

Solitons and Other Solutions to Complex-Valued Klein-Gordon Equation in Φ -4 Field Theory

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Abstract: The integrability aspects of complex-valued Klein-Gordon equation is studied in this paper. Three integration algorithms applied. Several solutions are obtained using such integration machineries. Solutions range from plane waves to shock waves and solitons as well as singular periodic solutions.

Keywords: Nonlinear evolution equations, Integrability, Solitons

1 Introduction

The study of nonlinear evolution equations (NLEEs) is growing at an alarming rate [1]-[29]. There are several aspects of these equations are constantly being studied. These are integrability issues, perturbation theory, asymptotic analysis and other aspects. There are several tools of integration of these equations are reported [26]-[29] The results of these integration techniques dwarfed the, once upon a time, monopoly of inverse scattering transform (IST). Several NLEEs, that are proved to be not integrable using IST, due to failure of Painleve test of integrability, can produce a plethora of solutions by these modern algorithms.

This paper will focus on one such NLEE that appears in the study of Φ -4 field theory. This is the complex-valued Klein-Gordon equation (cKGE). This equation will be studied in this paper with two forms of nonlinearity. They are the cubic law and the power law. There are three integration tools that will be applied to obtain several forms of solutions, such as soliton solutions, shock waves, plane waves, singular periodic solutions and others. These three integration tools are the extended tanh-function

method, functional variable scheme and first integral approach. The ansatz method was applied earlier to cKGE to obtain soliton solutions and these solutions computed conservation laws for this equation [7].

The dimensionless form of cKGE is given by [7]

$$q_{tt} - k^2 q_{xx} = aq + bF(|q|^2)q. \quad (1)$$

In Eq. (1), dependent variable $q(x,t)$ is a complex-valued function. Here a , b and k are all real-valued constants. Therefore the left hand side of (1) gives the wave operator. Eq. (1) is studied in the context of Φ -4 field theory that appears in particles and fields. Moreover, complex-valued KGE is a special case of Higgs equation that appears in the context of interaction of scalar nucleons and mesons in particle physics. So, $q(x,t)$ represents the complex scalar nucleon field.

The study of equation (1) will be divided into next three different and independent sections. Each section will focus on the integrability of the equation with the two forms of nonlinearity. The results will then be displayed into two subsections based on the type of nonlinearity in study.

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2 The extended tanh-function method

The extended tanh function method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding topological and non-topological 1-soliton solutions of NLEEs. Recently, this useful method was developed successfully by many authors [11, 15, 16, 17, 18, 23].

2.1 Introduction to the scheme

Consider the NLEE in the form

$$P_1(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0. \quad (2)$$

Using the wave variable $\xi = x - vt$ carries Eq. (2) into the following ordinary differential equation (ODE):

$$P_2(U, U', U'', \dots) = 0, \quad (3)$$

where prime denotes the derivative with respect to the same variable ξ .

Then, the solution of Eq. (3) we are looking for is expressed in the form of a finite series of tanh functions

$$U = \sum_{l=0}^N a_l (G(\xi))^l, \quad (4)$$

where $G^l = \tanh^l(\xi)$, N is a positive integer that can be determined by balancing the highest order derivative with the highest nonlinear terms in equation.

The parameters v, a_0, \dots, a_N are to be determined. The crucial step of the method is to take full advantage of a Riccati equation that the tanh function satisfies and use its solutions G . The required Riccati equation is written as

$$G'(\xi) = B + G^2(\xi). \quad (5)$$

where $G' = dG/d\xi$ and B is a constant. The Riccati equation has the general solutions

If $B < 0$

$$\begin{aligned} G(\xi) &= -\sqrt{-B} \tanh(\sqrt{-B}\xi), \\ G(\xi) &= -\sqrt{-B} \coth(\sqrt{-B}\xi). \end{aligned} \quad (6)$$

If $B = 0$

$$G(\xi) = -\frac{1}{\xi}. \quad (7)$$

If $B > 0$

$$\begin{aligned} G(\xi) &= \sqrt{B} \tan(\sqrt{B}\xi), \\ G(\xi) &= -\sqrt{B} \cot(\sqrt{B}\xi). \end{aligned} \quad (8)$$

Substituting Eq. (4) into Eq. (3) by using Eq. (5) yields a set of algebraic equations for G^l , and all coefficients of G^l have to vanish. After this separated algebraic equations, we can find coefficients v, B, a_0, \dots, a_N .

