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A Robust Finite Difference Method for Two-Parameter Parabolic Convection-Diffusion Problems

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Abstract: We consider a class of singularly perturbed parabolic differential equations with two small parameters affecting the derivatives. The solution to such problems typically has parabolic layers. We discretize the time variable by means of the classical backward Euler method. At each time level a two-point boundary value problem is obtained. These problems are, in turn, discretized in space on a uniform mesh following the nonstandard methodology of Mickens. We prove that the underlying discrete operator satisfies a minimum principle. We use this result in the error analysis. We show that the method is uniformly convergent with respect to the perturbation parameters. This is contradictory with the assertion [G.I. Shishkin, A difference scheme for a singularly perturbed equation of parabolic type with discontinuous initial condition, *Soviet Math. Dokl.* **37** (1988) 792-796] that parameter-uniform numerical methods cannot be designed on a uniform mesh for problems whose solution exhibits parabolic layers. Finally we give numerical results to attest the parameter-uniform convergence. Moreover, comparison with some existing methods in the literature proves the competitiveness of our method.

Keywords: Parabolic reaction-convection-diffusion problems, Singular perturbations, fitted operator finite difference methods, error bounds, uniform convergence

1 Introduction

The research field of singular perturbation problems (SPPs), which originated in the early nineteens [5] with the development of the boundary-layer idea in viscous flow [24] has flourished over the last few decades. Despite the large amount of work accomplished in this field, there is still avenue for relevantly more opportune research.

A small parameter affecting the highest derivative in the governing equation of a given SPP gives rise to large gradients in the solution over narrow regions of the domain, notwithstanding that large gradients could also result from possible discontinuities in the data of the problem. For more information about SPPs, readers are referred to [4], [13], [25] and the references therein.

The determination of the analytical solution is often a difficult task. The search of numerical approximations via classical methods have shown limited success in that they require the use of a large number of mesh points which increases both the round-off error and the computational cost. Several alternatives which provide acceptable numerical approximations exist in the literature (See for example [2], [6], [11], [15], [16], [17] and the survey article [8]).

All the work cited above concerns SPPs in which the perturbation parameter affects the highest derivative terms. While it is known that physical problems often involve several parameters [20], as far as numerical methods are concerned, not much is known. In particular, since the pioneering work of O'Malley on two-parameter SPPs [19], very few researchers have followed suit in this direction. Below we give some examples.

O'Riordan *et al.* [21] derive parameter-explicit theoretical bounds on the derivatives of the solutions to two-parameter singularly perturbed BVPs. They also construct a finite difference method. For the same family of problems, Roos and Uzelac [26] design a streamline-diffusion finite element method. Linß and Roos [10] use a barrier-function technique to derive bounds on the derivatives and use these bounds to analyse an upwind-difference scheme. Gracia *et al.* [7] develop a monotone numerical method and establish an asymptotic error bound whose error constants are independent of the singular perturbation parameters.

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The class of singularly perturbed two-parameter parabolic reaction-diffusion-convection problem is studied by O'Riordan *et al.* [22] and Kabadalbajoo and Yadaw [9]. The former authors derive bounds on the derivatives of the solutions and use them to analyse an upwind finite difference method while the latter design and a finite element method.

We note that all the works above use Shishkin type of meshes. Up to the best of our knowledge, no publication reports the use of uniform meshes except Patidar's [23] where he solves two-parameter singularly perturbed BVPs using a fitted operator approach.

The aim of the present paper is to solve a class of time-dependent singularly perturbed convection-diffusion problems with two parameters affecting the diffusion and the convection terms. More precisely, we are concerned with the problem of finding $u(x,t)$ such that for all $(x,t) \in Q = \Omega \times (0,T]$, $\Omega = (0,1)$,

$$L_{x,t}u \equiv \varepsilon \frac{\partial^2 u}{\partial x^2} + \mu a(x)\frac{\partial u}{\partial x} - b(x)u(x) - \frac{\partial u}{\partial t} = f(x,t), \tag{1.1}$$

$(x,t) \in D$, subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \;\; x \in \bar{\Omega}, \tag{1.2}$$

$$u(0,t) = u(1,t) = 0, t \in [0,T], \tag{1.3}$$

with two small parameters $0 < \varepsilon, \mu \leq 1$. The functions $a(x)$, $b(x)$ and $f(x,t)$ are sufficiently smooth and satisfy $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$; α and β are real numbers. The solution to (1.1)-(1.3) has parabolic layers. Such problems are encountered in several contexts including chemical flow reactor theory [1], [27], [29] and lubrication theory [3].

Note that the continuous operator $L_{x,t}$ satisfies the following minimum principle.

Lemma 11(Continuous Minimum Principle) *If* $\psi(x,t) \in C^2(Q) \cap C^0(\bar{Q})$ *is such that* $\psi(x,t)|_{\partial Q} \geq 0$ *and* $L_{x,t}\psi(x,t)|_Q \leq 0$*, then* $\psi(x,t)|_{\bar{Q}} \geq 0$.

