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# Lie Symmetry Analysis and Conservation Laws for a Fisher Equation with Variable Coefficients

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**Abstract:** In this paper, we study a generalized Fisher equation with variable coefficients which has applications governing the spatiotemporal dynamics of the bacterial population and tumor growth. Conservation laws for this equation are constructed for the first time by using the new conservation theorem due to Ibragimov as well as the Lie symmetries. Furthermore, some conservation laws are derived by employing the direct multipliers method of Anco and Bluman.

**Keywords:** Lie symmetries, Partial differential equation, Conservation Law, Fisher equation

## 1 Introduction

The classic simplest case of a nonlinear reaction-diffusion is the Fisher-Kolmogorov equation. This equation was introduced by Fisher in 1937 to model the advance of an advantageous gene through a geographic region. It is given by:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} + \rho\left(1-\frac{P}{\tilde{P}}\right), \tag{1}$$

where $P(x,t)$ represents the gene frequency at location $x$ and time $t$, $D$ is the diffusion coefficient, $\tilde{P}$ is the carrying capacity, that is, the saturation value beyond which the population cannot grow anymore and $\rho$ is the proliferation. $D$, $\tilde{P}$ and $\rho$ are positives constants. Reaction-diffusion equations such as Fisher equation appear in a variety of problems ranging from population genetics to neurobiology and pattern formation [3]. The Fisher equation and its extensions are a family of reaction-diffusion models arising in population dynamics problems [1,2], most prominently in cancer modelling [4, 5], applications to brain tumor dynamics [6], in the description of propagating crystallization/polymerization fronts [7], chemical kinetics [8], geochemistry [9] and many others fields.

Lie symmetry analysis of differential equations provides a powerful and fundamental framework to the exploitation of systematic procedures leading to the integration by quadrature of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems and to the derivation of conservation laws.

The equation analyzed in this paper is a generalized Fisher equation with variable coefficients

$$u_t = f(u) + \frac{1}{c(x)}\left(c(x)g(u)u_x\right)_x, \tag{2}$$

where $g(u)$ is the diffusion coefficient depending on the variable $u$, being $x$ and $t$ the independent variables, $f(u)$ an arbitrary function and $c(x)$ an arbitrary function depending on the space variable $x$. Let $u(x,t)$ denote the density of tumor cells. In some particular cases this equation has been studied by other authors.
The Kolmogorov-Petrovskii-Piskunov equation [10] given by

$$u_t = u_{xx} + f(u), \tag{3}$$

provides a different generalization to the Fisher equation. This equation is reduced to the well known Huxley equation for $f(u) = u^2(1-u)$ and has been studied by Hodgkin and Huxley and Kolmogorov. Another important equation of this class is for $f(u) = u(1-u^2)$, the Fitzhugh-Nagumo equation which arises in the study of nerve cells [11,12]. Over the last two decades a lot of attention has been paid on using Lie point symmetry methods to exploit the invariance of the generalized equation

$$u_t = (A(u)u_x)_x + B(u)u_x + C(u). \tag{4}$$

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A complete Lie symmetry classification for the non-linear heat equation (4) with $B = C = 0$ was described by [13]. Equation (4) with $B(u) = 0$ becomes the so called density dependent equation

$$u_t = f(u) + (g(u)u_x)_x, \tag{5}$$

which has been considered in [1]. The integrability properties of a generalized Fisher equation as well as explicit and numerical solutions for specific cases of nonintegrable systems were derived in [14]. In [15] the authors have considered equation (2) with $c(x) = x$, however they stated that a classification of (2) can only be achieved when $g$ is linear in $u$. In [16] Eq. (2) with $c(x) = x$ has been studied from the point of view of the theory of symmetry reductions in partial differential equations and a group classification was obtained. All the reductions were derived from the optimal system of subalgebras. Some of the reduced equations admit Lie symmetries which yield to further reductions.

The idea of a conservation law has its origin in mechanics and physics. Since a large number of physical theories, including some of the 'laws of nature', are usually expressed as systems of nonlinear differential equations, it follows that conservation laws are useful in both general theory and the analysis of concrete systems. The classical Noether's theorem provides an elegant and constructive way to obtain conservation laws of PDEs which admit a variational principle. However, a limitation of Noether's theorem is that it restricts to variational systems, or says it depends on existence of Lagrangian, where the majority of the PDEs arising in applications do not hold this property. In [17] Anco and Bluman gave a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy-Kovaleskaya form, in particular for evolution equations

$$u_t = G(x, u, u_x, u_{xx}, \dots, u_{nx}).$$

In [18] a general theorem on conservation laws for arbitrary differential equations which do not require the existence of Lagrangians was proved. Eq. (2) cannot be derived from a variational principle and Noether's theorem cannot be applied.

Due to the great interest in getting conservation laws and conserved quantities in recent papers [19,20,21,22,23,24,25,26] applying the concept of nonlinear self-adjointness and a theorem related to conservation laws due to Ibragimov [27], conservation laws have been derived for some generalized Fisher equations.

The aim of this paper is to derive nontrivial conservation laws for Eq. (2) by using two different approaches. First, we determine for equation (2) the subclasses of equations which are nonlinearly self-adjoint and we construct, by using the Lie generators admitted by (2) and the theorem in conservation laws [18], some non-trivial conservation laws. Then, taking into account that the multipliers of Anco and Bluman method without containing derivatives are just the substitutions of nonlinear self-adjointness we also derive some conservation laws of Eq. (2) by using the multipliers direct method [17].