2.2 Application to cKGE

This extended tanh-function approach will be applied to cKGE. The study will be split into two subsections for cubic and power law nonlinearity. The results will be exposed in details in the following two subsections.

2.2.1 Cubic nonlinearity

For cubic nonlinearity, $F(s) = s$. For cubic nonlinearity, the considered complex-valued Klein-Gordon equation with cubic nonlinearity is given by

$$q_{tt} - k^2 q_{xx} = aq + b|q|^2 q. \quad (9)$$

In order to solve Eq. (9), we use the following wave transformation

$$q(x, t) = U(\xi) e^{i\Phi(x, t)} \quad (10)$$

where $U(\xi)$ represents the shape of the pulse and

$$\xi = x - vt, \quad (11)$$

$$\Phi(x, t) = -\kappa x + \omega t + \theta. \quad (12)$$

In Eq. (10), the function $\Phi(x, t)$ is the phase component of the soliton. Then, in Eq. (12), κ is the soliton frequency, while ω is the wave number of the soliton and θ is the phase constant. Finally in Eq. (11), v is the velocity of the soliton. By replacing Eq. (10) into Eq. (9) and separating the real and imaginary parts of the result, we have

$$v = \frac{\kappa k^2}{\omega}, \quad (13)$$

and

$$(v^2 - k^2)U'' - (a + \omega^2 - \kappa^2 k^2)U - bU^3 = 0. \quad (14)$$

Balancing U'' with U^3 in Eq. (14) gives $N = 1$. Therefore, we may choose

$$U(\xi) = a_0 + a_1 G(\xi), \quad (15)$$

where $G = G(\xi)$ satisfies the Riccati equation

$$G'(\xi) = B + G^2(\xi). \quad (16)$$

Substituting Eq. (15) along with Eq. (16) in Eq. (14) and equating all the coefficients of powers of $G(\xi)$ to be zero, we obtain

$$G^3 : 2(v^2 - k^2)a_1 - ba_1^3 = 0, \quad (17)$$

$$G^2 : 3ba_0a_1^2 = 0, \quad (18)$$

$$\begin{aligned} G^1 : 3ba_0^2a_1 + (a + \omega^2 - \kappa^2 k^2)a_1 \\ - 2(v^2 - k^2)a_1B = 0, \end{aligned} \quad (19)$$

$$G^0 : ba_0^3 + (a + \omega^2 - \kappa^2 k^2)a_0 = 0. \quad (20)$$

With the aid of Maple, we shall find the special solution of the above system

$$\begin{aligned}
 a_0 &= 0 \\
 a_1 &= \pm \sqrt{\frac{2(v^2 - k^2)}{b}}, \\
 B &= \frac{a + \omega^2 - \kappa^2 k^2}{2(v^2 - k^2)}, \tag{21}
 \end{aligned}$$

where v, ω, κ and θ are arbitrary constants. Therefore, using solutions (6)-(8) of Eq. (5), ansatz (15), we obtain the following exact solutions of the cKGE with cubic nonlinearity:

Type-1: When $(v^2 - k^2)(a + \omega^2 - \kappa^2 k^2) < 0$, we have

(1) Topological 1-soliton solution:

$$\begin{aligned}
 q(x,t) &= \pm \sqrt{\frac{\kappa^2 k^2 - a - \omega^2}{b}} \\
 &\times \tanh \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(k^2 - v^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \tag{22}
 \end{aligned}$$

(2) Singular 1-soliton solution:

$$\begin{aligned}
 q(x,t) &= \pm \sqrt{\frac{\kappa^2 k^2 - a - \omega^2}{b}} \\
 &\times \coth \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(k^2 - v^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \tag{23}
 \end{aligned}$$

where v is given by (13).