Proof. Let us assume that there is a point $(x^*,t^*) \in \bar{Q}$ such that $\psi(x^*,t^*) < 0$ and $\psi(x^*,t^*) = \min_{(x,t)\in\bar{Q}} \psi(x,t)$. Clearly $(x^*,t^*) \notin \partial Q$. It follows that $\psi_x(x^*,t^*) = 0$, $\psi_x(x^*,t^*) = 0$ and $\psi_{xx}(x^*,t^*) \geq 0$ and this implies that $L_{x,t}\psi(x^*,t^*) > 0$, which is a contradiction. Thus $\psi(x^*,t^*) \geq 0$. It follows that $\psi(x,t) \geq 0$, $\forall (x,t) \in \bar{Q}$.

We propose a numerical scheme which we design in two steps. Firstly the time variable is discretized using the classical backward Euler method. This leads to a system of boundary value problems. These problems are, in turn, discretized in space on a uniform mesh following the nonstandard methodology of Mickens [12]. We refer to this scheme as a fitted operator finite difference method (FOFDM).

After proving that the fitted operator of the fully discrete problem satisfies a minimum principle, thus replicating this property of the continuous operator $L_{x,t}$, we establish its uniform stability in the maximum norm. Then we show that the proposed FOFDM is uniformly convergent with respect to the perturbation parameters ε and μ. This is in contradiction with the claim which was made in [28] and relayed in [9], [22] that parameter-uniform numerical methods cannot be designed on a uniform mesh for problems whose solution exhibits parabolic layers.

Finally we provide numerical results to attest the parameter-uniform convergence. Moreover, comparison with some existing methods in the literature proves the competitiveness of the method.

The rest of the paper is organised as follows. Section 2 is devoted to temporal discretization. We present the local and global error estimates. Spatial discretization is the subject of Section 3. We prove that the fitted operator satisfies a minimum principle. We use this fact to establish a stability result. In section 4, we show that the method is uniformly convergent with respect to the perturbation parameters. Numerical results confirming our findings are provided in Section 5. We also test the method for accuracy in comparison with some existing methods. Finally, some conclusions are drawn in Section 6.

2 Temporal discretization

We use the implicit Euler method to discretize the time variable with a uniform step size τ, so that the time interval $[0,T]$ is partitioned as

$$\bar{\omega}^K = \{t_k = k\tau, \;\; 0 \leq k \leq K, \;\; \tau = T/K\}. \tag{2.4}$$

We obtain the following linear system in space at each time level

$$L_x z \equiv \varepsilon z_{xx}(x,t_k) + \mu a(x) z_x(x,t_k) - \left(b(x) + \frac{1}{\tau}\right) z(x,t_k)$$
$$= f(x,t_k) - \frac{1}{\tau} z(x,t_{k-1}), \tag{2.5}$$

subject to the conditions

$$z(x,0) = u_0, \; \forall x \in (0,1); \;\; z(0,t_k) = z(1,t_k) = 0. \tag{2.6}$$

The local truncation error of the time discretization (2.5) is defined by $e_k = u(x,t_k) - \hat{z}(x,t_k)$, where $\hat{z}(x,t_k)$ is the solution of the two-point boundary value problem

$$\varepsilon z_{xx}(x,t_k) + \mu a(x) z_x(x,t_k) - \left(b(x) + \frac{1}{\tau}\right) z(x,t_k)$$
$$= f(x,t_k) - \frac{1}{\tau} u(x,t_{k-1}), \tag{2.7}$$

along with $z(0,t_k) = z(1,t_k) = 0$. The global error at time level t_k, which we denote by E_k, is the sum of all local errors anterior and up to time level t_k:

$$E_k = \sum_{l=0}^{k} e_l, \quad k \leq T/\tau.$$

We now state results for the bounds on these errors. Readers are referred to [2] for a proof.

Lemma 21(Local error estimate) *The local error estimate of the time discretization is given by*

$$||e_k||_\infty \le M\tau^2,\ 1 \le k \le K. \tag{2.8}$$

In the lemma above and in the rest of the paper, M represents a generic constant which is independent of the parameters ε and μ as well as of the time and space discretization parameters and which may assume differents values in equations or inequalities where it is involved.

Lemma 22(Global error estimate) *The global error estimate of the time discretization is given by*

$$||E_k||_\infty \le C\tau. \tag{2.9}$$

Proof. See [18].

