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A Decomposition Method for Solving q-Difference Equations

Hossein Jafari[1,2,*], *Sarah Jane Johnston*[1], *Sajad Mahmood Sani*[2] *and Dumitru Baleanu*[3,4]

[1] Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, South Africa.
[2] Department of Mathematics, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran.
[3] Department of Mathematics and Computer Science Çankaya University, Ankara, Turkey
[4] Institute of Space Sciences,P.O.BOX, MG-23, R 76900, Magurele-Bucharest, Romania

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Abstract: The q-difference equations are important in q-calculus. In this paper, we apply the iterative method which is suggested by Daftardar and Jafari, hereafter called the Daftardar-Jafari method, for solving a type of q-partial differential equations. We discuss the convergency of this method. In the implementation of this technique according to other iterative methods such as Adomian decomposition and homotopy perturbation methods, one does not need the calculation of the Adomian's polynomials for nonlinear terms. It is proven that under a special constraint, the given result by this method converges to exact solution of nonlinear q-ordinary or q-partial differential equations.

Keywords: q-calculus; Daftardar-Jafari method; iterative method; q-partial differential equations.

1 Introduction

The q-difference has many applications in different mathematical areas and appears in connections between physics and mathematics, such as statistical physics [18], fractal geometry [8,9], quantum mechanics, number theory, combinatorics, orthogonal polynomials [11] and other sciences including quantum theory, mechanics and theory of relativity [2,**?**].

Wu has applied the variational iteration method for solving q-diffusion equations and q-difference equations of second order [20,**?**,21]. In [17], Qin and Zeng have extended the homotopy perturbation method to obtain the exact solution of q-diffusion equations. The one-dimensional q-differential transformation (qDTM) has been used by Jing and Fan (cf. [14]) for solving the q-differential equations. In [7], El-Shahed and Gaber applied the two-dimensional q-differential transformation to solve the q-diffusion and q-wave equations. After that, Jefari et. al. used reduced q-differential transformation method for q-partial differential equations (cf. [12]).

Recently Daftardar and Jafari introduced an iterative method to obtain solution of functional equations (cf. [6]). It was proved that this method is convergent in [6,3]. This method has been used used for solving nonlinear time-fractional partial differential equations [16], singular boundary value problems, fifth and sixth order nonlinear boundary value problems, Laplace equation [22,23,24] and other type of equations [6,3].

This iterative method solves nonlinear equations without using Adomian polynomials and it is advantage over the ADM and the HPM.

In the paper we have used this method to obtain exact/ approximate solution of q-partial differential equations. The present paper is organized as follows. After this introduction Section 2 reviews the properties of q-calculus. In Section 3, we recall the DJM for solving q-PDEs, Also we investigate the convergence of this method. Furthermore in Section 4 by using the DJM, we solve few examples. Finally Section 5 is devoted to conclusions.

2 Properties of q-calculus

In this section we briefly review some notation and basic definitions and theorems of q-calculus.

* Corresponding author e-mail: jafari@umz.ac.ir

•***q*-Calculus**

The q-derivative of a real continuous function $f(x)$ is defined as follows

$$D_q^x f(x) = \frac{f(qx)-f(x)}{(q-1)x} \quad x\in\mathbb{R}\setminus\{0\},$$

where q in $(0,1)$ is a fixed number. The derivative at 0 is shown by $f'(0)$, that means $f'(0)$ is exists. . The partial q-derivatives for a multivariable continuous function $f(x;y;\ldots)$ are defined by Jackson [15] and are given by

$$\partial_q^x f(x;y;\ldots) = \frac{f(qx;y;\ldots)-f(x;y;\ldots)}{(q-1)x}, \quad q\in(0,1)$$

$$\partial_q^x f(x;y;\ldots)|_{x=0} = \lim_{n\to\infty}\frac{f(xq^n;y;\ldots)-f(0;y;\ldots)}{xq^n}.$$