Type-2: When $(v^2 - k^2)(a + \omega^2 - \kappa^2 k^2) > 0$, we obtain

(3,4) Singular periodic solutions:

$$\begin{aligned}
 q(x,t) &= \pm \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{b}} \\
 &\times \tan \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(v^2 - k^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \tag{24}
 \end{aligned}$$

and

$$\begin{aligned}
 q(x,t) &= \pm \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{b}} \\
 &\times \cot \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(v^2 - k^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \tag{25}
 \end{aligned}$$

where v is given by (13).

The following figure shows the profiles of a topological soliton for the chosen parameters.

Figure1:|q(x,t)| with n=1, a=1.5, b=0.5, k=1, K=1, w=0.5, v=0.5

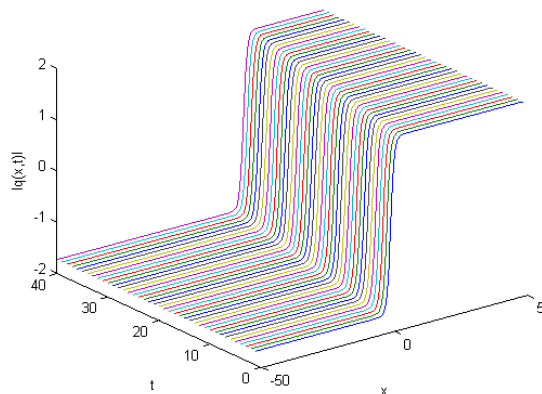


Fig. 1: The profiles of a topological soliton for the chosen parameters.

2.2.2 Power law nonlinearity

Power law nonlinearity arises when $F(s) = s^n$, where the parameter n is referred to as the nonlinearity parameter. For power law nonlinearity, the cKGE takes the form

$$q_{tt} - k^2 q_{xx} = aq + b|q|^{2n} q. \tag{26}$$

For searching the one-soliton solution for the above model, we use the same wave transformation

$$q(x,t) = U(\xi) e^{i\Phi(x,t)} \tag{27}$$

where $U(\xi)$ represents the shape of the pulse and

$$\xi = x - vt, \tag{28}$$

$$\Phi(x,t) = -\kappa x + \omega t + \theta. \tag{29}$$

By replacing Eq. (27) into Eq. (26) and separating the real and imaginary parts of the result, we have

$$v = \frac{\kappa k^2}{\omega}, \tag{30}$$

and

$$(v^2 - k^2)U'' - (a + \omega^2 - \kappa^2 k^2)U - bU^{2n+1} = 0. \tag{31}$$

Balancing U'' with U^{2n+1} in Eq. (31) gives $N = 1/n$.

To obtain an analytic solution, we use the transformation $U = V^{\frac{1}{2n}}$ in Eq. (31) one finds

$$\begin{aligned}
 (v^2 - k^2) \{ (1 - 2n)(V')^2 + 2nVV'' \} \\
 - 4(a + \omega^2 - \kappa^2 k^2)n^2V^2 - 4bn^2V^3 = 0. \tag{32}
 \end{aligned}$$

Balancing the order of VV'' and V^3 in Eq. (32), we have $N = 2$. Therefore, one chooses

$$V(\xi) = a_0 + a_1 G(\xi) + a_2 G^2(\xi), \tag{33}$$

where $G = G(\xi)$ satisfies the Riccati equation

$$G'(\xi) = B + G^2(\xi). \tag{34}$$

Substituting Eq. (33) along with Eq. (34) in Eq. (32) and equating all the coefficients of powers of $G(\xi)$ to be zero, we obtain

$$G^6 : -4bn^2a_2^3 + 4(v^2 - k^2)(1 + n)a_2^2 = 0, \tag{35}$$

$$G^5 : -12bn^2a_1a_2^2 + 4(v^2 - k^2)(1 - 2n)a_1 + 16(v^2 - k^2)na_1a_2 = 0, \tag{36}$$