For the sake of simplicity, we ignore the second argument of the function z as, in fact, z is a function of the single variable x. The differential operator L_x in boundary value problems (2.5)-(2.6) satisfies the following discrete minimum principle:

Lemma 23(Continuous Minimum Principle II) *If $\xi(x)$ is any sufficiently smooth function satisfying $\xi(0) \ge 0$, $\xi(1) \ge 0$ and $L_x\xi(x) \le 0$ for all $x \in \Omega$. Then $\xi(x) \ge 0$, for all $x \in \bar{\Omega}$.*

Proof. See the proof of Lemma 11 and ignore the time variable.

The convergence analysis of the numerical scheme which we propose requires that we use some bounds on the solution and its derivatives. The solutions $\lambda_0(x) < 0$ and $\lambda_1(x) > 0$ of the charateristic equation

$$\varepsilon\lambda^2(x) + \mu a(x)\lambda(x) - \left(b(x) + \frac{1}{\tau}\right) = 0 \tag{2.10}$$

are used to describe the boundary layers at $x = 0$ and $x = 1$, respectively. The following Lemma was proved in [9].

Lemma 24*For any $p \in (0,1)$ we have, up to a certain order q that depends on the smoothness of the functions $a(x)$, $b(x)$ and $f(x,t)$,*

$$\left|\frac{d^j z}{dx^j}\right| \le C\left(1 + \mu_0^j e^{-p\mu_0 x} + \mu_1^j e^{-p\mu_1(1-x)}\right), \tag{2.11}$$

for $0 \le j \le q$.

The quantities μ_0 and μ_1 are defined as $\mu_0 = -\max_{[0,1]} \lambda_0(x)$ and $\mu_1 = \min_{[0,1]} \lambda_1(x)$.

Remark 25*It is to be noted that, if $\mu^2 << \varepsilon$ i.e $\mu^2/\varepsilon \to 0$ as $\varepsilon \to 0$, then $\mu_0 \approx \mu_1 \approx \min\sqrt{(b(x)+1/\tau)\varepsilon^{-1}}$ and we have boundary layers at $x = 0$ and $x = 1$. The situation of one external layer (at $x = 0$) is encountered in the case where $\varepsilon << \mu^2$ i.e $\varepsilon/\mu^2 \to 0$ as $\mu \to 0$. In this case, $\mu_1 \approx 0$ and $\mu_0 \approx \min_{x\in[0,1]} \frac{\mu a(x)}{\varepsilon}$.*

3 The fully discrete problem

Let $\bar{\Omega}^N$ denote the following partition of $[0,1]$ into N subintervals such that

$$x_0 = 0,\ x_i = x_0 + ih,\ i = 1(1)n,\ h = x_i - x_{i-1}, x_N = 1,$$

$\bar{Q}^{N,K} = \bar{\Omega}^N \times \bar{\omega}^K$ be the grid for the x,t-variables, and $Q^{N,K} = \bar{Q}^{N,K} \cap Q$. With reference to the nonstandard finite difference (NSFD) methodology of Mickens [12], we construct the following scheme to solve (2.7) along with suitable boundary conditions.

$$\begin{aligned} L_\varepsilon^{N,K} U(x_j,t_k) \equiv\ & \varepsilon\frac{U(x_{j+1},t_k) - 2U(x_j,t_k) + U(x_{j-1},t_k)}{\phi_j^2} \\ & + \mu a(x_j)\frac{U(x_{j+1},t_k) - U(x_j,t_k)}{h} \\ & - \left(b(x_j) + \frac{1}{\tau}\right)U(x_j,t_k) \\ & = f(x_j,t_k) - \frac{1}{\tau}U(x_j,t_{k-1}) \end{aligned} \tag{3.12}$$

subject to the discrete initial and boundary conditions

$$\begin{aligned} & U(x_j,0) = u_0(x_j),\ \ x_j \in \bar{\Omega}^N; \\ & U(0,t_k) = U(1,t_k) = 0,\ \ t_k \in \bar{\omega}^K. \end{aligned} \tag{3.13}$$

The denominator function ϕ_j^2 is given by

$$\phi_j^2(h,\varepsilon,\mu) \equiv \phi_j^2 = \frac{h\varepsilon}{\mu a(x_j)}\left(\exp\left(\frac{\mu a(x_j)h}{\varepsilon}\right) - 1\right). \tag{3.14}$$

The scheme above results in a system of linear equations

$$AU = F. \tag{3.15}$$

The entries of the tridiagonal matrix A and column-vector F are

$$A_{ij} = r_j^-,\ \ i = j+1;\ j = 1(1)(n-2), \tag{3.16}$$

$$A_{ij} = r_j^c,\ \ i = j;\ j = 1(1)(n-1), \tag{3.17}$$

$$A_{ij} = r_j^+,\ \ i = j-1;\ j = 12(1)(n-1), \tag{3.18}$$

$$F_j = f(x_j,t_k) - \frac{1}{\tau}U(x_j,t^{k-1});\ j = 1(1)(n-1), \tag{3.19}$$

where

$$\begin{aligned} & r_j^- = \frac{\varepsilon}{\phi_j^2},\ \ r_j^c = -\left(\frac{2\varepsilon}{\phi_j^2} + \frac{\mu a(x_j)}{h} + \left(b(x_j) + \frac{1}{\tau}\right)\right), \\ & r_j^+ = \frac{\varepsilon}{\phi_j^2} + \frac{\mu a(x_j)}{h}. \end{aligned} \tag{3.20}$$

The scheme (3.12)-(3.14) is a fitted operator finite difference method (FOFDM) to solve (1.1)-(1.3) . It is an extension of the method presented in [23] to the class of two-parameter time-dependent singularly perturbed problems (1.1)-(1.3).