## 2 Nonlinearly self-adjoint equations

We will prove now by explicit constructions that equation (2) is nonlinearly self-adjoint in the sense of [27].
Equation (2) is written in the form

$$F \equiv u_t - f(u) - \frac{1}{c(x)}\left[c(x)g(u)u_x\right]_x = 0. \tag{6}$$

The adjoint equation to equation (6) has the form

$$F^* \equiv \frac{\delta(vF)}{\delta u} = 0, \tag{7}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t(\frac{\partial}{\partial u_t}) - D_x(\frac{\partial}{\partial u_x}) + D_x^2(\frac{\partial}{\partial u_{xx}}) - \cdots, \tag{8}$$

denotes the variational derivatives (the Euler-Lagrange operator) and $v$ is a new dependent variable. Here $D_t$, $D_x$ are the total differentiations.

**Definition 1.** *Equation (6) is said to be* nonlinearly self-adjoint *if the equation obtained from its adjoint equation by the substitution $v = h(x,t,u,u_x,\dots)$, with a certain function $h(x,t,u,u_x,\dots)$ such that $h(x,t,u,u_x,\dots) \neq 0$ is identical to the original equation (6), i.e:*

$$F^*_{|v=h} = \lambda F.$$

*In the particular case, in which $v = h(x,t,u)$ with $h_u \neq 0$, equation (6) is called weak self-adjoint [28].*
*If $v = h(u)$ with $h_u \neq 0$, equation (6) is called quasi self-adjoint [29].*

### *2.1 The subclass of nonlinearly self-adjoint equations*

Let us single out nonlinearly self-adjoint equations from the equations of the form (2). Eq. (7) yields

$$\begin{aligned} F^* &\equiv \frac{\delta}{\delta u}[v(u_t - f(u) - \frac{1}{c(x)}(c(x)g(u)u_x)_x)] \\ &= -g\,v_{xx} + \frac{c_x\,g\,v_x}{c} - v_t + \frac{c_{xx}\,g\,v}{c} \\ &\quad - \frac{(c_x)^2\,g\,v}{c^2} - f_u\,v. \end{aligned} \tag{9}$$

Setting $v = h(x,t,u)$ in (9), we get

$$\begin{aligned} &-g\,h_u\,u_{xx} - g\,h_{uu}\,(u_x)^2 - 2g\,h_{ux}\,u_x + \frac{c_x\,g\,h_u\,u_x}{c} - h_u\,u_t \\ &-g\,h_{xx} + \frac{c_x\,g\,h_x}{c} - h_t + \frac{c_{xx}\,g\,h}{c} - \frac{(c_x)^2\,g\,h}{c^2} - f_u\,h = 0, \end{aligned}$$

which yields:

$$
\begin{aligned}
&F^* - \lambda (u_t - f(u) - \frac{1}{c(x)}(c(x)g(u)u_x)_x) = \\
&g u_{xx} \lambda + g_u (u_x)^2 \lambda + \frac{c_x g u_x \lambda}{c} - u_t \lambda + f \lambda - g h_u u_{xx} \\
&- g h_{uu} (u_x)^2 - 2 g h_{ux} u_x + \frac{c_x g h_u u_x}{c} - h_u u_t - g h_{xx} \\
&+ \frac{c_x g h_x}{c} - h_t + \frac{c_{xx} g h}{c} - \frac{(c_x)^2 g h}{c^2} - f_u h.
\end{aligned}
$$

From the coefficients for the different derivatives of $u$ we obtain that the following conditions must be satisfied:

$$
\begin{aligned}
-h_u - \lambda &= 0, \\
\lambda g - g h_u &= 0, \\
-2 g h_{ux} + \frac{c_x g h_u}{c} + \frac{\lambda c_x g}{c} &= 0, \\
\lambda g_u - g h_{uu} &= 0, \\
-g h_{xx} + \frac{c_x g h_x}{c} - h_t + \frac{c_{xx} g h}{c} - \frac{(c_x)^2 g h}{c^2} & \\
- f_u h + \lambda f &= 0.
\end{aligned}
\tag{10}
$$

and solving this system, we can state the following result:

**Theorem 1.** *Eq. (2) is neither quasi self-adjoint nor weak self-adjoint, however Eq. (2) is nonlinearly self-adjoint, upon the substitution*

$$h = h(x,t)$$

*for any functions $f = f(u)$, $g = g(u)$ and $c = c(x)$ with $h = h(x,t)$ verifying the following equation*

$$g h_{xx} - \frac{c_x g h_x}{c} + h_t - \frac{c_{xx} g h}{c} + \frac{(c_x)^2 g h}{c^2} + f_u h = 0. \tag{11}$$

For $f$, $g$ and $c$ different from constant we can distinguish the following two cases:

**Case 1-**For $g = g(u)$ arbitrary, $c = c(x)$ arbitrary and $f(u) = au + b$. Then $h = h(x,t)$ must satisfy the following two conditions

$$
\begin{aligned}
&h_{xx} - \left(\frac{c_x}{c} h\right)_x = 0, \\
&h_t + a h = 0.
\end{aligned}
\tag{12}
$$

whose solution is

$$h = c\left(\alpha + \beta \int \frac{1}{c} dx\right) e^{-at},$$

where $\alpha$ and $\beta$ are integration constants.

**Case 2-** For $c = c(x)$ arbitrary, $g = g(u)$ arbitrary and $f = a \int g du$ with $a$ constant. Then $h = h(x,t)$ must satisfy the following conditions

$$
\begin{aligned}
&h_{xx} - \left(\frac{c_x}{c} h\right)_x + ah = 0, \\
&h_t = 0.
\end{aligned}
\tag{13}
$$

We point out that the substitutions $h(x,t)$ derived by using the condition of nonlinear self-adjointness correspond to the multipliers of Anco and Bluman method proposed in [17].