$$G^4 : (v^2 - k^2)(1 + 2n)a_1^2 - 12bn^2a_1^2a_2 + 8(v^2 - k^2)(1 - 2n)Ba_2^2 + 12(v^2 - k^2)na_0a_2 + 16(v^2 - k^2)nBa_2^2 - 4(a + \omega^2 - \kappa^2k^2)n^2a_2^2 - 12bn^2a_0a_2^2 = 0, \tag{37}$$

$$G^3 : 4(v^2 - k^2)na_0a_1 - 24bn^2a_0a_1a_2 + 20n(v^2 - k^2)a_1a_2B + 8(1 - 2n)(v^2 - k^2)Ba_1a_2 - 8(a + \omega^2 - \kappa^2k^2)n^2a_1a_2 - 4bn^2a_1^3 = 0, \tag{38}$$

$$G^2 : 4(1 - 2n)(v^2 - k^2)B^2a_2^2 + 4nB^2(v^2 - k^2)a_2^2 - 8(a + \omega^2 - \kappa^2k^2)n^2a_0a_2 - 12bn^2a_0^2a_2 + 16(v^2 - k^2)nBa_0a_2 + 2(v^2 - k^2)(1 - 2n)Ba_1^2 + 4nB(v^2 - k^2)a_1^2 - 4(a + \omega^2 - \kappa^2k^2)n^2a_1^2 - 12bn^2a_0a_1^2 = 0, \tag{39}$$

$$G^1 : 4(1 - 2n)(v^2 - k^2)a_1B^2 + 4n(v^2 - k^2)B^2a_1a_2 + 4(v^2 - k^2)na_0a_1B - 12bn^2a_0^2a_1 - 8(a + \omega^2 - \kappa^2k^2)n^2a_0a_1 = 0, \tag{40}$$

$$G^0 : -4bn^2a_0^3 - 4(a + \omega^2 - \kappa^2k^2)n^2a_0^2 + 4(v^2 - k^2)nB^2a_0a_2 + (v^2 - k^2)(1 - 2n)B^2a_1^2 = 0 \tag{41}$$

Solving the system (Eqs. (35)-(41)) simultaneously, we get the solutions set

$$\begin{aligned} a_0 &= \frac{B(v^2 - k^2)(1 + n)}{bn^2} \\ a_1 &= 0, \\ a_2 &= \frac{(v^2 - k^2)(1 + n)}{bn^2}, \\ \omega &= \pm \frac{\sqrt{n^2(\kappa^2k^2 - a) + B(k^2 - v^2)}}{n}, \end{aligned} \tag{42}$$

where v, B, κ and θ are arbitrary constants. Therefore, using solutions (6)-(8) of Eq. (5), ansatz (33), we obtain the following exact solutions of the cKGE with cubic nonlinearity:

(1) Soliton solutions

$$q(x,t) = \left[\frac{B(v^2 - k^2)(1 + n)}{bn^2} \operatorname{sech}^2(\sqrt{-B}(x - vt)) \right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm \left\{ \frac{\sqrt{n^2(\kappa^2k^2 - a) + B(k^2 - v^2)}}{n} \right\} t + \theta\right)}, \tag{43}$$

Figure2: $|q(x,t)|$ with $n=1, B=1, v=0.5, \kappa=1, b=-1$

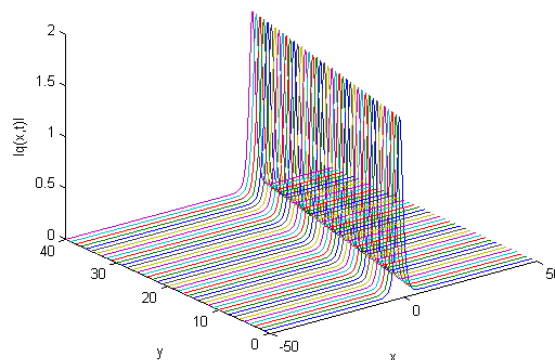


Fig. 2: The profile of a solitary wave for $a =, b =, k = 1$ and $n = 1$.

$$q(x,t) = \left[\frac{B(k^2 - v^2)(1 + n)}{bn^2} \operatorname{csch}^2(\sqrt{-B}(x - vt)) \right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm \left\{ \frac{\sqrt{n^2(\kappa^2k^2 - a) + B(k^2 - v^2)}}{n} \right\} t + \theta\right)}, \tag{44}$$

where v is given by (30).