We adopt the notation $w_j^k = w(x_j,t_k)$ for ease of exposition in the rest of the paper.

The discrete operator of the scheme (3.12)-(3.13) safisfies the following minimum principle.

Lemma 31(Discrete minimum principle) *Assume that* $L^{N,K}$ *is the discrete operator given in (3.12) and* Ψ_j^k *is any mesh function satisfying* $\Psi_j^0 \geq 0,\ \ 0 \leq j \leq N, \Psi_0^k \geq 0,\ \Psi_N^k \geq 0,\ \ 0 \leq k \leq K$. *If* $L^{N,K}\Psi_j^k \geq 0$ *in* Q_N^K, *then* $\Psi_j^k \geq 0$ *in* $\bar{Q}_j^k$.

Proof. Let s and l be indices sucht that $\Psi_s^l = \min_{(j,k)} \Psi_j^k$, for $\Psi_j^k \in \bar{Q}^{N,K}$. Assume that $\Psi_s^l < 0$. It is easy to see that $(s,l) \in \{1,2,\cdots,N\} \times \{1,2,\cdots,K\}$, because otherwise $\Psi_s^l \geq 0$. It follows that $\Psi_{s+1}^l - \Psi_s^l > 0$ and $\Psi_{s-1}^l - \Psi_s^l > 0$. Thus,$L^{N,K}\Psi_s^l < 0$, which is a contradiction. Therefore $\Psi_s^l \geq 0$. The indices s and l being arbitrary, we obtain $\Psi_j^k \geq 0$ in $\bar{Q}_j^k$.

In the next section, we analyze the proposed method for convergence. In the analysis, we will evoke the following stability result.

Lemma 32(Uniform stability estimate) *At any time level* t_k, *if* Z_j^k *is any mesh function such that* $Z_0^k = Z_N^k = 0$, *then*

$$|Z_i^k| \leq \frac{1}{\alpha} \max_{1\leq j\leq N-1} |L^{N,K} Z_j^k|, \text{ for } 0 < i < N. \tag{3.21}$$

In other words, the operator $L^{N,K}$ *is uniformly stable.*

Proof. Let $(\Psi^{\pm})_j^k$ be the mesh function defined by

$$(\Psi^{\pm})_j^k = P \pm Z_j^k$$

where

$$P = \frac{1}{\beta} \max_{1\leq j\leq N-1} |L^{N,K} Z_j^k|.$$

We have $(\Psi^{\pm})_0^k = (\Psi^{\pm})_N^k = P \geq 0$. Moreover, for $1 \leq j \leq N-1$, we have

$$L^{N,K}(\Psi^{\pm})_j^k = -\frac{(b_j + 1/\tau)}{\beta} \max_{1\leq j\leq N-1} |L^{N,K} Z_j^k| \pm L^{N,K} Z_j^k.$$

Using the fact that $0 < \beta \leq b_j < b_j + 1/\tau$, we have $L^{N,K}(\Psi^{\pm})_j^k \leq 0$. By the discrete minimum principle above, we obtain

$$(\Psi^{\pm})_j^k \geq 0, \text{ for } 0 \leq j \leq N$$

and this ends the proof.

4 Convergence analysis

The following analysis concerns the space variable x. We will thus drop the time level indices (for now) for the sake of simplicity. The local truncation error of FOFDM (3.12)-(3.13) is

$$\begin{aligned} L^{N,K}(U_j - z_j) &= (L_x - L^{N,K}) z_j \\ &= \varepsilon z_j'' + \mu a_j z_j' \\ &\quad -\varepsilon \frac{z_{j+1} - 2z_j + z_{j-1}}{\phi_j^2} - \mu a_j \frac{z_{j+1} - z_j}{h} \end{aligned} \tag{4.22}$$

