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The Homomorphism Maps between Variable Threshold Concept Lattice and AFS Algebras*

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Variable threshold concept lattice (VTCL) was proposed by Ma et al.(2006), which provide a new parameterized way to obtain formal concepts from data with fuzzy attributes. Axiomatic Fuzzy Set (AFS) algebras were proposed by Liu (1998), which are new semantic methodology relating to the fuzzy theory. In this paper, in order to explore the relationship between the AFS algebras and VTCL, three algebra homomorphism maps are established, by which one can find that AFS algebras have similar properties to VTCL.

Keywords: AFS algebra, variable threshold concept lattice, completely distributive lattice, homomorphism map

1 Introduction

Formal concept analysis (FCA) originally proposed by Wille [6], which is an important theory for data analysis and knowledge discovery. In the past decades, FCA has great development in theory, and has become a powerful tool to deal with data. In artificial intelligence, FCA is used as a knowledge representation mechanism and as a conceptual clustering method [5, 7, 16]. In database theory, FCA has extensively been used for design and management of class hierarchies [4, 8, 18, 21, 22]. Concept lattice, or Galois lattice, is the core of the mathematical theory of FCA. Concept lattice is a form of a hierarchy in which each node (formal concept) represents a subset of objects (extent) with their common attributes (intent). The characteristic of concept lattice theory lies in reasoning on the possible attributes of data sets [25]. The classical concept lattices only reflect the accurate relationships between objects and attributes, while the fuzzy concept lattices [1,2,15] show the uncertain relationships between objects and attributes. Since there exists a great of uncertain in real world, it is important and interesting to study the fuzzy concept lattice. While the huge number of fuzzy formal concepts is a drawback, in order to track this problem, Ma and Zhang proposed fuzzy concept lattices with a variable threshold [15]. Compared to

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the classical concept lattice and the fuzzy concept lattice, the fuzzy formal context with a variable threshold will be simpler in terms of the number of formal concepts. The process of generating the variable threshold concept lattice can be viewed as a process of choosing representative concepts from generation of a concept lattice.

AFS algebras were proposed by Liu [10,11], which are new approach relating to the semantic interpretations of fuzzy attribute. An AFS algebra is a family of completely distributive lattice [19]. Recently, AFS algebras have been further developed and applied to fuzzy clustering analysis [14], fuzzy decision trees [12] and concept representations [9, 20, 23], etc. About the detail properties of AFS algebras, please see [9–11, 13].

The main purpose of this paper is to explore the homomorphism relationship between VTCL and AFS algebras. The remain of paper is organized as follows: In Section 2, some basic notions pertinent to this paper are introduced. In Section 3, three algebra homomorphism maps between AFS algebras and VTCL are established to show that AFS algebras have similar properties VTCL. Finally, conclusions are drawn in Section 4.

2 Preliminaries of the AFS algebras and VTCL

In this section, we recall some definitions and present several pertinent results of VTCL and AFS algebras, i.e., EI, EII, $E^{\#}I$ algebra. There exist few different definitions about VTCL [3, 15, 24], we adopt the definition introduced by [15].

2.1 Variable threshold concept lattice (VTCL)

Definition 2.1. ([15]) (Fuzzy Formal Context). A fuzzy formal context is a triple $K = (X, M, I = \phi(X, M))$, where X is a set of objects, M is a set of attributes, and I is a fuzzy set on domain $X \times M$. Each relation $(x, m) \in I$ has a membership value $\mu(x, m)$ in [0,1].

Definition 2.2. ([15]) Let $(X, M, \mathcal{I})$ be a fuzzy formal context and $\delta \in (0, 1]$. A pair (A, B) is referred to as a variable threshold formal concept, for short, a variable threshold concept, of (X, M, I), if and only if $A \subseteq X$, $B \subseteq M$, $A^{'\delta} = B$ and $A = B^{'\delta}$. A is referred to as the extent and B the intent of (A, B). We denote by $\mathcal{B}_\delta(X, M, \mathcal{I})$ the set of all variable threshold concepts of a fuzzy formal context $(X, M, \mathcal{I})$, where
$A^{'\delta} = \{b \in B | (a, b) \in \mathcal{I}_\delta, \text{for all } a \in A\}$,
$B^{'\delta} = \{a \in A | (a, b) \in \mathcal{I}_\delta, \text{for all } b \in B\}$,
$(a, b) \in \mathcal{I}_\delta$ denotes the degree that the object a has the attribute b, or the degree that b is possessed by a no less than δ. i.e. $\mu(a, b) \geq \delta$.