(2) Periodic singular solutions

$$q(x,t) = \left[\frac{B(v^2 - k^2)(1 + n)}{bn^2} \sec^2(\sqrt{B}(x - vt)) \right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm \left\{ \frac{\sqrt{n^2(\kappa^2k^2 - a) + B(k^2 - v^2)}}{n} \right\} t + \theta\right)}, \tag{45}$$

$$q(x,t) = \left[\frac{B(v^2 - k^2)(1 + n)}{bn^2} \operatorname{csc}^2(\sqrt{B}(x - vt)) \right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm \left\{ \frac{\sqrt{n^2(\kappa^2k^2 - a) + B(k^2 - v^2)}}{n} \right\} t + \theta\right)}, \tag{46}$$

where v is given by (30).

(3) Plane wave solution

$$q(x,t) = \left[\frac{(v^2 - k^2)(1 + n)}{bn^2} \frac{1}{(x - vt)^2} \right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm \sqrt{\kappa^2k^2 - a} t + \theta\right)}, \tag{47}$$

where v is given by (30).

The following figure shows the profile of a solitary wave for $a =, b =, k = 1$ and $n = 1$.

3 Functional variable method

This is the second integration algorithm that will be implemented to integrate cKGE. This method will integrate cKGE with power-law nonlinearity so that results for cubic nonlinearity will fall out as a special case. The first subsection will be a succinct introduction to this algorithm, followed by the application to the equation.

3.1 Preview of the algorithm

The functional variable method, which is a direct and effective algebraic method for the computation of compactons, solitons, solitary patterns and periodic solutions, was first proposed by Zerarka et al [24]. This method was further developed by many authors in [9, 10, 20, 25]. We now summarize the functional variable method, established by Zerarka et al [24], the details of which can be found in [9, 10, 20, 25] among many others. Consider a general NLEE in the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \dots\right) = 0, \quad (48)$$

where P is a polynomial in u and its partial derivatives. Using a wave variable $\xi = x - vt$ so that

$$u(x, t) = U(\xi), \quad (49)$$

Eq. (48) can be converted to an ordinary differential equation (ODE) as

$$Q(U, U', U'', U''', \dots) = 0, \quad (50)$$

where Q is a polynomial in $U = U(\xi)$ and prime denotes derivative with respect to ξ . If all terms contain derivatives, then Eq. (50) is integrated where integration constants are considered zeros.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U) \quad (51)$$

and some successively derivatives of U are

$$U_{\xi\xi} = \frac{1}{2}(F^2)', \quad (52)$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'],$$

where $' = d/dU$.

The ODE (50) can be reduced in terms of U , F and its

derivatives upon using the expressions of Eq. (52) into Eq. (50) gives

$$R(U, F, F', F'', F''', \dots) = 0. \quad (53)$$

The key idea of this particular form Eq. (53) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Eq. (53) provides the expression of F , and this in turn together with Eq. (51) give the relevant solutions to the original problem.

Remark-I: The functional variable method definitely can be applied to nonlinear NLEEs which can be converted to a second order ordinary differential equations (ODE) through the traveling wave transformation.

3.2 Application to cKGE

While functional variable method is not applicable to cKGE with cubic nonlinearity, this algorithm will integrate the NLEE with power law nonlinearity. Subsequently, the results for cubic nonlinearity will fall out as a special case upon setting $n = 1$. These are discussed in the subsections below.

3.2.1 Power law nonlinearity

In this section we study the complex-valued Klein-Gordon equation with power law nonlinearity in the following form:

$$q_{tt} - k^2 q_{xx} = aq + b|q|^{2n}q. \quad (54)$$

We use the transformation

$$q(x, t) = U(\xi)e^{i(-\kappa x + \omega t + \theta)}, \quad \xi = x - vt, \quad (55)$$

where κ , ω , θ and v are constants to be determined later.