The q-integral [15] is

$$\int_0^x f(t)d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(xq^n).$$

•***q*-Leibniz Product law**

$$D_q^x[f(x)g(x)] = g(qx)D_q^x[f(x)] + f(x)D_q^x[g(x)].$$

•***q*-Integration by parts**

We may write

$$\int_a^b g(qx)D_q^x f(x)d_qx = f(x)g(x)|_a^b - \int_a^b f(x)D_q^x g(x)d_qx. \quad (1)$$

From (1) we have the following relation:

$$\int_0^x D_q^x f(x)d_qx = f(x)-f(0).$$

For more information about q-calculus, readers are referred to [15,10,4,2,11].

3 Daftardar-Jafari method

In this section, we adopt the Daftardar-Jafari method for solving q-difference equations. Consider the following type of q-functional equation:

$$u = g + N_q(u), \quad (2)$$

where g is a known function and N_q is a nonlinear operator. In this technique we decompose y as following infinite series form:

$$u = \sum_{i=0}^{\infty} u_i. \quad (3)$$

Also, the nonlinear operator in (2) decompose as

$$N_q\left(\sum_{i=0}^{\infty} u_i\right) = N_q(u_0) + \sum_{i=1}^{\infty}\left\{N_q(\sum_{j=0}^{i} u_j) - N_q(\sum_{j=0}^{i-1} u_j)\right\}. \quad (4)$$

Substituting equations (3) and (4) into (2) leads to

$$\sum_{i=1}^{\infty} u_i = f + N_q(u_0) + \sum_{i=1}^{\infty}\left\{N_q(\sum_{j=0}^{i} u_j) - N_q(\sum_{j=0}^{i-1} u_j)\right\}.$$

To compute the components of u_i, $i\geq 0$ in series (3), we use the following recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N_q(u_0), \\ u_{m+1} = N_q(u_0+\ldots+u_m) - N_q(u_0+\ldots+u_{m-1}), \\ \qquad m = 1,2,\ldots \end{cases} \quad (5)$$

Theorem 1.*If N is a contraction, then the defined series in (3) is absolutely convergent.*

*Proof.*If N is a contraction, i.e. $\|N_q(x)-N_q(y)\| \leqslant k\|x-y\|$, $0<k<1$, Then in view of (5) we have

$$\|u_{m+1}\| = \|N_q(u_0+\ldots+u_m) - N_q(u_0+\ldots+u_{m-1}\| \leqslant k\|u_m\| \leqslant k^m\|u_0\|, \quad m = 0,1,2,\ldots$$

So the series $\sum_{i=0}^{\infty} u_i$ converges (absolutely and uniformly) to a solution of equation (2) (see [5]).

Theorem 2.*If the defined series in (3) is convergent, then it gives an exact solution of the nonlinear problem (2).*

*Proof.*If the series (3) is convergent, then in view of (5) we have

$$u_0+u_1+\ldots+u_{m+1} = f + N_q(u_0+\ldots+u_m).$$

When m tends to infinity we have

$$\lim_{m\to\infty}\sum_{i=0}^{m+1} u_i = \lim_{m\to\infty}\left(f + N_q\left(\sum_{i=0}^{m} u_i\right)\right) = f + N_q\left(\lim_{m\to\infty}\sum_{i=0}^{m} u_i\right) = f + N_q(y) = y.$$

4 Examples

*Example 1.*Consider the following Riccati type nonlinear q-differential equation

$$\frac{d_q}{d_qx}y(x) = 1 + y^2(x); \quad y(0) = 0. \quad (6)$$

Equation (6) is equivalent to the following q-integral equation

$$y(x) = x + \int_0^x y^2(t)d_qt$$

Following the algorithm given in (5) we have

$$\begin{aligned} u_0(x) &= f(x) = x, \\ u_1(x) &= N_q(u_0) = \int_0^x u_0^2(t) d_q t = \int_0^x t^2 d_q t = \frac{1-q}{1-q^3}x^3, \\ u_2(x) &= N_q(u_0+u_1) - N_q(u_0) = \\ &\quad \frac{1-q}{1-q^3}x^5\left(2\frac{1-q}{1-q^5} + \frac{1-q}{1-q^7}x^2\right), \\ &\vdots \end{aligned} \tag{7}$$