Using Taylor series expansions and taking into account the truncated Taylor expansion

$$\frac{1}{\phi_j^2} = \frac{1}{h^2} - \frac{\mu a_j}{2\varepsilon h} + \frac{\mu^2 a_j^2}{12\varepsilon^2}, \tag{4.23}$$

we obtain

$$\begin{aligned} L^{N,K}(U_j - z_j) &= \left(-\frac{\mu a_j h}{2} + \frac{\mu a_j h^2}{2\varepsilon} - \frac{\mu^2 a_j^2 h^2}{12\varepsilon}\right) z_j'' - \frac{\mu a_j h^2}{6} z_j''' \\ &\quad + \left(\frac{\mu a_j h^4}{48\varepsilon} - \frac{\varepsilon^2}{24} - \frac{\mu^2 a_j^2 h^4}{288\varepsilon}\right) \\ &\quad \times \left(z^{(iv)}(\xi_1) + z^{(iv)}(\xi_2)\right) \\ &\quad - \frac{\mu a_j h^2}{24} z^{(iv)}(\xi_3) \end{aligned} \tag{4.24}$$

where $\xi_i \in (x_j, x_{j+1}),\ i \in \{1,3\}$ and $\xi_2 \in (x_{j-1}, x_j)$. Using bounds on derivatives of z (Lemma 24), the fact that for small h, $h^4 < h^3 < h^2 < h$ and noticing that (see [14] for a proof) both $\mu_0^l \exp(-p\mu_0 x_j)$ and $\mu_1^l \exp(-p\mu_1(1-x_j))$ approach zero as $\varepsilon \to 0$ for all $l \in \{0,1,2,\cdots\}$, we obtain

$$|L^{N,K}(U_j - z_j)| \leq Mh. \tag{4.25}$$

Now, invoking the uniform stability estimate (Lemma 32) yields

$$\max_{0\leq j\leq N} |U_j^k - z_j^k| \leq Mh. \tag{4.26}$$

Note that we have re-instate the dropped time level index. Since by Lemma 22 we have $\max_{0\leq k\leq K} |z_j^k - u_j^k| \leq M\tau$, we obtain the main and final result of this work.

Theorem 41*Let* $u(x,t)$ *be the solution of (1.1)-(1.3) and* $U(x_j, t_k)$ *its numerical approximation obtained via* (3.12)-(3.14). *Then there exists a constant* M *independent of* ε, μ, τ *and* h *such that*

$$\max_{0\leq j\leq N, 0\leq k\leq K} |U_j^k - u_j^k| \leq M(h + \tau). \tag{4.27}$$

This result indicates that the numerical method developed in this work is first order convergent, independently of the parameters ε and μ. In the next section, we test this method and compare the numerical results obtained with the works in [9], [22].

5 Numerical results

The maximum errors at all the mesh points are evaluated using the formula

$$E_{\varepsilon,\mu}^{N,K} = \max_{0\leq j\leq N; 0\leq k\leq K} \left| (U_{\varepsilon,\mu}^{N,K})_{j,k} - (u_{\varepsilon,\mu}^{N,K})_{j,k} \right|$$

if the exact solution $u(x,t)$ is available. However, since the exact solution for Example 51 below is not known, we use a variant of the double mesh principle

$$E_{\varepsilon,\mu}^{N,K} = \max_{0\le j\le N; 0\le k\le K} \left|(U_{\varepsilon,\mu}^{N,K})_{j,k} - (U_{\varepsilon,\mu}^{2N,2K})_{2j,2k}\right|.$$

In the above, $(u_{\varepsilon,\mu}^{N,K})_{j,k}$ and $(U_{\varepsilon,\mu}^{N,K})_{j,k}$ are the exact and approximate solutions obtained using a constant time step τ and space step h. Likewise, $(U_{\varepsilon,\mu}^{2N,2K})_{2j,2k}$ is computed using the constant time step $\tau/2$ and space step $h/2$. Also, we compute the numerical rates of convergence as follows [4]:

$$r_l \equiv r_{\varepsilon,\mu,l} := \log_2(E_{\varepsilon,\mu}^{N_l,K_l}/E_{\varepsilon,\mu}^{2N_l,2K_l}), \quad l = 1,2,\cdots.$$

Example 51*We consider problem*

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \mu(1+x)\frac{\partial u}{\partial x} - u(x) - \frac{\partial u}{\partial t} = 16x^2(1-x)^2,$$
$$(x,t) \in (0,1)\times(0,1]$$
$$u(x,0) = 0,\ x \in \bar{\Omega},$$
$$u(0,t) = 0,\ u(1,t) = 0,\ t \in [0,1].$$

In Fig. 1 we plot the profile of the numerical solution obtained via the proposed FOFDM for $N = 128, K = 64$ and various values of ε and μ. We note the very high gradients near $x = 0$ for $\varepsilon = 2^{-40}$ and $\mu = 1$.

The maximum pointwise errors of our method when implemented on the example given above are provided in tables 1 and 3 for $\mu = 2^{-2}$ and $\mu = 2^{-10}$. We see that, for fixed h and τ, the maximum error is constant as ε approaches zero. This confirms that the method implemented is ε-uniformly convergent. The μ-uniform convergence is shown in Table 5 where for fixed ε, h and τ, the maximum error is constant as μ tends to zero. Tables 2, 4 and 6 give the the orders of local convergence.