Lemma 2.1. *([15]) For a fuzzy formal context $(X, M, \mathcal{I})$, the following properties hold: for all $A_1, A_2, A \subseteq X$, $B_1, B_2, B \subseteq M$ and $\delta \in (0, 1]$,*

1. $A_1 \subseteq A_2 \Rightarrow A_2^{'\delta} \subseteq A_1^{'\delta}$, $B_1 \subseteq B_2 \Rightarrow B_2^{'\delta} \subseteq B_1^{'\delta}$.
2. $A \subseteq A^{'\delta'\delta}$, $B \subseteq B^{'\delta'\delta}$.
3. $A = A^{'\delta'\delta'\delta}$, $B = B^{'\delta'\delta'\delta}$.
4. $A \subseteq B^{'\delta}$, $B \subseteq A^{'\delta}$.
5. $(A_1 \bigcup A_2)^{'\delta} = A_1^{'\delta} \bigcap A_2^{'\delta}$, $(B_1 \bigcup B_2)^{'\delta} = B_1^{'\delta} \bigcap B_2^{'\delta}$.
6.$(A_1 \bigcap A_2)^{'\delta} \supseteq A_1^{'\delta} \bigcup A_2^{'\delta}$, $(B_1 \bigcap B_2)^{'\delta} \supseteq B_1^{'\delta} \bigcup B_2^{'\delta}$.
7. $(A^{'\delta'\delta}, A^{'\delta})$ *and* $(B^{'\delta}, B^{'\delta'\delta})$ *are variable threshold concepts.*

Table 2.1: Descriptions of features [17]

	outlook			temperature	humidity	windy?	
x_i	rain	sunny	overcast	temp ($°F$)	humid ($\%$)	yes	no
x_1	1	0	0	71	96	1	0
x_2	0	0	1	72	90	1	0
x_3	0	0	1	83	78	0	1
x_4	1	0	0	75	80	0	1
x_5	0	1	0	75	70	1	0
x_6	0	1	0	85	85	0	1

Lemma 2.2. *([15]) Let $(X, M, \mathcal{I})$ be a fuzzy formal context, $\delta_1,\ \delta_2 \in (0, 1]$ and $\delta_1 < \delta_2$, Then for all $A \subseteq X$, $B \subseteq M$, the following properties hold:*
1. $A^{'\delta_1 '\delta_2} \subseteq A^{'\delta_1 '\delta_1} \subseteq A^{'\delta_2 '\delta_1},\ \ B^{'\delta_1 '\delta_2} \subseteq B^{'\delta_1 '\delta_1} \subseteq B^{'\delta_2 '\delta_1}$.
2. $A^{'\delta_1 '\delta_2} \subseteq A^{'\delta_2 '\delta_2} \subseteq A^{'\delta_2 '\delta_1},\ \ B^{'\delta_1 '\delta_2} \subseteq B^{'\delta_2 '\delta_2} \subseteq B^{'\delta_2 '\delta_1}$.

Lemma 2.3. *([15]) Let $(X, M, \mathcal{I})$ be a fuzzy formal context, $\delta \in (0, 1]$, (A_1, B_1), $(A_2, B_2) \in B_\delta(X, M, \mathcal{I})$ are ordered by $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_2 \subseteq B_1)$, Then $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ is a complete distribute lattice, and conjunction and disjunction given by:*
1. $(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)^{'\delta '\delta})$,
2. $(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)^{'\delta '\delta}, B_1 \cap B_2)$.

2.2 A review of the AFS algebras

In this section, we recall some notations and present several pertinent results of AFS algebras. The following example, which employs the features table from [17], serves as an introductory illustration of the AFS algebras.

Example 2.1. Let $X = \{x_1, x_2, ..., x_6\}$ be a set of 6 cases and their features which are described by real numbers (temperature, humidity), Boolean values (outlook, windy). Let $M = \{m_1, m_2,\ ...,\ m_{10}\}$ be the set of fuzzy or crisp attributes on X and each $m \in M$ associates to a single feature. Where m_1 : *"rain"*, m_2 : *"sunny"*, m_3 : *"overcast"*, m_4 : *"hot"*, m_5 : *"cool"*, m_6 :*"about 80°F"*, m_7 : *"humid"*, m_8 : *"dry"*, m_9 : *"windy"*, m_{10} : *"no windy"*. The elements of M are viewed as "*elementary*" attributes.

Many new attributes can be generated by Boolean conjunction and disjunction of the attributes in M. For instance, $A = \{m_1, m_6\} \subseteq M$, it implies a new fuzzy attribute (*"complex attribute"*) "*the rain day which temperature is about 80°F*", which associates to the features sunny and temperature. $\sum_{i\in I} A_i$, which is a formal sum of the attributes $A_i \subseteq M, i \in I$. For example, we may have $\gamma = m_1m_6 + m_1m_9$ which translates as "*the rain day which temperature is about 80°F*" or "*windy rain day*" (the "+" denotes here a disjunction of attributes). For $A_i \subseteq M, i \in I$, $\sum_{i\in I} A_i$ has a well-defined meaning such as the one we have discussed above. By a straightforward comparison of

$$\gamma_1 = m_1m_6 + m_1m_9 \quad and \quad \gamma_2 = m_1m_6 + m_1m_9 + m_1m_5m_9, \tag{2.1}$$

we conclude that the expressions of γ_1 and γ_2 are equivalent in semantics. Considering the terms of γ_2, for any x, if x satisfies the condition $m_1m_5m_9$, then it must satisfies m_1m_9.