## 3 Lie symmetries

In this section, we perform Lie symmetry analysis for Eq. (2). Let us consider a one-parameter Lie group of infinitesimal transformations in $(x,t,u)$ given by

$$
\begin{aligned}
x^* &= x + \varepsilon \xi(x,t,u) + \mathscr{O}(\varepsilon^2), \\
t^* &= t + \varepsilon \tau(x,t,u) + \mathscr{O}(\varepsilon^2), \\
u^* &= u + \varepsilon \phi(x,t,u) + \mathscr{O}(\varepsilon^2),
\end{aligned}
\tag{14}
$$

where $\varepsilon$ is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of Eq. (2) . This yields to the overdetermined, linear system of eleven equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}. \tag{15}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \phi = 0. \tag{16}$$

After solving the determining equations, we can distinguish nine different cases in which the symmetries are admitted by Eq. (2). The functional forms of $c(x)$, $f(u)$ and $g(u)$ as well as the corresponding generators are given in Table 1.

**Table 1:** Functions $c_i$, $g_i$, $f_i$ with $i = 1 \cdots 9$ and generators $\mathbf{v_k}$ with $k = 1 \cdots 9$.

| $i$ | $c_i$ | $g_i$ | $f_i$ | $\mathbf{v}_k$ |
|---|---|---|---|---|
| 1 | *arbitrary* | *arbitrary* | *arbitrary* | $\mathbf{v}_1$ |
| 2 | $k_1 x^r$ | $k_2 u^q$ | $k_3 u^p$ | $\mathbf{v}_1, \mathbf{v}_2$ |
| 3 | $k_1 x^r$ | $k_2 u^q$ | $k_3 u$ | $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ |
| 4 | $k_1 e^{rx}$ | $k_2 u^q$ | $k_3 u^p$ | $\mathbf{v}_1, \mathbf{v}_4$ |
| 5 | $k_1 e^{rx}$ | $\frac{k_2}{u^2}$ | $k_3 u$ | $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ |
| 6 | $k_1 e^{rx}$ | $k_2$ | $k_3 u^p$ | $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_6$ |
| 7 | $k_1 e^{rx}$ | $k_2 e^{qu}$ | $k_3 e^{pu}$ | $\mathbf{v}_1, \mathbf{v}_4 \, (p \neq q)$ |
| 8 | $k_1 e^{rx}$ | $k_2 e^{qu}$ | $\frac{2k_2 e^{qu} r^2}{9q}$ | $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8$ |
| 9 | $k_1 e^{rx}$ | $k_2$ | $k_3 e^{pu}$ | $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_9$ |

where,

$$\begin{aligned}
&\mathbf{v}_1 = \partial_t,\\
&\mathbf{v}_2 = (q+1-p)x\partial_x + 2(1-p)t\partial_t + 2u\partial_u,\\
&\mathbf{v}_3 = e^{-k_3qt}\partial_t + k_3e^{-k_3qt}u\partial_u,\\
&\mathbf{v}_4 = \partial_x,\\
&\mathbf{v}_5 = \frac{e^{-rx}}{r}\partial_x + e^{-rx}u\partial_u,\\
&\mathbf{v}_6 = \frac{(-x+k_2rt)(p-1)}{2}\partial_x - (p-1)t\partial_t + u\partial_u,\\
&\mathbf{v}_7 = \frac{-3qe^{\frac{-rx}{3}}}{2r}\partial_x + e^{\frac{-rx}{3}}\partial_u,\\
&\mathbf{v}_8 = -qt\partial_t + \partial_u,\\
&\mathbf{v}_9 = \frac{p(-x+k_2rt)}{2}\partial_x - pt\partial_t + \partial_u.
\end{aligned}$$

## 4 Conservation laws

### 4.1 General theorem for Equation (2)

We use the following theorem on conservation laws proved in [18].

**Theorem 2.** *Any Lie point, Lie-Bäcklund or non-local symmetry*

$$\mathbf{v} = \xi^i(x,u,u_{(1)},\dots)\frac{\partial}{\partial x^i} + \eta(x,u,u_{(1)},\dots)\frac{\partial}{\partial u} \qquad (17)$$

*of equation (6) provides a conservation law $D_i(C^i) = 0$ for the system (6), (7). The conserved vector is given by*

$$\begin{aligned}
C^i = {}& \xi^i\mathscr{L} + W\left[\frac{\partial\mathscr{L}}{\partial u_i} - D_j\left(\frac{\partial\mathscr{L}}{\partial u_{ij}}\right) + \cdots\right]\\
&+ D_j(W)\left[\frac{\partial\mathscr{L}}{\partial u_{ij}} - + \cdots\right]
\end{aligned} \qquad (18)$$

*where $W$ and $\mathscr{L}$ are defined as follows:*

$$W = \eta - \xi^j u_j, \quad \mathscr{L} = vF\left(x,u,u_{(1)},\dots,u_{(s)}\right). \qquad (19)$$

We will write generators of point transformation group admitted by Eq. (2) in the form

$$\mathbf{v} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u},$$

by setting $t = x^1$, $x = x^2$. The conservation law will be written

$$D_t(C^1) + D_x(C^2) = 0. \qquad (20)$$

We will obtain the conservation laws for $g(u) \neq constant$ and by using the generators given in table 1.

**1-** Let us apply Theorem 2 to the nonlinearly self-adjoint Eq. (2) with $g(u)$, $c(x)$ arbitrary functions and $f = a\int g\,du$, that is

$$u_t = f(u) + \frac{1}{c(x)}\left(c(x)g(u)u_x\right)_x,$$

where $h = h(x)$ must satisfy (13).
We get the conservation law (20) provided by generator

$$\mathbf{v}_1 = \partial_t,$$

with

$$\begin{aligned}
C^1 = {}& -g_u h_x u u_x + \frac{c_x g_u h u u_x}{c} + \frac{g h u}{k} - f h + D_x(B),\\
C^2 = {}& g g_u h_x u u_{xx} - \frac{c_x g g_u h u u_{xx}}{c} + (g_u)^2 h_x u\,(u_x)^2\\
&- \frac{c_x\,(g_u)^2 h u\,(u_x)^2}{c} + \frac{c_x g g_u h_x u u_x}{c} - \frac{(c_x)^2 g g_u h u u_x}{c^2}\\
&+ f g_u h_x u - \frac{c_x f g_u h u}{c} - D_t(B),
\end{aligned}$$