A comparison of results in tables 2 and 4 with those in tables 1 and 2 of [22] suggests that our method converges faster than the one presented in that work. Similar comparative data can be drawn for the method in [9] including with respect to accuracy since the maximum error produced by our method is much smaller (see Table 7).

6 Conclusion

We treated a class of singularly perturbed parabolic differential equations with two small parameters affecting the derivatives. A temporal discretization by means of the classical backward Euler method and spatial discretization on a uniform mesh following the nonstandard methodology of Mickens led to a fully discrete problem whose underlying operator safisfied a minimum principle. A convergence analysis based on this fact showed that the proposed method is robust with respect to the perturbation parameters in the sense that the

Table 1: The maximum pointwise errors for $\mu = 2^{-2}$ for various values of ε and $N(=K)$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^0	8.12E-3	4.39E-3	2.32E-3	1.22E-3	6.24E-4	3.16E-4	1.59E-4
2^{-2}	1.72E-2	7.10E-3	3.11E-3	1.44E-3	6.89E-4	3.37E-4	1.67E-4
2^{-4}	3.01E-2	1.16E-2	4.50E-3	1.90E-3	8.66E-4	4.12E-4	2.01E-4
2^{-6}	5.11E-2	1.95E-2	7.29E-3	2.91E-3	1.27E-3	5.89E-4	2.84E-4
2^{-8}	6.26E-2	3.21E-2	1.31E-2	4.91E-3	1.92E-3	8.19E-4	3.75E-4
2^{-10}	6.28E-2	3.43E-2	1.75E-2	8.20E-3	3.28E-3	1.24E-3	4.99E-4
2^{-12}	6.28E-2	3.43E-2	1.76E-2	8.85E-3	4.40E-3	2.04E-3	8.18E-4
2^{-14}	6.28E-2	3.43E-2	1.76E-2	8.85E-3	4.42E-3	2.20E-3	1.10E-3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	6.28E-2	3.43E-2	1.76E-2	8.85E-3	4.42E-3	2.20E-3	1.10E-3

Table 2: The orders of convergence for $\mu = 2^{-2}$ for various values of ε and $N(=K)$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^0	0.88	0.92	0.93	0.96	0.98	0.99	0.99
2^{-2}	1.28	1.19	1.11	1.06	1.03	1.02	1.01
2^{-4}	1.38	1.36	1.24	1.14	1.07	1.04	1.02
2^{-6}	1.39	1.42	1.33	1.20	1.10	1.05	1.03
2^{-8}	0.97	1.29	1.42	1.35	1.23	1.13	1.06
2^{-10}	0.87	0.97	1.09	1.32	1.40	1.31	1.19
2^{-12}	0.87	0.97	0.99	1.01	1.11	1.32	1.40
2^{-14}	0.87	0.97	0.99	1.00	1.00	1.01	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	0.87	0.97	0.99	1.00	1.00	1.00	1.00

Table 3: The maximum pointwise errors for $\mu = 2^{-10}$ for various values of ε and $N(=K)$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^0	8.13E-3	4.41E-3	2.32E-3	1.21E-3	6.22E-4	3.16E-4	1.59E-4
2^{-2}	1.77E-2	7.18E-3	3.09E-3	1.41E-3	6.75E-4	3.29E-4	1.63E-4
2^{-4}	3.34E-2	1.17E-2	4.24E-3	1.72E-3	7.58E-4	3.55E-4	1.71E-4
2^{-6}	4.63E-2	1.63E-2	5.65E-3	2.11E-3	8.78E-4	3.96E-4	1.87E-4
2^{-8}	5.24E-2	1.84E-2	6.24E-3	2.29E-3	9.33E-4	4.12E-4	1.93E-4
2^{-10}	5.42E-2	1.89E-2	6.38E-3	2.33E-3	9.43E-4	4.15E-4	1.94E-4
2^{-12}	5.46E-2	1.90E-2	6.41E-3	2.34E-3	9.45E-4	4.16E-4	1.94E-4
2^{-14}	5.48E-2	1.91E-2	6.43E-3	2.34E-3	9.46E-4	4.16E-4	1.94E-4
2^{-16}	5.49E-2	1.91E-2	6.45E-3	2.35E-3	9.48E-4	4.17E-4	1.94E-4
2^{-18}	5.49E-2	1.92E-2	6.47E-3	2.36E-3	9.53E-4	4.18E-4	1.94E-4
2^{-20}	5.49E-2	1.92E-2	6.47E-3	2.36E-3	9.58E-4	4.22E-4	1.96E-4
2^{-22}	5.49E-2	1.92E-2	6.47E-3	2.36E-3	9.58E-4	4.22E-4	1.97E-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	5.49E-2	1.92E-2	6.47E-3	2.36E-3	9.58E-4	4.22E-4	1.97E-4

