)\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t \le \frac{p}{q}\left[\int_1^b z(t)\left(\frac{\log t}{z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}} \times\left[\int_1^b z(t)\left(\Lambda(t)\right)^{p/q}\Delta t\right]^{\frac{p-q}{p}}.$$

which is the desired inequality (19). The proof is complete.

As special cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, we can establish from Theorem 2.1 some new differential and discrete inequalities. We begin with the case when $\mathbb{T}=\mathbb{R}$. In this case, (note that $\mu(t)=0$ and $\sigma(t)=t$) the inequality (19) reduces to

$$\int_1^b \frac{1}{t}\left(\int_t^b f(s)ds\right)^{p/q}dt \le \frac{p}{q}\left[\int_1^b \frac{1}{t}\left(t\log t f(t)\right)^{p/q}dt\right]^{\frac{q}{p}} \times\left[\int_1^b \frac{1}{t}\left(\int_t^b f(s)ds\right)^{p/q}dt\right]^{\frac{p-q}{p}}$$

and hence we have

$$\left[\int_1^b \tfrac{1}{t}\left(\int_t^b f(s)ds\right)^{p/q}dt\right]^{1-\frac{p-q}{p}} \le \tfrac{p}{q}\left[\int_1^b \tfrac{(t\log t)^{p/q}}{t}f^{p/q}(t)dt\right]^{\frac{q}{p}}.$$

This gives us after simplification and replacing p/q by $\lambda>1$ the following result.

Corollary 2.1. *Let $\lambda>1$ and assume that f is a nonnegative function on $[1,\infty)_{\mathbb{R}}$. Then for any $b>1$, we have*

$$\int_1^b\frac{1}{t}\left(\int_t^b f(s)ds\right)^\lambda dt \le \lambda^\lambda\int_1^b\frac{(t\log t)^\lambda}{t}f^\lambda(t)dt, \;\; \lambda>1. \tag{26}$$

*Remark.*One can see that this inequality (26) will be the same as the inequality (5) established by Chan [6] if $b\to\infty$.

When $\mathbb{T}=\mathbb{N}$, we have the following result from Theorem 2.1.

Corollary 2.2. *Let $b\in[1,\infty)_{\mathbb{N}}$ and $p,\ q>0$ such that $p/q>1$ and let $f(n)$ be a nonnegative sequence. Define*

$$\Lambda(n):=\sum_{s=n}^{b-1}f(s),\quad \text{for any } b\in[1,\infty)_{\mathbb{N}}. \tag{27}$$

Then for any $b>1$

$$\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n}^{b-1}f(s)\right)^{p/q} \le\left(\frac{p}{q}\right)^{p/q}\left[\sum_{n=1}^{b-1}\frac{((\log n)\,f(n))^{p/q}}{(\log(\frac{n+1}{n}))^{\frac{p}{q}-1}}\right]^{\frac{q}{p}} \times\left[\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n}^{b-1}f(s)\right)^{p/q}\right]^{\frac{p-q}{p}}.$$

One can use the function $Z(t)$ instead of the function $z(t)$ and prove the following result.

Theorem 2.2. *Let $\mathbb{T}$ be a time scale with $b\in[1,\infty)_{\mathbb{T}}$ and $p,\ q>0$ such that $p/q>1$ and $g(t)$ is a nonnegative function such that $\frac{g^{\Delta}(t)}{g(t)}\in\mathscr{R}$. Let $\Lambda(t)$ is defined as in (18). Then for any $b>1$*

$$\int_1^b Z(t)\,(\Lambda^{\sigma}(t))^{p/q}\,\Delta t\le\frac{p}{q}\left[\int_1^b Z(t)\left(\frac{\log g(t)}{Z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}} \times\left[\int_1^b Z(t)\,(\Lambda(t))^{p/q}\,\Delta t\right]^{\frac{p-q}{p}}. \tag{28}$$

From the chain rule formula (10), we have

$$(\Lambda^{p/q}(t))^{\Delta}=\frac{p}{q}\int_0^1[h\Lambda^{\sigma}+(1-h)\Lambda]^{\frac{p}{q}-1}dh\Lambda^{\Delta}(t). \tag{29}$$