where

$$B = \left(g h_x k_1 - \frac{c_x g h k_1}{c}\right)u - g h k_1 u_x.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$\begin{aligned}
C^1 = {}& -g_u h_x u u_x + \frac{c_x g_u h u u_x}{c} + \frac{g h u}{k} - f h,\\
C^2 = {}& g g_u h_x u u_{xx} - \frac{c_x g g_u h u u_{xx},}{c} + (g_u)^2 h_x u\,(u_x)^2\\
&- \frac{c_x\,(g_u)^2 h u\,(u_x)^2}{c} + \frac{c_x g g_u h_x u u_x}{c} - \frac{(c_x)^2 g g_u h u u_x}{c^2}\\
&+ f g_u h_x u - \frac{c_x f g_u h u}{c}.
\end{aligned}$$

This conservation law has been published in [20].

**2.** Let us apply Theorem 2 to the nonlinearly self-adjoint equation

$$u_t = k_3u^p + \frac{1}{x^r}\left[x^r k_2 u^{p-1} u_x\right]_x, \qquad (21)$$

where $g = k_2u^{p-1}$, $f = k_3u^p$ and $c = x^r$. In this case $h = h(x)$ must satisfy

$$-\frac{h_x r}{x} + \frac{h r}{x^2} + \frac{h k_3 p}{k_2} + h_{xx} = 0,$$

whose solution in terms of Bessel is:

$$\begin{aligned}
h = {}& c_1 x^{\frac{1}{2}+\frac{1}{2}r}\mathrm{J}\left(\tfrac{1}{2}\sqrt{(r-1)^2},\ \sqrt{\tfrac{k_3 p}{k_2}}x\right)\\
&+ c_2 x^{\frac{1}{2}+\frac{1}{2}r}\mathrm{Y}\left(\tfrac{1}{2}\sqrt{(r-1)^2},\ \sqrt{\tfrac{k_3 p}{k_2}}x\right).
\end{aligned}$$

Let us find the conservation law provided by generator

$$\mathbf{v}_2 = 2(1-p)t\partial_t + 2u\partial u,$$

in this case we have

$$W = 2u - 2(1-p)tu_t$$

and Eqs. (18) yield the conservation law (20) with

$$C^1 = 2hu + D_x(B),$$

$$C^2 = -\frac{2hk_2ru^p}{px} - 2hk_2u^{p-1}u_x + \frac{2h_xk_2u^p}{p} - D_t(B)$$

and

$$B = -\frac{2hk_2rtu^p}{px} + \frac{2hk_2rtu^p}{x} + 2hk_2ptu^{p-1}u_x$$
$$-2hk_2tu^{p-1}u_x + \frac{2h_xk_2tu^p}{p} - 2h_xk_2tu^p.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = 2hu,$$

$$C^2 = -\frac{2hk_2ru^p}{px} - 2hk_2u^{p-1}u_x + \frac{2h_xk_2u^p}{p}.$$

**3.** Let us apply Theorem 2 to the nonlinearly self-adjoint equation

$$u_t = k_3u + \frac{1}{x^r}\left[x^rk_2u^qu_x\right]_x, \tag{22}$$

where $f = k_3u$, $g = k_2u^q$, $c = x^r$ and $h = h(x)$ satisfies

$$h_t + k_3h = 0,$$

$$-\frac{h_xr}{x} + \frac{hr}{x^2} + h_{xx},$$

whose solution is

$$h = (c_1x + c_2x^r)e^{-k_3t}.$$

• For generator

$$\mathbf{v_2} = qx\partial_x + 2u\partial_u,$$

we have

$$W = 2u - qxu_x$$

and Eqs. (18) yield the conservation law (20) with

$$C^1 = h_xqux + hqu + 2hu + D_x(B),$$

$$C^2 = -h_xk_2qu^qu_xx - \frac{2hk_2ru^{q+1}}{x} - hk_2qu^qu_x$$
$$-2hk_2u^qu_x + 2h_xk_2u^{q+1} - D_t(B)$$

and

$$B = -hqux.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = h_xqux + hqu + 2hu,$$

$$C^2 = -h_xk_2qu^qu_xx - \frac{2hk_2ru^{q+1}}{x} - hk_2qu^qu_x$$
$$-2hk_2u^qu_x + 2h_xk_2u^{q+1}.$$

• For generator

$$\mathbf{v_3} = e^{-k_3qt}\partial_t + k_3e^{-k_3qt}u\partial_u,$$

we obtain a trivial conservation law.

**4.** Let us apply Theorem 2 to the nonlinearly self-adjoint equation

$$u_t = k_3u^p + \frac{1}{e^{rx}}\left[e^{rx}k_2u^{p-1}u_x\right]_x, \tag{23}$$

where $f = k_3u^p$, $g = k_2u^{p-1}$ and $c = e^{rx}$. Here $h = h(x)$ must satisfy $h = h(x)$ satisfies (13)

$$-h_xr + \frac{hk_3p}{k_2} + h_{xx} = 0,$$

whose solution is

$$h = \begin{cases} (c_1x + c_2)\,e^{\frac{rx}{2}} & \Delta = 0 \\ c_1e^{\frac{(r+\sqrt{\Delta})x}{2}} + c_2e^{\frac{(r-\sqrt{\Delta})x}{2}} & \Delta > 0 \\ e^{\frac{rx}{2}}\left(c_1\sin\left(\frac{\sqrt{\Delta}x}{2}\right) + c_2\cos\left(\frac{\sqrt{-\Delta}x}{2}\right)\right) & \Delta < 0 \end{cases}$$

with $\Delta = r^2 - \frac{4k_3p}{k_2}$. Let us find the conservation law provided by generator $\mathbf{v}_4 = \partial_x$, we have $W = -u_x$ and Eqs. (18) yield the conservation law (20) with