Table 4: The orders of convergence for $\mu = 2^{-10}$ for various values of ε and $N(=K)$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^0	0.88	0.93	0.93	0.96	0.98	0.99	0.99
2^{-2}	1.30	1.22	1.13	1.07	1.04	1.02	1.01
2^{-4}	1.51	1.46	1.31	1.18	1.10	1.05	1.03
2^{-6}	1.51	1.53	1.42	1.27	1.15	1.08	1.04
2^{-8}	1.51	1.56	1.44	1.30	1.18	1.10	1.05
2^{-10}	1.52	1.57	1.45	1.30	1.18	1.10	1.05
2^{-12}	1.52	1.57	1.46	1.31	1.18	1.10	1.05
2^{-14}	1.52	1.57	1.46	1.31	1.18	1.10	1.05
2^{-16}	1.52	1.57	1.46	1.31	1.19	1.10	1.05
2^{-18}	1.52	1.57	1.45	1.31	1.19	1.11	1.06
2^{-20}	1.52	1.57	1.45	1.30	1.18	1.11	1.06
2^{-22}	1.52	1.57	1.45	1.30	1.18	1.10	1.06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	1.52	1.57	1.45	1.30	1.18	1.10	1.06

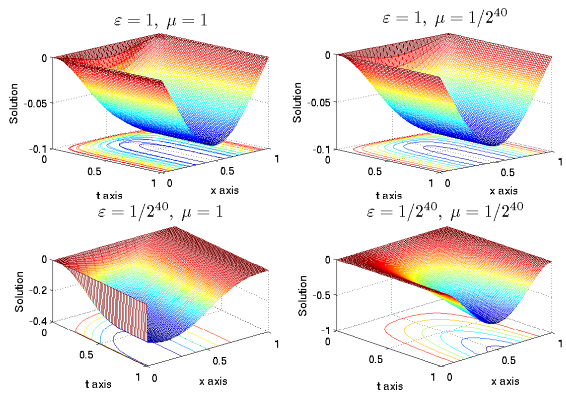

Fig. 1: Solution profile of Example 51 via the proposed scheme for $N=128, K=64$ and various values of ε and μ.

Table 5: The maximum pointwise errors $\varepsilon=2^{-5}$ for various values of μ and $N(=2K)$

μ	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{0}	1.15E-2	6.33E-3	3.38E-3	1.76E-3	9.02E-4	4.57E-4
2^{-2}	8.63E-3	3.95E-3	1.88E-3	9.20E-4	4.54E-4	2.26E-4
2^{-4}	7.52E-3	3.29E-3	1.53E-3	7.35E-4	3.61E-4	1.78E-4
2^{-6}	7.44E-3	3.23E-3	1.49E-3	7.18E-4	3.51E-4	1.74E-4
2^{-8}	7.43E-3	3.23E-3	1.49E-3	7.16E-4	3.51E-4	1.73E-4
2^{-10}	7.43E-3	3.22E-3	1.49E-3	7.16E-4	3.50E-4	1.73E-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	7.43E-3	3.22E-3	1.49E-3	7.16E-4	3.50E-4	1.73E-4

Table 6: The orders of convergence for $\varepsilon=2^{-5}$ for various values of μ and $N(=2K)$

μ	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{0}	0.86	0.90	0.94	0.97	0.98	0.99
2^{-2}	1.13	1.07	1.03	1.02	1.01	1.00
2^{-4}	1.19	1.11	1.06	1.03	1.01	1.01
2^{-6}	1.20	1.11	1.06	1.03	1.02	1.01
2^{-8}	1.20	1.11	1.06	1.03	1.02	1.01
2^{-10}	1.20	1.11	1.06	1.03	1.02	1.01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	1.20	1.11	1.06	1.03	1.02	1.01

method converges uniformly independently of the parameters. We have therefore challenged the current understanding that parameter-uniform numerical methods cannot be designed on a uniform mesh for problems whose solution has parabolic layers [28]. We tested the proposed method on a numerical example to attest the parameter-uniform convergence. We also compared our method with some existing methods in the literature.

Table 7: Comparison of numerical results via the proposed scheme and the scheme in [9]

	Scheme	$N=128$	$N=256$	$N=512$	$N=1024$
Maximum errors	Proposed scheme	1.49E-03	7.16E-04	3.50E-04	1.73E-04
	Scheme in [9]	3.09E-03	1.86E-03	1.02E-03	5.37E-03
Orders of Convergence	Proposed scheme	1.06	1.03	1.02	
	Scheme in [9]	0.73	0.86	0.93	