Using this formula and the inequality

$$a^{\lambda}+b^{\lambda}\le(a+b)^{\lambda}\le 2^{\lambda-1}(a^{\lambda}+b^{\lambda}),\ \text{if } a,\ b\ge 0,\ \ \lambda\ge 1, \tag{30}$$

one can prove several new results, and the details are left to the interested reader. For example, using the fact that $x\le 2^{x-1}$ for $x\ge 2$ we have from Theorem 2.2 the follwoing results.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale with $b\in[1,\infty)_{\mathbb{T}}$ and $p,\ q>0$ such that $p/q\ge 2$. Let $\Lambda(t)$ is defined as in (18). Then for any $b>1$*

$$\int_1^b z(t)\,(\Lambda^{\sigma}(t))^{p/q}\,\Delta t\le 2^{\frac{p}{q}-1}\left[\int_1^b z(t)\left(\frac{\log t}{z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}} \times\left[\int_1^b z(t)\,(\Lambda(t))^{p/q}\,\Delta t\right]^{\frac{p-q}{p}}. \tag{31}$$

One can also use the function $Z(t)$ instead of the function $z(t)$ to prove the following result.

Theorem 2.4. *Let $\mathbb{T}$ be a time scale with $b\in[1,\infty)_{\mathbb{T}}$ and $p,\ q>0$ such that $p/q\ge 2$, and $g(t)$ is a nonnegative function such that $\frac{g^{\Delta}(t)}{g(t)}\in\mathscr{R}$. Let $\Lambda(t)$ is defined as in (18). Then for any $b>1$*

$$\int_1^b Z(t)\,(\Lambda^{\sigma}(t))^{p/q}\,\Delta t\le 2^{\frac{p}{q}-1}\left[\int_1^b Z(t)\left(\frac{\log g(t)}{Z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}} \times\left[\int_1^b Z(t)\,(\Lambda(t))^{p/q}\,\Delta t\right]^{\frac{p-q}{p}}. \tag{32}$$

As special cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, we can establish from Theorem 2.3 some new differential and discrete inequalities. We begin with the case when $\mathbb{T}=\mathbb{R}$. In this case, (note that $\mu(t)=0$ and $\sigma(t)=t$) the inequality (31) reduces to

$$\int_1^b\frac{1}{t}(\Lambda(t))^{p/q}\,dt\le 2^{\frac{p}{q}-1}\left[\int_1^b\frac{1}{t}(t\log t\,f(t))^{p/q}dt\right]^{\frac{q}{p}} \times\left[\int_1^b\frac{1}{t}(\Lambda(t))^{p/q}\,dt\right]^{\frac{p-q}{p}}$$

and hence we have

$$\left[\int_1^b\tfrac{1}{t}\Lambda^{p/q}(t)dt\right]^{1-\frac{p-q}{p}}\le 2^{\frac{p}{q}-1}\left[\int_1^b\tfrac{1}{t}(t\log t)^{p/q}f^{p/q}(t)dt\right]^{\frac{q}{p}}.$$

This gives us after simplification and replacing p/q by $\lambda\ge 2$ the following result.

Corollary 2.3. *Let $\lambda\ge 2$ and f is a nonnegative function on $[1,\infty)_{\mathbb{R}}$. Then for any $b>1$, we have*

$$\int_1^b\frac{1}{t}\left[\int_t^b f(s)ds\right]^{\lambda}dt\le\left(2^{\lambda-1}\right)^{\lambda} \left[\int_1^b\frac{1}{t}(t\log t)^{\lambda}f^{\lambda}(t)dt\right],\ \lambda\ge 2. \tag{33}$$

*Remark.*Note the difference between the constant $\left(2^{\lambda-1}\right)^{\lambda}$ in (33) and p^p in (5), where the later is the best constant in the inequality (5). This in fact arose since the chain rule that we applied is different from the classical chain rule.

When $\mathbb{T}=\mathbb{N}$, we have the following result as a special case of Theorem 2.3.

