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Singular Values of One Parameter Family of Generalized Generating Function of Bernoulli's Numbers

Mohammad Sajid∗

College of Engineering, Qassim University, Buraidah, AlQassim, Saudi Arabia

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Abstract: The goal of this paper is to describe the singular values of one parameter family of generalized generating function of Bernoulli's numbers, $f_\lambda(z)=\lambda\frac{z}{b^z-1}$, $f_\lambda(0)=\frac{\lambda}{\ln b}$ for $\lambda\in\mathbb{R}\backslash\{0\}$, $z\in\mathbb{C}$ and $b>0$ except $b=1$. It is found that the function $f_\lambda(z)$ has an infinite number of singular values for all $b>0$ except $b=1$. Further, it is shown that all the critical values of $f_\lambda(z)$ belongs to the exterior of the disk centered at origin and having radius $|\frac{\lambda}{\ln b}|$ in the right half plane for $0<b<1$ and in the left half plane for $b>1$ respectively.

Keywords: Critical values, singular values, meromorphic function

1 Introduction

The singular values play a very special role in the dynamics of functions in the complex plane. The dynamics of functions, which have finite singular values, are studied by many researchers [1,4,6,17]. But, in the presence of infinite number of singular values, it is very crucial to investigate the dynamical properties of such functions. These investigations are enormously applicable for the description of Julia sets and Fatou sets in the dynamics of functions [2,5,8,9,10,11,12].

In this work, the singular values of one parameter family of generalized generating function of Bernoulli's numbers $\frac{z}{b^z-1}$ are described which is a generalization of one parameter family of function $\frac{z}{e^z-1}$ [15]. Let us consider

$$\mathscr{F}=\{f_\lambda(z)=\lambda\frac{z}{b^z-1}\text{ and }f_\lambda(0)=\frac{\lambda}{\ln b}:\lambda\in\mathbb{R}\backslash\{0\},\ z\in\mathbb{C},b>0,b\neq 1\}$$

The function $f_\lambda\in\mathscr{F}$ is a transcendental meromorphic function with infinite number of poles; it is neither even nor odd and not periodic. The function $f_\lambda\in\mathscr{F}$ is also related on base b to generalized Apostol-Bernoulli's generating function $\left(\frac{z}{\lambda e^z-1}\right)^\alpha e^{tz}=\sum_{k=0}^{\infty}B_k^{(\alpha)}(t;\lambda)\frac{z^k}{k!}$ by choosing $\alpha=1$, $\lambda=1$ and $t=0$.

The point $z^*\in\mathbb{C}$ is said to be a critical point of $f(z)$ if $f'(z^*)=0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. The point $w\in\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma:[0,\infty)\to\hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty}\gamma(t)=\infty$ and $\lim_{t\to\infty}f(\gamma(t))=w$. A singular value of f is defined to be either a critical value or an asymptotic value of f.

The organization of the present paper is as follows: In Theorem 2.1, it is found that the function $f_\lambda\in\mathscr{F}$ has infinitely many singular values. It is shown that, in Theorem 2.2, the function $f'_\lambda(z)$ has no roots in (i) the left half plane for $0<b<1$ (ii) the right half plane for $b>1$. Moreover, it is proved that all the critical values of $f_\lambda\in\mathscr{F}$ belongs to the exterior of the open disk in Theorem 2.3.

It is observed that the singular values of one parameter families of functions are bounded or inside the open disk in [13,16], but the singular values are found outside the open disk in [14]. The singular values of one special class of functions is found by Eremenko [3]. Some more results on singular values can be seen in [7,18].

∗ Corresponding author e-mail: msajid@qec.edu.sa

2 Singular Values of $f_\lambda \in \mathscr{F}$

The following theorem shows that the function $f_\lambda \in \mathscr{F}$ has infinitely many singular values:

Theorem 2.1. Let $f_\lambda \in \mathscr{F}$. Then, the function $f_\lambda(z)$ possesses infinitely many singular values.