$$C^1 = h_xu + D_x(B),$$

$$C^2 = -h_xk_2u^{p-1}u_x - hk_3u^p - D_t(B)$$

and

$$B = -hu.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = h_xu,$$

$$C^2 = -h_xk_2u^{p-1}u_x - hk_3u^p.$$

**5.** Let us apply Theorem 2 to the nonlinearly self-adjoint equation

$$u_t = k_3u + \frac{1}{e^{rx}}\left[e^{rx}\frac{k_2}{u^2}u_x\right]_x, \tag{24}$$

where $f = k_3 u$, $g = \frac{k_2}{u^2}$ and $c = e^{rx}$. Here $h(x,t)$ satisfies

$$h_t + k_3 h = 0,$$
$$h_{xx} - r h_x = 0,$$

whose solution is

$$h = (c_1 e^{rx} + c_2)\, e^{-k_3 t}.$$

• For generator $\mathbf{v_4} = \partial_x$, we have $W = -u_x$ and Eqs. (18) yield the conservation law (20) with

$$C^1 = h_x u + D_x(B),$$

$$C^2 = -\frac{h_x k_2 u_x}{u^2} - D_t(B)$$

and

$$B = -h u.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = h_x u,$$

$$C^2 = -\frac{h_x k_2 u_x}{u^2}.$$

• For generator $\mathbf{v_5} = \dfrac{e^{-rx}}{r}\partial_x + e^{-rx} u \partial_u$, we have

$$W = u e^{-rx} - \frac{e^{-rx}}{r} u_x$$

and Eqs. (18) yield the conservation law (20) with

$$C^1 = \frac{h_x u e^{-rx}}{r} + D_x(B),$$

$$C^2 = \frac{h_x k_2 e^{-rx}}{u} - \frac{h_x k_2 u_x e^{-rx}}{r u^2} - D_t(B)$$

and

$$B = -\frac{h u e^{-rx}}{r}.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = \frac{h_x u e^{-rx}}{r},$$

$$C^2 = \frac{h_x k_2 e^{-rx}}{u} - \frac{h_x k_2 u_x e^{-rx}}{r u^2}.$$

**6.** Let us apply Theorem 2 to the nonlinearly self-adjoint equation

$$u_t = k e^{qu} + \frac{1}{e^{rx}} \left[e^{rx} e^{qu} u_x\right]_x, \tag{25}$$

where $f = \frac{2 k_2 r^2 e^{qu}}{9q}$, $g = k_2 e^{qu}$ and $c = e^{rx}$. Here $h(x)$ satisfies

$$2 h r^2 - 9 h_x r + 9 h_{xx} = 0,$$

whose solution is

$$h = c_1 e^{\frac{2rx}{3}} + c_2 e^{\frac{rx}{3}}.$$

• For generator $\mathbf{v_4} = \partial_x$, we have $W = -u_x$ and Eqs. (18) yield the conservation law (20) with

$$C^1 = h_x u + D_x(B),$$

$$C^2 = -h_x k_2 e^{qu} u_x - \frac{2 h k_2 r^2 e^{qu}}{9q} - D_t(B)$$

and

$$B = -h u.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = h_x u,$$

$$C^2 = -h_x k_2 e^{qu} u_x - \frac{2 h k_2 r^2 e^{qu}}{9q}.$$

• For generator $\mathbf{v_7} = \dfrac{-3q e^{\frac{-rx}{3}}}{2r}\partial_x + e^{\frac{-rx}{3}}\partial_u$, we have

$$W = \frac{3 q e^{-\frac{rx}{3}}}{2r} u_x + e^{-\frac{rx}{3}}$$

and Eqs. (18) yield the conservation law (20) with

$$C^1 = -\frac{3 h_x q u e^{-\frac{rx}{3}}}{2r} + \frac{h q u e^{-\frac{rx}{3}}}{2} + h e^{-\frac{rx}{3}} + D_x(B),$$

$$C^2 = \frac{3 h_x k_2 q u_x e^{qu - \frac{rx}{3}}}{2r} - \frac{h k_2 q u_x e^{qu-\frac{rx}{3}}}{2} - \frac{h k_2 r e^{qu-\frac{rx}{3}}}{3} + h_x k_2 e^{qu-\frac{rx}{3}} - D_t(B)$$

and

$$B = \frac{3 h q u e^{-\frac{rx}{3}}}{2r}.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = -\frac{3 h_x q u e^{-\frac{rx}{3}}}{2r} + \frac{h q u e^{-\frac{rx}{3}}}{2} + h e^{-\frac{rx}{3}},$$

$$C^2 = \frac{3 h_x k_2 q u_x e^{qu - \frac{rx}{3}}}{2r} - \frac{h k_2 q u_x e^{qu-\frac{rx}{3}}}{2} - \frac{h k_2 r e^{qu-\frac{rx}{3}}}{3} + h_x k_2 e^{qu-\frac{rx}{3}}.$$