Therefore, the term $m_1m_5m_9$ is redundant in semantics when forming the fuzzy attribute γ_2.

2.2.1 EI algebra

Let M be non-empty set. The set EM^* is defined by

$$EM^* = \{\sum_{i\in I} A_i | A_i \in 2^M, i \in I, I \text{ is any non-empty indexing set}\}.$$

Definition 2.3. ([10]) Let M be a non-empty set A binary relation R^I on EM^* defined as follows: for $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in EM^*$, $(\sum_{i\in I} A_i)R^I(\sum_{j\in J} B_j) \Longleftrightarrow$ (i) $\forall A_i\ (i \in I)$, $\exists B_h\ (h \in J)$ such that $A_i \supseteq B_h$; (ii) $\forall B_j\ (j \in J)$, $\exists\ A_k\ (k \in I)$, such that $B_j \supseteq A_k$.

It's obvious that R^I is an equivalence relation. The quotient set EM^*/R^I is denoted by EM. Indeed, any element of EM is an equivalence class. Let $[\sum_{i\in I} A_i]_{R^I} \in EM$ be the set of all elements which are equivalent to $\sum_{i\in I} A_i \in EM^*$. For the sake of convenience, in the following, $[\sum_{i\in I} A_i]_{R^I}$ is denoted as $\sum_{i\in I} A_i$, if $\sum_{i\in I} A_i \in EM^*$ is not specified. That is to say, when $\sum_{i\in I} A_i \in EM^*$ is specified, $\sum_{i\in I} A_i$ only denote an element of the EM^*, otherwise $\sum_{i\in I} A_i$ always means the equivalence class $[\sum_{i\in I} A_i]_{R^I}$. For $\sum_{i\in I} A_i, \sum_{j\in J} B_j \in EM^*$, $\sum_{i\in I} A_i$ and $\sum_{j\in J} B_j$ are equivalent under the equivalence relation R^I means $[\sum_{i\in I} A_i]_{R^I} = [\sum_{j\in J} b_j]_{R^I}$.

Definition 2.4. ([19]) A complete lattice L is called completely distributive lattices, if one of the following conditions hold

$$\bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{ij}) = \bigvee_{f\in\prod_{i\in I} J_i}(\bigwedge_{i\in I} a_{if(i)}), \quad \bigvee_{i\in I}(\bigwedge_{j\in J_i} a_{ij}) = \bigwedge_{f\in\prod_{i\in I} J_i}(\bigvee_{i\in I} a_{if(i)})$$

where $\forall i \in I, \forall j \in J_i$, $a_{ij} \in L$, and $f \in \prod_{i\in I} J_i$ means f is a mapping $f : I \to \bigcup_{i\in I} J_i$ such that $f(i) \in J_i$.

Theorem 2.1. *([10]) For any $\sum_{i\in I} A_i$, $\sum_{j\in J} B_j \in EM$, then $(EM, \vee, \wedge)$ forms a completely distributive lattice under the binary compositions $\vee$ and $\wedge$ defined as follows,*

$$\sum_{i\in I} A_i \vee \sum_{j\in J} B_j = \sum_{k\in I\sqcup J} C_k, \quad \sum_{i\in I} A_i \wedge \sum_{j\in J} B_j = \sum_{i\in I, j\in J} (A_i \cup B_j), \tag{2.2}$$

where for any $k \in I \sqcup J$ (the disjoint union of I and J), $C_k = A_k$ if $k \in I$, and $C_k = B_k$ if $k \in J$.

$(EM, \vee, \wedge)$ is called the EI algebra over M. For $\alpha = \sum_{i\in I} A_i$, $\beta = \sum_{j\in J} B_j \in EM$, $\alpha \leq \beta \Longleftrightarrow \alpha \vee \beta = \beta \Leftrightarrow \forall A_i\ (i \in I), \exists B_h\ (h \in J)$ such that $A_i \supseteq B_h$. $M, \emptyset$ are the minimum and maximum element in EM, respectively.