Proof. For critical points, $f'_\lambda(z) = \lambda \frac{(1-z\ln b)b^z - 1}{(b^z-1)^2} = 0$. This gives

$$(z\ln b - 1)b^z + 1 = 0.$$

The real and imaginary parts of this equation are

$$\frac{y\ln b}{\sin(y\ln b)} - b^{y\cot(y\ln b) - \frac{1}{\ln b}} = 0 \quad (1)$$

$$x = \frac{1}{\ln b} - y\cot(y\ln b) \quad (2)$$

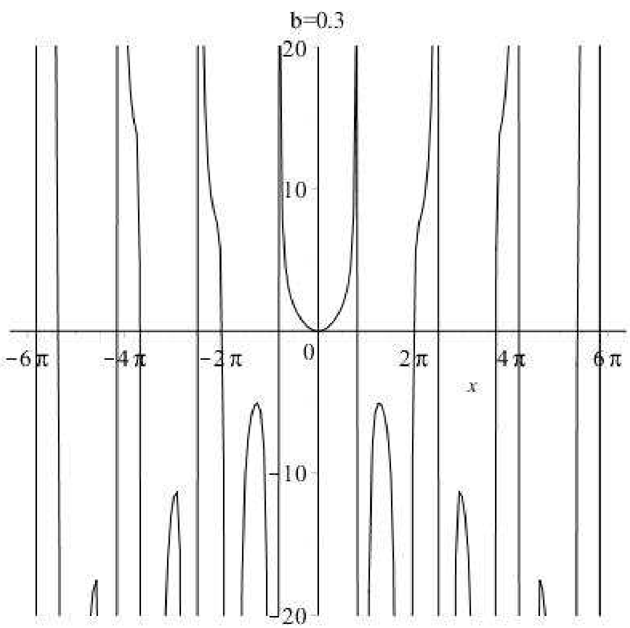

Fig. 1: Graph of $\frac{y\ln 0.3}{\sin(y\ln 0.3)} - 0.3^{y\cot(y\ln 0.3) - \frac{1}{\ln 0.3}}$

From the Figure 1 for $b = 0.3$ and Figure 2 for $b = 3$, it is observed that Equation (1) has infinitely many solutions. Also, Equation (2) has infinitely many values. The rest of proof is similar as [Theorem 2.3, [16]]. Hence, the function $f_\lambda \in \mathscr{F}$ has infinitely many critical values.

The finite asymptotic value of $f_\lambda(z)$ is 0 since $f_\lambda(z) \to 0$ as $z \to \infty$ along (i) negative real axis for $0 < b < 1$ and (ii) positive real axis for $b > 1$.

Thus, it proves that the function $f_\lambda \in \mathscr{F}$ possesses infinitely many singular values. □

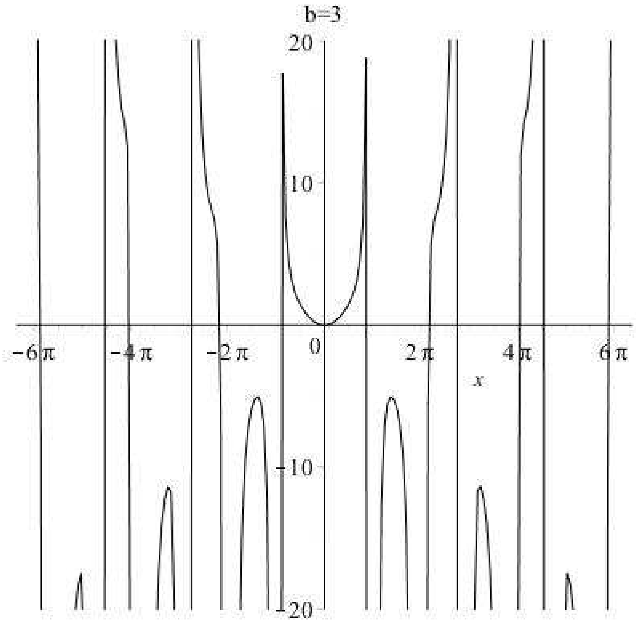

Fig. 2: Graphs of $\frac{y\ln 3}{\sin(y\ln 3)} - 3^{y\cot(y\ln 3) - \frac{1}{\ln 3}}$

The left half plane and the right half plane are denoted, respectively, by

$$H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$$

and

$$H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}.$$

In the following theorem, it is found that $f'_\lambda(z)$ has no zeros in the left half plane for $0 < b < 1$ and the right half plane for $b > 1$:

Theorem 2.2. Let $f_\lambda \in \mathscr{F}$. Then, the function $f'_\lambda(z)$ has no roots in (i) the left half plane H^- for $0 < b < 1$ (ii) the right half plane H^+ for $b > 1$.