• For generator $\mathbf{v_8} = -qt\partial_t + \partial_u$, we have

$$W = q t u_t + 1$$

and Eqs. (18) yield the conservation law (20) with

$$C^1 = -\frac{2hk_2qr^2te^{qu}u_xx}{9} - \frac{2h_xk_2r^2te^{qu}x}{9} - h_xx + hk_2qrte^{qu}u_x - h_xk_2qte^{qu}u_x + D_x(B),$$

$$\begin{aligned}C^2 &= \frac{2hk_2{}^2qr^2te^{2qu}u_{xx}x}{9} + \frac{2hk_2{}^2q^2r^2te^{2qu}(u_x)^2x}{9} \\ &+ \frac{2hk_2{}^2qr^3te^{2qu}u_xx}{9} + \frac{4hk_2{}^2r^4te^{2qu}x}{81} + \frac{2hk_2r^2e^{qu}x}{9} \\ &- hk_2{}^2qrte^{2qu}u_{xx} + h_xk_2{}^2qte^{2qu}u_{xx} - hk_2{}^2q^2rte^{2qu}(u_x)^2 \\ &+ h_xk_2{}^2q^2te^{2qu}(u_x)^2 - hk_2{}^2qr^2te^{2qu}u_x + h_xk_2{}^2qrte^{2qu}u_x \\ &- \frac{2hk_2{}^2r^3te^{2qu}}{9} + \frac{2h_xk_2{}^2r^2te^{2qu}}{9} - hk_2re^{qu} + h_xk_2e^{qu} \\ &- D_t(B)\end{aligned}$$

and

$$B = \frac{2hk_2r^2te^{qu}x}{9} + hx + hk_2qte^{qu}u_x.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\dots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = -\frac{2hk_2qr^2te^{qu}u_xx}{9} - \frac{2h_xk_2r^2te^{qu}x}{9} - h_xx + hk_2qrte^{qu}u_x - h_xk_2qte^{qu}u_x,$$

$$\begin{aligned}C^2 &= \frac{2hk_2{}^2qr^2te^{2qu}u_{xx}x}{9} + \frac{2hk_2{}^2q^2r^2te^{2qu}(u_x)^2x}{9} \\ &+ \frac{2hk_2{}^2qr^3te^{2qu}u_xx}{9} + \frac{4hk_2{}^2r^4te^{2qu}x}{81} + \frac{2hk_2r^2e^{qu}x}{9} \\ &- hk_2{}^2qrte^{2qu}u_{xx} + h_xk_2{}^2qte^{2qu}u_{xx} - hk_2{}^2q^2rte^{2qu}(u_x)^2 \\ &+ h_xk_2{}^2q^2te^{2qu}(u_x)^2 - hk_2{}^2qr^2te^{2qu}u_x + h_xk_2{}^2qrte^{2qu}u_x \\ &- \frac{2hk_2{}^2r^3te^{2qu}}{9} + \frac{2h_xk_2{}^2r^2te^{2qu}}{9} - hk_2re^{qu} + h_xk_2e^{qu}.\end{aligned}$$

### *4.2 Direct multipliers method for Equation (2)*

In [17] Anco and Bluman gave a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy-Kovaleskaya form, in particular for evolution equations

$$u_t = G(x, u, u_x, u_{xx}, \dots, u_{nx}).$$

The nontrivial conservation laws are characterized by a multiplier $\lambda$ with no dependence on $u_t$ satisfying

$$\hat{E}[u]\left(\Lambda u_t - \Lambda G(x, u, u_x, u_{xx}, \dots, u_{nx})\right) = 0.$$

Here

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + \dots.$$

The conserved current must satisfy

$$\Lambda = \hat{E}[u]\Phi^t$$

and the flux $\Phi^x$ is given by [30]

$$\Phi^x = -D_x^{-1}(\Lambda G) - \frac{\partial \Phi^t}{\partial u_x}G + GD_x\left(\frac{\partial \Phi^t}{\partial u_{xx}}\right) + \dots.$$

The conservation law will be written

$$D_t(\Phi^t) + D_x(\Phi^x) = 0.$$

For Eq. (2) by using the Maple package GeM, we get the following multipliers. Each multiplier determines a corresponding conserved density and flux.

**1.** For $f(u) = u,\ g(u) = u^q,\ c(x) = e^x$.

We get that multiplier $\Lambda$ must satisfy (12) and the solution is

$$\Lambda = k_1e^{x-t} + k_2e^{-t}.$$

We get the the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases}\Lambda = e^{x-t}, & \Lambda = e^{-t}, \\ \phi^t = e^{x-t}u, & \phi^t = e^{-t}u, \\ \phi^x = -e^{x-t}u^qu_x, & \phi^x = -e^{-t}u^qu_x - \frac{e^{-t}u^{q+1}}{q+1}.\end{cases}$$

**2.** For $f(u) = u,\ g(u) = u^q,\ c(x) = x^r$. We get that multiplier $\Lambda$ must satisfy (12) and the solution is

$$\Lambda = k_1x^re^{-t} + k_2xe^{-t}.$$

We get the the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases}\Lambda = x^re^{-t}, & \Lambda = xe^{-t}, \\ \phi^t = x^re^{-t}u, & \phi^t = xe^{-t}u, \\ \phi^x = -e^{-t}u^qu_xx^r, & \phi^x = -e^{-t}u^qu_xx - \frac{(r-1)e^{-t}u^{q+1}}{q+1}.\end{cases}$$

**3.** For $f(u) = au^p,\ g(u) = pu^{(p-1)},\ c(x) = e^{rx}$

We get that $\Lambda$ must satisfy condition (13)

$$\Lambda_{xx} - r\Lambda_x + a\Lambda = 0, \tag{26}$$

the solutions with $\Delta = r^2 - 4a$ are:

$$\Lambda = \begin{cases}k_1xe^{rx/2} + k_2e^{rx/2} & \Delta = 0 \\ k_1\,e^{\frac{(\sqrt{\Delta}+r)x}{2}} + k_2\,e^{\frac{(r-\sqrt{\Delta})x}{2}} & \Delta > 0 \\ e^{\frac{rx}{2}}\left(k_1\sin\left(\frac{\sqrt{-\Delta}x}{2}\right) + k_2\cos\left(\frac{\sqrt{-\Delta}x}{2}\right)\right) & \Delta < 0\end{cases}$$