Corollary 2.4. *Let $b\in[1,\infty)_{\mathbb{N}}$ and $p/q\ge 2$. Assume that $f(n)$ is a nonnegative sequence and $\Lambda(n)$ be as defined as in (27). Then for any $b>1$*

$$\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n+1}^{b-1}f(s)\right)^{p/q}\le 2^{\frac{p}{q}-1}\left[\sum_{n=1}^{b-1}\frac{((\log n)\,f(n))^{p/q}}{(\log(\frac{n+1}{n}))^{\frac{p}{q}-1}}\right]^{\frac{q}{p}} \times\left[\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n}^{b-1}f(s)\right)^{p/q}\right]^{\frac{p-q}{p}}.$$

*Remark.*One can use Theorems 2.2 and 2.4 to obtain new results for the time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$ for a general function $\log g(t)$ instead of $\log(t)$. The details are left to the interested reader.

In the following, we consider the case when $p/q\leq 2$ and prove new inequalities of Hardy's type on time scales. To prove these results, we need the inequality

$$2^{r-1}\left(a^{r}+b^{r}\right)\leq(a+b)^{r}\leq\left(a^{r}+b^{r}\right),\ \text{where } a,b\geq 0,\ 0\leq r\leq 1. \tag{34}$$

Applying this inequality when $0\leq p/q-1<1$ on the term $[h\Lambda^{\sigma}+(1-h)\Lambda]^{\frac{p}{q}-1}$, we see that

$$\begin{aligned}-(\Lambda^{p/q}(t))^{\Delta}&=-\frac{p}{q}\int_{0}^{1}[h\Lambda^{\sigma}+(1-h)\Lambda]^{\frac{p}{q}-1}dh\Lambda^{\Delta}(t)\\&=\frac{p}{q}\int_{0}^{1}[h\Lambda^{\sigma}+(1-h)\Lambda]^{\frac{p}{q}-1}dhf(t)\\&\leq\left[[\Lambda^{\sigma}]^{\frac{p}{q}-1}+[\Lambda]^{\frac{p}{q}-1}\right]f(t),\end{aligned}$$

and then since $\Lambda^{\Delta}(t)=-f(t)<0$ and $\sigma(t)\geq t$, we get that

$$-(\Lambda^{p/q}(t))^{\Delta}\leq 2\left[\Lambda(t)\right]^{\frac{p}{q}-1}f^{\Delta}(t).$$

This gives us the following result.

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $b\in[1,\infty)_{\mathbb{T}}$ and $p,q>0$ such that $p/q\leq 2$. Let $\Lambda(t)$ is defined as in (18). Then for any $b>1$*

$$\int_{1}^{b}z(t)\left(\Lambda^{\sigma}(t)\right)^{p/q}\Delta t\leq 2\left[\int_{1}^{b}z(t)\left(\frac{\log t}{z(t)}f(t)\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\times\left[\int_{1}^{b}z(t)\left(\Lambda(t)\right)^{p/q}\Delta t\right]^{\frac{p-q}{p}}.$$

When $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, we have from Theorems 2.5 the following results.

Corollary 2.5. *Assume that f is a nonnegative function on $[1,\infty)_{\mathbb{R}}$ and $0<\lambda\leq 2$. Then for any $b>1$*

$$\int_{1}^{b}\frac{1}{t}\left[\int_{t}^{b}f(s)ds\right]^{\lambda}dt\leq 2^{\lambda}\left[\int_{1}^{b}\frac{1}{t}\left(t\log t\right)^{\lambda}f^{\lambda}(t)dt\right],\ \lambda\leq 2. \tag{35}$$

Corollary 2.6. *Let $b\in[1,\infty)_{\mathbb{N}}$ and $0<p/q\leq 2$. Let f be a nonnegative sequences and $\Lambda(n)$ is defined as in (27) . Then for any $b>1$*

$$\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n+1}^{b-1}f(s)\right)^{p/q}\leq 2\left[\sum_{n=1}^{b-1}\frac{\left((\log n)\,f(n)\right)^{p/q}}{\left(\log\left(\frac{n+1}{n}\right)\right)^{\frac{p}{q}-1}}\right]^{\frac{q}{p}}\times\left[\sum_{n=1}^{b-1}\log\left(\frac{n+1}{n}\right)\left(\sum_{s=n}^{b-1}f(s)\right)^{p/q}\right]^{\frac{p-q}{p}}.$$