Proof. For roots of $f'_\lambda(z) = 0$, we have $b^{-z} = 1 - z\ln b$. The real and imaginary parts of this equation are

$$b^{-x}\cos(y\ln b) = 1 - x\ln b$$

$$b^{-x}\sin(y\ln b) = y\ln b$$

The rest proof of this theorem is similar as [Theorem 2.1 (a), [16]] for $0 < b < 1$ and [Theorem 2.2 (i), [16]] for $b > 1$. □

It is proved, in the following theorem, that the function $f_\lambda \in \mathscr{F}$ has all the critical values into the exterior of the open disk centered at origin and having radius $\frac{|\lambda|}{\ln b}$:

Theorem 2.3. Let $f_\lambda \in \mathscr{F}$. Then, all the critical values of $f_\lambda(z)$ belongs to the exterior of the open disk centered at origin and having radius $|\frac{\lambda}{\ln b}|$ in the right half plane H^+

for $0 < b < 1$ and in the left half plane H^- for $b > 1$ respectively.

Proof. At first, we show that $f_\lambda(z)$ maps the right half plane H^+ onto the exterior of the open disk. Suppose that the line segment γ is given by $\gamma(t) = tz$, $t \in [0,1]$. Let $\zeta(z) = b^z$ for an arbitrary fixed $z \in \mathbb{C}$. Now

$$\int_\gamma \zeta(z)dz = \int_0^1 \zeta(\gamma(t))\gamma'(t)dt = z\int_0^1 b^{tz}dt = \frac{1}{\ln b}(b^z - 1) \quad (3)$$

(a)For $0 < b < 1$
Since $M_1 \equiv \max_{t\in[0,1]} |\zeta(\gamma(t))| = \max_{t\in[0,1]} |b^{tz}| < 1$ for $z \in H^+$, then, by Equation (3),

$$|b^z - 1| = \left|\ln b \int_\gamma \zeta(z)dz\right| \leq M_1|z||\ln b| < |z||\ln b|$$

$$\left|\frac{z}{b^z - 1}\right| > \left|\frac{1}{\ln b}\right| \quad \text{for all } z \in H^+.$$

It follows that

$$|f_\lambda(z)| = \left|\lambda \frac{z}{b^z - 1}\right| > \left|\frac{\lambda}{\ln b}\right| \quad \text{for all } z \in H^+.$$

It shows that $f_\lambda(z)$ maps H^+ onto the exterior of the open disk centered at origin and having radius $|\frac{\lambda}{\ln b}|$. By Theorem 2.2 (i), the function $f'_\lambda(z)$ has no zeros in the left half plane H^-. It follows that all the critical points lie in the right half plane H^+. Consequently, all the critical values of $f_\lambda \in \mathscr{F}$ belongs to the exterior of the open disk centered at origin and having radius $|\frac{\lambda}{\ln b}|$ in the right half plane H^+ for $0 < b < 1$.

(b)For $b > 1$
Since $M_2 \equiv \max_{t\in[0,1]} |\zeta(\gamma(t))| = \max_{t\in[0,1]} |b^{tz}| < 1$ for $z \in H^-$, then, using Equation (3),

$$|b^z - 1| = \left|\ln b \int_\gamma \zeta(z)dz\right| \leq M_2|z|\ln b < |z|\ln b$$

$$\left|\frac{z}{b^z - 1}\right| > \frac{1}{\ln b} \quad \text{for all } z \in H^-.$$

It gives that

$$|f_\lambda(z)| = \left|\lambda \frac{z}{b^z - 1}\right| > \frac{|\lambda|}{\ln b} \quad \text{for all } z \in H^-.$$

It proves that $f_\lambda(z)$ maps H^- onto the exterior of the open disk centered at origin and having radius $\frac{|\lambda|}{\ln b}$. Using similar arguments as above, by Theorem 2.2 (ii), it gives that all the critical values of $f_\lambda \in \mathscr{F}$ belong to the exterior of the open disk centered at origin and having radius $\frac{|\lambda|}{\ln b}$ in the left half plane for $b > 1$. □

3 Conclusion

In this paper, the singular values of one parameter family of generalized generating function of Bernoulli's numbers were described. It was found that this family of functions has an infinite number of singular values. It was also shown that all the critical values of this family belongs to the exterior of the open disk in the right half plane for positive base less than one while, for more than one, lie in the left half plane.

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Mohammad Sajid received the PhD degree in Mathematics at Indian Institute of Technology Kanpur, India. His current research interests are in the fields Engineering Mathematics, Real and Complex Dynamics including chaos and Fractals. He has published research articles in international refereed journals of mathematical and engineering sciences. He is referee and editorial board member of several mathematical and Engineering sciences journals.