By replacing Eq. (55) into Eq. (54) and separating the real and imaginary parts of the result, we have

$$v = \frac{\kappa k^2}{\omega}, \quad (56)$$

and

$$(v^2 - k^2)U'' - (a + \omega^2 - \kappa^2 k^2)U - bU^{2n+1} = 0. \quad (57)$$

Then we use the transformation

$$U_\xi = F(U), \quad (58)$$

that will convert Eq. (57) to

$$\frac{(v^2 - k^2)}{2}(F^2(U))' - (a + \omega^2 - \kappa^2 k^2)U - bU^{2n+1} = 0. \quad (59)$$

Thus, we get from Eq. (59) the expression of the function $F(U)$ reads

$$F(U) = \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} U \times \sqrt{1 + \frac{b}{(n+1)(a + \omega^2 - \kappa^2 k^2)}} U^{2n} \quad (60)$$

After making the change of variables

$$Z = -\frac{b}{(n+1)(a + \omega^2 - \kappa^2 k^2)} U^{2n}, \quad (61)$$

and using the relation (58), the solution of the Eq. (57) is in the following form

$$U(\xi) = \left\{ -\frac{(n+1)(a + \omega^2 - \kappa^2 k^2)}{b} \right. \\ \left. \times \operatorname{sech}^2 \left[n \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} \xi \right] \right\}^{\frac{1}{2n}}. \quad (62)$$

We can easily obtain the following hyperbolic solutions:

$$q(x, t) = \left\{ -\frac{(n+1)(a + \omega^2 - \kappa^2 k^2)}{b} \right. \\ \left. \times \operatorname{sech}^2 \left[n \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] \right\}^{\frac{1}{2n}} \\ \times e^{i(-\kappa x + \omega t + \theta)}, \quad (63)$$

and

$$q(x, t) = \left\{ \frac{(n+1)(a + \omega^2 - \kappa^2 k^2)}{b} \right. \\ \left. \times \operatorname{csch}^2 \left[n \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] \right\}^{\frac{1}{2n}} \\ \times e^{i(-\kappa x + \omega t + \theta)}, \quad (64)$$

where v is given by (56).

Remark-II: The exact solutions (63) and (64) obtained for the complex-valued Klein-Gordon equation with power law nonlinearity is the same as the solution obtained by Biswas et al. [7] who used the solitary wave ansatz method.

It is easy to see that solutions (63) and (64) can reduce to singular periodic solutions as follows:

$$q(x, t) = \left\{ -\frac{(n+1)(a + \omega^2 - \kappa^2 k^2)}{b} \right. \\ \left. \times \sec^2 \left[n \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] \right\}^{\frac{1}{2n}} \\ \times e^{i(-\kappa x + \omega t + \theta)}, \quad (65)$$

and

$$q(x, t) = \left\{ -\frac{(n+1)(a + \omega^2 - \kappa^2 k^2)}{b} \right. \\ \left. \times \csc^2 \left[n \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] \right\}^{\frac{1}{2n}} \\ \times e^{i(-\kappa x + \omega t + \theta)}, \quad (66)$$

where v is given by (56).

3.2.2 Cubic nonlinearity

For this integration algorithm, the results of cKGE with cubic nonlinearity will fall out as a special case of power law nonlinearity upon setting $n = 1$. In Eqs. (63)-(64), If we take $n = 1$, then we obtain the following solitary wave solutions

$$q(x, t) = \sqrt{-\frac{2(a + \omega^2 - \kappa^2 k^2)}{b}} \\ \times \operatorname{sech} \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (67)$$

and

$$q(x, t) = \sqrt{\frac{2(a + \omega^2 - \kappa^2 k^2)}{b}} \\ \times \operatorname{csch} \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}. \quad (68)$$