*Remark.*It is worth mentioning here that the constant b can be replaced by ∞ in Theorems 2.1- 2.5. Also one can replace $z(t)$ by $Z(t)$ in Theorem 2.5 to obtain general results with $\log g(t)$ instead of $\log t$.

In the following, we prove a new class of inequalities similar to the inequality (6) on time scales by using the operator

$$\Omega(t):=\int_{0}^{t}f(s)\Delta s,\ \ \textit{for any}\ \ t\in\mathbb{T}. \tag{36}$$

Theorem 2.6. *Let $\mathbb{T}$ be a time scale $p,q>0$ such that $p/q\geq 2$. Assume that f is an nonnegative function on $\mathbb{T}$ and $\Omega(t)$ is defined as in (36). Then*

$$\int_{0}^{1}z(t)\left(\Omega^{\sigma}(t)\right)^{p/q}\Delta t\leq(2^{\frac{p}{q}-1})^{p/q}\int_{0}^{1}z(t)\left(\frac{|\log t|}{z(t)}f(t)\right)^{p/q}\Delta t. \tag{37}$$

Proof. As in the proof of Theorem 2.3 we have

$$\int_{0}^{1}z(t)(\Omega^{\sigma}(t))^{p/q}\Delta t=(p/q)\int_{0}^{1}(|\log t|)\int_{0}^{1}[h\Omega^{\sigma}+(1-h)\Omega)]^{\frac{p}{q}-1}dh(f(t))\Delta t. \tag{38}$$

Applying the inequality (30) on the term $[h\Omega^{\sigma}+(1-h)\Omega)]^{\frac{p}{q}-1}$, we see (note that $\Omega^{\Delta}(t)=f(t)>0$) that

$$\begin{aligned}&\frac{p}{q}\int_{0}^{1}[h\Omega^{\sigma}+(1-h)\Omega)]^{\frac{p}{q}-1}dh\\&\leq\frac{p}{q}2^{\frac{p}{q}-2}\int_{0}^{1}\left[h^{\frac{p}{q}-1}(\Omega^{\sigma})^{\frac{p}{q}-1}+(1-h)^{\frac{p}{q}-1}\Omega^{\frac{p}{q}-1}\right]dh\\&<2^{\frac{p}{q}-1}(\Omega^{\sigma})^{\frac{p}{q}-1}.\end{aligned} \tag{39}$$

Substituting (39) into (38), we have

$$\begin{aligned}&\int_{0}^{1}z(t)\left(\Omega^{\sigma}(t)\right)^{p/q}\Delta t\leq 2^{\frac{p}{q}-1}\int_{0}^{1}|\log t|\left(\Omega^{\sigma}(t)\right)^{\frac{p}{q}-1}f(t)\Delta t\\&=2^{\frac{p}{q}-1}\int_{1}^{b}\left[(z(t))^{-(p-q)/p}(\log t\, f(t))\right]\left[(z(t))^{(p-q)/p}(\Omega^{\sigma}(t))^{(p-q)/q}\right]\Delta t.\end{aligned}$$

Applying the Hölder inequality (20) on the right hand side with indices p/q and $p/(p-q)$ and proceeding as in the proof of Theorem 2.3, we get the desired inequality (37). The proof is complete.