Hence

$$y(x) = x + \frac{1-q}{1-q^3}x^3 + \frac{1-q}{1-q^3}x^5(2\frac{1-q}{1-q^5} + \frac{1-q}{1-q^7}x^2) + \ldots.$$

*Example 2.*Consider the following q-partial derivative difference equation [20,17]

$$\frac{\partial_q}{\partial_q t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial x}(x u(x,t)), \quad u(x,0) = x^2 \tag{8}$$

By applying q-integral operator on both side of (8), the equation converts to the following q-integral equation

$$u(x,t) = x^2 + \int_0^t \left[\frac{\partial^2}{\partial x^2}u(x,k) + \frac{\partial}{\partial x}(x u(x,k))\right] d_q k.$$

In view of the given algorithm in (5) the components of $u_i(x,t), i \geq 0$ will compute as follows:

$$\begin{aligned} u_0(x,t) &= f = x^2, \\ u_1(x,t) &= N_q(u_0) = \int_0^t [2+3x^2] d_q k = [2+3x^2]t, \\ u_2(x,t) &= N_q(u_0+u_1) - N_q(u_0) = (8+9x^2)\frac{t^2}{[2]_q!}, \\ u_3(x,t) &= (26+27x^2)\frac{t^3}{[3]_q!}, \\ &\vdots \\ u_n(x,t) &= (3^n - 1 + 3^n x^2)\frac{t^n}{[n]_q!}, \quad n \geq 1 \end{aligned}$$

Hence

$$\begin{aligned} u(x,t) &= x^2 + (2+3x^2)\frac{t}{[1]_q!} + (8+9x^2)\frac{t^2}{[2]_q!} + \\ &\quad (26+27x^2)\frac{t^3}{[3]_q!} + \ldots + \\ &\quad (3^n - 1 + 3^n x^2)\frac{t^n}{[n]_q!} + \ldots \\ &= x^2 + \sum_{i=1}^{\infty}(3^n - 1 + 3^n x^2)\frac{t^n}{[n]_q!}. \end{aligned}$$

This example has been solved using the VIM [20] and also the HPM [17].

5 Conclusions

The Daftardar-Jafari iterative method has been applied to give very reliable and accurate solutions to the q-difference equations. This method solves nonlinear problems without using Adomians polynomials which can be considered as preferable over the ADM and the HPM. Expanding of our work to q-fractional order derivative cases will be interesting.

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Hossein Jafari is Associate Professor of Mathematics at University of Mazandaran. His research interests are in the areas of applied mathematics and mathematical physics including the mathematical methods and models for fractional differential equations. He is referee and editor of mathematical journals.

Sarah Jane Jonston is an Associate Professor of Mathematics at the University of South Africa. Her research interests are in special functions, orthogonal polynomials and hypergeometric functions and polynomials. She is an active member of the South African Mathematical Society in the portfolio of Scientific Activities and has refereed articles for the Journal of Computational and Applied Mathematics, the Journal of Approximation Theory, Mathematical and Computer Modelling, Applied Mathematics Letters, Advances in Pure Mathematics and Miskolc Mathematical Notes

Sajad Mahmood Sani has completed M.Sc. in Applied Mathematics at University of Mazandaran. His research interests are in the areas of applied mathematic

Dumitru Baleanu is Professor at the Institute of Space Sciences, Magurele-Bucharest, Romania and visiting staff member at the Department of Mathematics and Computer Sciences, Cankaya University, Ankara, Turkey. His research interests include fractional dynamics and its applications, fractional differential equations, discrete mathematics, dynamic systems on time scales and the wavelet method and its applications. He is referee and editor of mathematical journals.