It is easy to see that solutions (67) and (68) can reduce to singular periodic solutions as follows:

$$q(x, t) = \sqrt{-\frac{2(a + \omega^2 - \kappa^2 k^2)}{b}} \\ \times \sec \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (69)$$

and

$$q(x, t) = \sqrt{\frac{2(a + \omega^2 - \kappa^2 k^2)}{b}} \\ \times \csc \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}. \quad (70)$$

4 First integral approach

One of the most effective direct methods to develop the traveling wave solution of NLEEs is the first integral method [12]. This method has been successfully applied to obtain exact solutions for a variety of NLEEs [2, 3, 4, 19]. Different from other traditional methods, the first

integral method has many advantages, which is mainly embodied in that it could avoid a great deal of complicated and tedious calculation and provide more exact and explicit traveling solitary solutions with high accuracy.

4.1 Overview of the method

Bekir et al. [16] summarized the main steps for using the first integral method, as follows:

Step-I: Suppose a NLEE

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{71}$$

can be converted to an ODE

$$Q(U, -\omega U', kU', \omega^2 U'', -k\omega U'', k^2 U'', \dots) = 0, \tag{72}$$

using a traveling wave variable $u(x, t) = U(\xi)$, $\xi = kx - \omega t$, where the prime denotes the derivation with respect to ξ . If all terms contain derivatives, then Eq. (72) is integrated where integration constants are considered zeros.

Step-II: Suppose that the solution of ODE (72) can be written as follows:

$$u(x, t) = U(\xi) = f(\xi). \tag{73}$$

Step-III: We introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y(\xi) = f'(\xi), \tag{74}$$

which leads a system of

$$X'(\xi) = Y(\xi), \tag{75}$$

$$Y'(\xi) = F(X(\xi), Y(\xi)).$$

Step-IV: By using the Division Theorem for two variables in the complex domain C which is based on the Hilbert-Nullstellensatz Theorem [13], we can obtain one first integral to Eq. (75) which can reduce Eq. (72) to a first-order integrable ordinary differential equation. An exact solution to Eq. (71) is then obtained by solving this equation directly.

Division Theorem: Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$; and $P(w, z)$ is irreducible in $C[w, v]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that

$$Q(w, z) = P(w, z)G(w, z).$$

4.2 Application to cKGE with cubic nonlinearity

In this subsection, we would like to extend the first integral method to solve the complex-valued Klein-Gordon equation with cubic nonlinearity

$$q_{tt} - k^2 q_{xx} = aq + b|q|^2 q. \tag{76}$$

Substituting the traveling wave transformation, we use

$$q(x, t) = U(\xi)e^{i(-\kappa x + \omega t + \theta)}, \quad \xi = x - vt, \tag{77}$$

where κ , ω , θ and v are constants to be determined later.

By replacing Eq. (77) into Eq. (76) and separating the real and imaginary parts of the result, we have

$$v = \frac{\kappa k^2}{\omega}, \tag{78}$$

and

$$(v^2 - k^2)U'' - (a + \omega^2 - \kappa^2 k^2)U - bU^3 = 0. \tag{79}$$

If we let $X(\xi) = U(\xi)$, $Y(\xi) = \frac{dU(\xi)}{d\xi}$, the Eq. (79) is equivalent to the two dimensional autonomous system

$$\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= \left(\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2} \right) X(\xi) + \frac{b}{v^2 - k^2} X^3(\xi). \end{aligned} \tag{80}$$

Now, we apply the above division theorem to look for the first integral of system (80). Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions to system (80), and $Q(X, Y) = \sum_{l=0}^m a_l(X)Y^l$ is an irreducible polynomial in the complex domain C such that

$$Q(X(\xi), Y(\xi)) = \sum_{l=0}^m a_l(X(\xi))Y^l(\xi) = 0, \tag{81}$$

where $a_l(X)$ ($l = 0, 1, \dots, m$) are polynomials of X and $a_m(X) \neq 0$. Eq. (81) is a first integral of system (80). We note that $dQ/d\xi$ is a polynomial in X and Y , and $Q(X(\xi), Y(\xi)) = 0$ implies that $dQ/d\xi|_{(80)} = 0$. According to the division theorem, there exists a polynomial $T(X, Y) = g(X) + h(X)Y$ in the complex domain C such that

$$\begin{aligned} \frac{dQ}{d\xi} &= \frac{dQ}{dX} \frac{dX}{d\xi} + \frac{dQ}{dY} \frac{dY}{d\xi} \\ &= (g(X) + h(X)Y) \sum_{l=0}^m a_l(X)Y^l. \end{aligned} \tag{82}$$