Applying the chain rule ([4, Theorem 1.87])

$$f^{\Delta}(g(t))=f^{'}(g(c))g^{\Delta}(t),\ \text{where } c\in[t,\sigma(t)],$$

we see that there exists $c\in[t,\sigma(t)]$ such that

$$\left(\Omega^{p/q}(t)\right)^{\Delta}=(\frac{p}{q})\Omega^{\frac{p}{q}-1}(c)\Omega^{\Delta}(t). \tag{40}$$

Using (36), we see that $\Omega^{\Delta}(t) = f(t) > 0$. This implies that $\Omega^{\sigma}(t) \geq \Omega(c)$, since $\sigma(t) \geq c$. Substituting this into (40), we have

$$\left(\Omega^{p/q}(t)\right)^{\Delta} \leq \frac{p}{q}(\Omega^{\sigma}(t))^{\frac{p}{q}-1} f(t). \tag{41}$$

Proceeding as in the proof of Theorem 2.6, we have the following theorem.

Theorem 2.7. *Let $\mathbb{T}$ be a time scale $p,\, q > 0$ such that $p/q \geq 2$. Assume that f is an nonnegative function on $\mathbb{T}$ and $\Omega(t)$ is defined as in (36). Then*

$$\int_0^1 z(t)\,(\Omega^{\sigma}(t))^{p/q}\,\Delta t \leq (\frac{p}{q})^{p/q} \int_0^1 z(t)\left(\frac{|\log t|}{z(t)} f(t)\right)^{p/q} \Delta t.$$

Applying the inequality (34) on the term $[h\Omega^{\sigma} + (1-h)\Omega)]^{\frac{p}{q}-1}$ when $p/q \leq 2$, we see that

$$\begin{aligned} &\frac{p}{q}\int_0^1 [h\Omega^{\sigma} + (1-h)\Omega)]^{\frac{p}{q}-1}\, dh \\ &\leq \frac{p}{q}\int_0^1 \left[h^{\frac{p}{q}-1}(\Omega^{\sigma})^{\frac{p}{q}-1} + (1-h)^{\frac{p}{q}-1}\Omega^{\frac{p}{q}-1}\right] dh \\ &= \left[(\Omega^{\sigma})^{\frac{p}{q}-1} + \Omega^{\frac{p}{q}-1}\right] \leq 2\,(\Omega^{\sigma})^{\frac{p}{q}-1}. \end{aligned} \tag{42}$$

Proceeding as in the proof of Theorem 2.6, we have the following theorem.

Theorem 2.8. *Let $\mathbb{T}$ be a time scale $p,\, q > 0$ such that $p/q \leq 2$. Assume that f is an nonnegative function on $\mathbb{T}$ and $\Omega(t)$ is defined as in (36). Then*

$$\int_0^1 z(t)\,(\Omega^{\sigma}(t))^{p/q}\,\Delta t \leq 2^{p/q} \int_0^1 z(t)\left(\frac{|\log t|}{z(t)} f(t)\right)^{p/q} \Delta t.$$

In Theorems 2.6-2.8 one can use $Z(t)$ instead of $z(t)$ and get the following results.

Theorem 2.9. *Let $\mathbb{T}$ be a time scale and $p,\, q > 0$ such that $p/q \geq 2$. Assume that f is an nonnegative function on $\mathbb{T}$ and $\Omega(t)$ is defined as in (36). Then*

$$\int_0^1 Z(t)\,(\Omega^{\sigma}(t))^{p/q}\,\Delta t \leq (2^{\frac{p}{q}-1})^{p/q} \int_0^1 Z(t)\left(\frac{|\log g(t)|}{Z(t)} f(t)\right)^{p/q} \Delta t.$$

Theorem 2.10. *Let $\mathbb{T}$ be a time scale $p,\, q > 0$ such that $p/q \geq 2$. Assume that f is an nonnegative function on $\mathbb{T}$ and $\Omega(t)$ is defined as in (36). Then*

$$\int_0^1 Z(t)\,(\Omega^{\sigma}(t))^{p/q}\,\Delta t \leq (\frac{p}{q})^{p/q} \int_0^1 Z(t)\left(\frac{|\log t|}{z(t)} f(t)\right)^{p/q} \Delta t.$$

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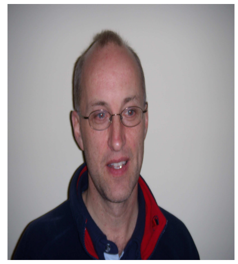

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