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References

[1] J. Chen, R.E. O'Malley, Jr., On the asymptotic solution of a two-parameter boundary value problem of chemical reactor theory, *SIAM J. Appl. Math.* **26(4)** (1974) 717-729.
[2] C. Clavero, J.C. Jorge, F. Lisbona, G.I. Shishkin, An alternating direction scheme on a nonuniform mesh for reaction-diffusion parabolic problem, *IMA J. Numer. Anal.* **20** (2000) 263-280.
[3] R.C. DiPrima, Asumptotic methods for an infinitely long slider squeeze-film bearing, *Trans. ASME Ser. F. J. Lub. Tech.* **90** (1968) 173-183.
[4] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.
[5] M. van Dyke, Nineteenth-century roots of the boundary-layer idea, *SIAM Review* **36(3)** (1994), 415-424.
[6] P.W. Hemker, G.I. Shishkin, L.P. Shishkina, ε-uniform schemes with high-order time accuracy for parabolic singular perturbation problems, *IMA J. Numer. Anal.* **(20)** (2000) 99-121.
[7] J.L. Gracia, E. O'Riordan, M. L. Pickett, A parameter robust second order numerical method for a singularly perturbed two-parameter problem, *Appl. Numer. Math.* **56** (2006) 962-980.
[8] M.K. Kadalbajoo, V. Gupta, A brief survey on numerical methods for solving singularly perturbed problems, *Appl. Math. Comput.* **217** (2010) 3641-3716.
[9] M.K. Kadalbajoo, A. S. Yadaw, Parameter-uniform finite element method for two-parameter singularly perturbed parabolic reaction-diffusion problems, *Int. J. Comput. Methods* **9(4)** (2012) 1250047-1 - 1250047-16.
[10] T. Linß, H. -G. Roos, Analysis of a finite-difference scheme for a singularly perturbed problem with two small parameters, *J. Math. Anal. Appl.* **289** (2004) 355-366.
[11] T. Linß, N. Madden, Analysis of an alternating direction method applied to singularly perturbed reaction-diffusion problems, *Int. J. Numer. Anal. Model.* **7(3)** (2010) 507-519.
[12] R.E. Mickens, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.
[13] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

[14] J.B. Munyakazi, K.C. Patidar, Limitations of Richardson's extrapolation for a high order fitted mesh method for self-adjoint singularly perturbed problems, *J. Appl. Math. Comput.* **32** (2010) 219-236.
[15] J.B. Munyakazi, K.C. Patidar, Higher order numericalc methods for singularly perturbed elliptic problems, *Neural Parallel Sci. Comput.* **18** (2010) 75-88.
[16] J.B. Munyakazi, K.C. Patidar, Novel fitted operator finited difference methods for singularly perturbed elliptic convection-diffusion problems in two dimensions, *J. Differ. Equ. Appl.*,**18(5)** (2012) 799-813.
[17] J.B. Munyakazi, K.C. Patidar, A fitted numerical method for singularly perturbed parabolic reaction-diffusion problems, *Comp. Appl. Math.* **32** (2013) 509-519.
[18] J.B. Munyakazi, A uniformly convegent fitted numerical method for parabolic singular perturbation problems of convection-diffusion type, submitted.
[19] R.E. O'Malley, Jr, Two-parameter singular perturbation problems, doctoral thesis, Stanford University, 1965.
[20] R.E. O'Malley, Jr, Two-parameter singular perturbation problems for second-order equations, *J. Math. Mech.* **16(10)** (1967) 1143-1164.
[21] E. O'Riordan, M.L. Pickett, G.I. Shishkin, Singularly perturbed problems modeling reaction-convection-diffusion processes, *Comput. Methods Appl. Math.* **3(3)** (2003) 424-442.
[22] E. O'Riordan, M.L. Pickett, G.I. Shishkin, Parameter-uniform finite difference schemes for a singularly perturbed parabolic diffusion-convection-reaction problems, *Math. Comp.* **75(255)** (2006) 1135-1154.
[23] K.C. Patidar, A robust fitted operator finite difference method for a two-parameter singular perturbation problem, *J. Differ. Equ. Appl.* **14(12)** (2008) 1197-1214.
[24] L. Prandtl, Über Flussigkeitsbewegungen bei sehr kleiner Reinbung, In: *Verhandlungen des III. Internationalen Mathematiker Kongresses*, pp. 484-491, Heidelberg, Germany, 1904.
[25] H.-G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin, 1996.
[26] H. -G. Roos, Z. Uzelac, The SDFEM for a convection-diffusion problem with two small parameters, *Comput. Methods Appl. Math.*, **3(3)** (2003) 443-458.
[27] H. Schlichting, *Boundary layer theory*, 7th edition, McGraw-Hill, New York, 1979.
[28] G.I. Shishkin, A difference scheme for a singularly perturbed equation of parabolic type with discontinuous initial condition, *Soviet Math. Dokl.* **37** (1988) 792-796.
[29] A.B. Vasileva, Asymptotic methods in the theory of ordinary differential equations containing small parameters in front of the highest derivatives, *USSR Comput. Math. Phys.* **3** (1963) 823-836.

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