• For $\Delta = 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = xe^{\frac{rx}{2}}, \\ \phi^t = xe^{\frac{x}{2}}u, \\ \phi^x = -pu^{p-1}u_x xe^{\frac{rx}{2}} - \dfrac{2au^p xe^{\frac{rx}{2}}}{r} + \dfrac{4au^p e^{\frac{rx}{2}}}{r^2} \end{cases}.$$

and

$$\begin{cases} \Lambda = e^{\frac{rx}{2}}, \\ \phi^t = ue^{\frac{rx}{2}}, \\ \phi^x = -pu^{p-1}u_x e^{\frac{rx}{2}} - \dfrac{2au^p e^{\frac{rx}{2}}}{r}. \end{cases}$$

• For $\Delta > 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = e^{\frac{(\sqrt{\Delta}+r)x}{2}}, \\ \phi^t = e^{\frac{(\sqrt{\Delta}+r)x}{2}}u, \\ \phi^x = \dfrac{\left(\sqrt{\Delta}-r\right)u^p e^{\frac{\sqrt{\Delta}x}{2}+\frac{rx}{2}}}{2a} - \dfrac{pu^{p-1}u_x e^{\frac{\sqrt{\Delta}x}{2}+\frac{rx}{2}}}{a} \end{cases}$$

and

$$\begin{cases} \Lambda = e^{\frac{(r-\sqrt{\Delta})x}{2}}, \\ \phi^t = ue^{\frac{(r-\sqrt{\Delta})x}{2}}, \\ \phi^x = -pu^{p-1}u_x e^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}} - \dfrac{\sqrt{\Delta}u^p e^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}}}{2} - \dfrac{ru^p e^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}}}{2}. \end{cases}$$

• For $\Delta < 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = e^{\frac{rx}{2}} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^t = ue^{\frac{rx}{2}} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^x = -\dfrac{ru^p e^{\frac{rx}{2}} \sin\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} + \dfrac{\sqrt{-\Delta}u^p e^{\frac{rx}{2}} \cos\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} \\ \qquad -pu^{p-1}u_x e^{\frac{rx}{2}} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right) \end{cases}$$

and

$$\begin{cases} \Lambda = e^{\frac{rx}{2}} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^t = ue^{\frac{rx}{2}} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^x = -\dfrac{\sqrt{-\Delta}u^p e^{\frac{rx}{2}} \sin\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} - \dfrac{ru^p e^{\frac{rx}{2}} \cos\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} \\ \qquad -pu^{p-1}u_x e^{\frac{rx}{2}} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right). \end{cases}$$

**4.** For $f(u) = ae^{pu}$, $g(u) = pe^{pu}$, $c(x) = e^{rx}$.

We get that $\Lambda$ must satisfy condition (26) with $\Delta = r^2 - 4a$.

•For $\Delta = 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = xe^{\frac{rx}{2}}, \\ \phi^t = xe^{\frac{x}{2}}u, \\ \phi^x = -pu_x xe^{\frac{rx}{2}+pu} - \dfrac{rxe^{\frac{rx}{2}+pu}}{2} + e^{\frac{rx}{2}+pu} \end{cases}$$

and

$$\begin{cases} \Lambda = e^{\frac{rx}{2}}, \\ \phi^t = ue^{\frac{rx}{2}}, \\ \phi^x = -pu_x e^{\frac{rx}{2}+pu} - \dfrac{re^{\frac{rx}{2}+pu}}{2}. \end{cases}$$

• For $\Delta > 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = e^{\frac{(r+\sqrt{\Delta})x}{2}}, \\ \phi^t = e^{\frac{(\sqrt{\Delta}+r)x}{2}}u, \\ \phi^x = -pu_x e^{\frac{rx}{2}+\frac{\sqrt{\Delta}x}{2}+pu} - \dfrac{re^{\frac{rx}{2}+\frac{\sqrt{\Delta}x}{2}+pu}}{2} + \dfrac{\sqrt{\Delta}e^{\frac{rx}{2}+\frac{\sqrt{\Delta}x}{2}+pu}}{2} \end{cases}$$

and

$$\begin{cases} \Lambda = e^{\frac{(r-\sqrt{\Delta})x}{2}}, \\ \phi^t = e^{\frac{(r-\sqrt{\Delta})x}{2}}u, \\ \phi^x = -pu_x e^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}+pu} - \dfrac{re^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}+pu}}{2} - \dfrac{\sqrt{\Delta}e^{\frac{rx}{2}-\frac{\sqrt{\Delta}x}{2}+pu}}{2}. \end{cases}$$

• For $\Delta < 0$ we obtain the following multipliers and the corresponding conserved densities and fluxes:

$$\begin{cases} \Lambda = e^{\frac{rx}{2}} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^t = ue^{\frac{rx}{2}} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^x = -\dfrac{re^{\frac{rx}{2}+pu} \sin\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} + \dfrac{\sqrt{-\Delta}e^{\frac{rx}{2}+pu} \cos\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} \\ \qquad -pu_x e^{\frac{rx}{2}+pu} \sin\left(\dfrac{\sqrt{-\Delta}x}{2}\right) \end{cases}$$

and

$$\begin{cases} \Lambda = e^{\frac{rx}{2}} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^t = ue^{\frac{rx}{2}} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right), \\ \phi^x = -\dfrac{re^{\frac{rx}{2}+pu} \cos\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} - \dfrac{\sqrt{-\Delta}e^{\frac{rx}{2}+pu} \sin\left(\frac{\sqrt{-\Delta}x}{2}\right)}{2} \\ \qquad -pu_x e^{\frac{rx}{2}+pu} \cos\left(\dfrac{\sqrt{-\Delta}x}{2}\right). \end{cases}$$

**5.** For $f(u) = au^p$, $g(u) = pu^p$, $c(x) = x^r$ .

We get that $\Lambda$ must satisfy condition (13) and the solution is

$$\Lambda = k_1 x^{\frac{1}{2}+\frac{1}{2}r} \mathrm{J}\left(\frac{1}{2}\,|r-1|,\,\sqrt{a}x\right) + k_2 x^{\frac{1}{2}+\frac{1}{2}r} \mathrm{Y}\left(\frac{1}{2}\,|r-1|,\,\sqrt{a}x\right).$$