We assume that $m = 1$ in Eq. (81). Taking Eqs. (80) and (82) into account, we get

$$\begin{aligned} \sum_{l=0}^1 a_l'(X)Y^{l+1} + \left(\left(\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2} \right) X + \frac{b}{v^2 - k^2} X^3 \right) \\ \left(\sum_{l=0}^1 l a_l(X)Y^{l-1} \right) \\ = (g(X) + h(X)Y) \sum_{l=0}^1 a_l(X)Y^l, \end{aligned} \tag{83}$$

where the primes denote derivatives with respect to X . Equating the coefficients of Y^l ($l = 2, 1, 0$) in Eq. (83) leads to the system

$$a_1'(X) = h(X)a_1(X), \quad (84)$$

$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X), \quad (85)$$

$$a_1(X) \left\{ \left(\frac{a + \omega^2 - \kappa^2 k^2}{v^2 - k^2} \right) X + \frac{b}{v^2 - k^2} X^3 \right\} = g(X)a_0(X). \quad (86)$$

Since $a_l(X)$ ($l = 0, 1$) are polynomials, then from Eq. (84) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$.

$$a_0(X) = A_0 + B_0X + \frac{A_1}{2}X^2, \quad (87)$$

where A_0 is arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into Eq. (86) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0 = 0, \quad A_1 = \pm \sqrt{\frac{2b}{v^2 - k^2}}, \quad A_0 = \mp \frac{a + \omega^2 - \kappa^2 k^2}{\sqrt{2b(v^2 - k^2)}} \quad (88)$$

where κ , ω and θ are arbitrary constants.

Using the conditions (88) in Eq. (82), we obtain

$$Y(\xi) = \mp \sqrt{\frac{b}{2(v^2 - k^2)}} X^2(\xi) \pm \frac{a + \omega^2 - \kappa^2 k^2}{\sqrt{2b(v^2 - k^2)}}. \quad (89)$$

Combining (89) with (80), we obtain the exact solution to Eq. (79) and then exact solutions for the cKGE equation with cubic nonlinearity can be written as:

Type-1: When $(v^2 - k^2)(a + \omega^2 - \kappa^2 k^2) < 0$, we have

(1) Topological 1-soliton solution:

$$q(x, t) = \pm \sqrt{\frac{\kappa^2 k^2 - a - \omega^2}{b}} \times \tanh \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(k^2 - v^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (90)$$

(2) Singular 1-soliton solution:

$$q(x, t) = \pm \sqrt{\frac{\kappa^2 k^2 - a - \omega^2}{b}} \times \coth \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(k^2 - v^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (91)$$

where v is given by (78).

Type-2: When $(v^2 - k^2)(a + \omega^2 - \kappa^2 k^2) > 0$, we obtain

(3,4) Singular periodic solutions:

$$q(x, t) = \pm \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{b}} \times \tan \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(v^2 - k^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (92)$$

and

$$q(x, t) = \pm \sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{b}} \times \cot \left[\sqrt{\frac{a + \omega^2 - \kappa^2 k^2}{2(v^2 - k^2)}} (x - vt) \right] e^{i(-\kappa x + \omega t + \theta)}, \quad (93)$$

where v is given by (78).

5 Conclusions

This paper addressed the integrability aspects of cKGE. There are three integration tools applied to extract solutions to the equation. Solitons and other solutions are obtained. There are constraint conditions that are obtained for the existence of these solutions. The results of this paper will be of great importance in Φ -4 field theory.

In future, there are additional integration algorithms that will be implemented to obtain several other forms of solutions. The cKGE with perturbation terms will also be studied. The results of those research will give an edge over the current and former results.

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