We get the the following multipliers and the corresponding conserved densities and fluxes:

$$\Lambda = \mathrm{J}\left(\tfrac{|r-1|}{2},\sqrt{a}\,x\right) x^{\frac{r}{2}+\frac{1}{2}},$$

$$\phi^t = \mathrm{J}\left(\tfrac{|r-1|}{2},\sqrt{a}\,x\right) x^{\frac{r}{2}+\frac{1}{2}}\,u,$$

$$\begin{aligned}\phi^x =& \frac{1}{2}\sqrt{a}\,u^p x^{\frac{r}{2}+\frac{1}{2}} B_7 - \frac{1}{2}\sqrt{a}\,u^p x^{\frac{r}{2}+\frac{1}{2}} B_6 \\ & -p\,u^{p-1} u_x x^{\frac{r}{2}+\frac{1}{2}} B_4 - \frac{r}{2} u^p x^{\frac{r}{2}-\frac{1}{2}} B_4 + \frac{1}{2} u^p x^{\frac{r}{2}-\frac{1}{2}} B_4 \\ & +\frac{1}{4}\int x^{\frac{r+1}{2}} u^p\, a\,(-B_5 - 2B_4 - B_3)\,dx \\ & +\frac{1}{4}\int x^{\frac{r-3}{2}} u^p B_4\left(r^2 - 2r + 1\right) dx \\ & +\frac{1}{4}\int \sqrt{a}\,u^p x^{\frac{r}{2}-1}\,(2B_1 - 2J_2)\,dx,\end{aligned}$$

with

$$\begin{aligned}&B_1 = J\left(\tfrac{|r-1|+2}{2},\sqrt{a}\,x\right), \quad B_2 = J\left(\tfrac{|r-1|-2}{2},\sqrt{a}\,x\right),\\ &B_3 = J\left(\tfrac{|r-1|+4}{2},\sqrt{a}\,x\right), \quad B_4 = J\left(\tfrac{|r-1|}{2},\sqrt{a}\,x\right),\\ &B_5 = J\left(\tfrac{|r-1|-4}{2},\sqrt{a}\,x\right), \quad B_6 = J\left(\tfrac{|r-1|}{2}+1,\sqrt{a}\,x\right),\\ &B_7 = J\left(\tfrac{|r-1|}{2}-1,\sqrt{a}\,x\right).\end{aligned}$$

$$\Lambda = \mathrm{Y}\left(\tfrac{|r-1|}{2},\,\sqrt{a}x\right) x^{\frac{1}{2}+\frac{1}{2}r},$$

$$\phi^t = \mathrm{Y}\left(\tfrac{|r-1|}{2},\,\sqrt{a}x\right) x^{\frac{1}{2}+\frac{1}{2}r} u,$$

$$\begin{aligned}\phi^x =& \frac{1}{2}\sqrt{a}\,u^p x^{\frac{r}{2}+\frac{1}{2}} B_7 - \frac{1}{2}\sqrt{a}\,u^p x^{\frac{r}{2}+\frac{1}{2}} B_6 \\ & -p\,u^{p-1} u_x x^{\frac{r}{2}+\frac{1}{2}} B_4 - \frac{r}{2} u^p x^{\frac{r}{2}-\frac{1}{2}} B_4 + \frac{1}{2} u^p x^{\frac{r}{2}-\frac{1}{2}} B_4 \\ & +\frac{1}{4}\int x^{\frac{r+1}{2}} u^p\, a\,(-B_5 - 2B_4 - B_3)\,dx \\ & +\frac{1}{4}\int x^{\frac{r-3}{2}} u^p B_4\left(r^2 - 2r + 1\right) dx \\ & +\frac{1}{4}\int \sqrt{a}\,u^p x^{\frac{r}{2}-1}\,(2B_1 - 2B_2)\,dx\end{aligned}$$

with

$$\begin{aligned}&B_1 = Y\left(\tfrac{|r-1|+2}{2},\sqrt{a}\,x\right), \quad B_2 = Y\left(\tfrac{|r-1|-2}{2},\sqrt{a}\,x\right),\\ &B_3 = Y\left(\tfrac{|r-1|+4}{2},\sqrt{a}\,x\right), \quad B_4 = Y\left(\tfrac{|r-1|}{2},\sqrt{a}\,x\right),\\ &B_5 = Y\left(\tfrac{|r-1|-4}{2},\sqrt{a}\,x\right), \quad B_6 = Y\left(\tfrac{|r-1|}{2}+1,\sqrt{a}\,x\right),\\ &B_7 = Y\left(\tfrac{|r-1|}{2}-1,\sqrt{a}\,x\right).\end{aligned}$$

## 5 Conclusions

We have found for the Fisher equation with variable coefficients (2) conservation laws using two different approaches. First he have determined the subclasses of equations which are nonlinearly self-adjoint and in order to derive the conservation laws associated to symmetry generators we have applied the classical Lie method to Eq. (2). By using the property of nonlinear self-adjointness of (2) and the general theorem of conservation laws due to Ibragimov, we have constructed some nontrivial conservation laws for this equation associated with symmetries of the differential equation. It has been proved [31], that the multiplier $\Lambda(x,t,u)$ is identical to the substitution $h(x,t,u)$ of nonlinear self-adjointness, taking this fact into account, we have obtained the corresponding conserved densities and fluxes by using the direct method of the multipliers of Anco and Bluman. The conservation laws by multiplier method are obtained by integral formulae while the conservation laws via nonlinear self-adjointness method are constructed by the formulae in Theorem 2 which avoids the integrals of functions.


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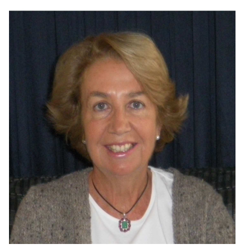

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