In Example 2.1, let $\psi_1 = m_1m_5 + m_2m_4m_9$, $\psi_2 = m_4m_9 + m_4m_8 \in EM$. By (2.2), the algebra operations of them are shown as follows:

$$\psi_2 \vee \psi_2 = m_1m_5 + m_4m_9 + m_4m_8,$$
$$\psi_1 \wedge \psi_2 = m_1m_4m_5m_9 + m_1m_4m_5m_8 + m_2m_4m_9.$$

As long as we can determine the algebra operations $\vee, \wedge$ of the few attributes in M, the logical operations $\vee$ ("*or*") and $\wedge$ ("*and*") of all complex attributes in EM can also be determined. A collection of a few attributes in M plays a similar role to the one of a "basis" used in linear vector spaces.

In the sequel, we denote the subsets of X with the lower case letters and the subsets of M with the capital letters, in order to distinguish the subsets of X from the subsets of M.

2.2.2 EII algebra

Definition 2.5. ([10]) Let X, M be non-empty sets. A binary relation R^{II} on the set $EXM^* = \{\sum_{i\in I} u_iA_i | A_i \in 2^M, u_i \in 2^X, i \in I, I$ is any non-empty indexing set$\}$ is defined as follows: for any $\sum_{i\in I} u_iA_i, \sum_{j\in J} v_jB_j \in EXM^*$, $(\sum_{i\in I} u_iA_i)R^{II}(\sum_{j\in J} v_jB_j) \Longleftrightarrow$ (i) $\forall u_iA_i\ (i \in I), \exists v_hB_h\ (h \in J)$ such that $A_i \supseteq B_h$, $u_i \subseteq v_h$; (ii) $\forall v_jB_j\ (j \in J), \exists\ u_kA_k\ (k \in I)$, such that $B_j \supseteq A_k$, $v_j \subseteq u_k$.

Obviously, R^{II} is an equivalence relation. The quotient set EXM^*/R^{II} is denoted by EXM. Similar to EI algebra, the equivalent class $[\sum_{i\in I} a_iA_i]_{R^{II}}$ is denoted as $\sum_{i\in I} a_iA_i$ in the sequel, if $\sum_{i\in I} a_iA_i \in EXM^*$ is not specified. For $\sum_{i\in I} a_iA_i, \sum_{j\in J} b_jB_j \in EX^*$, $\sum_{i\in I} a_iA_i$ and $\sum_{j\in J} b_jB_j$ are equivalent under the equivalence relation R^{II} means $[\sum_{i\in I} a_iA_i]_{R^{II}} = [\sum_{j\in J} b_jB_j]_{R^{II}}$.

Theorem 2.2. *For any $\sum_{i\in I} u_iA_i$, $\sum_{j\in J} v_jB_j \in EXM$, then $(EXM, \vee, \wedge)$ forms a completely distributive lattice under the binary compositions $\vee$ and $\wedge$ defined as follows,*

$$\sum_{i\in I} u_iA_i \vee \sum_{j\in J} v_jB_j = \sum_{k\in I\sqcup J} w_kC_k, \tag{2.3}$$

$$\sum_{i\in I} u_iA_i \wedge \sum_{j\in J} v_jB_j = \sum_{i\in I, j\in J} \left[(u_i \cap v_j)(A_i \cup B_j)\right]. \tag{2.4}$$

$(EXM, \vee, \wedge)$ is called the EII algebra over X and M. For $\alpha = \sum_{i\in I} u_iA_i$, $\beta = \sum_{j\in J} v_jB_j \in EXM$, $\alpha \le \beta \Longleftrightarrow \alpha \vee \beta = \beta \Leftrightarrow \forall u_iA_i\ (i \in I), \exists v_hB_h\ (h \in J)$ such that $A_i \supseteq B_h$, $u_i \subseteq v_h$. $\emptyset M$, $X\emptyset$ are the minimum and maximum element in EXM, respectively.

2.2.3 $E^{\#}I$ algebra

In order to better solve the real world problems, authors proposed an other AFS algebra, denoted as $E^{\#}I$ algebra [9].

Let X be non-empty set. The set EX^* is defined by

$$EX^* = \{\textstyle\sum_{i\in I} a_i | a_i \in 2^X,\ I \text{ is any non-empty indexing set}\}$$

Definition 2.6. ([9]) Let X be a non-empty set. A binary relation $R^{\#}$ on EX^* is defined as follows: for $\sum_{i\in I} a_i, \sum_{j\in J} b_j \in EX^*$, $(\sum_{i\in I} a_i)R^{\#}(\sum_{j\in J} b_j) \Leftrightarrow \forall a_i\ (i \in I), \exists b_h$ $(h \in J)$ such that $a_i \subseteq b_h$ and $\forall b_j\ (j \in J), \exists a_k\ (k \in I)$ such that $b_j \subseteq a_k$.

It is obvious that $R^{\#}$ is an equivalence relation on EX^*. The quotient set $EX^*/R^{\#}$ is denoted by $E^{\#}X$. Similar to EI algebra, equivalent class $[\sum_{i\in I} a_i]_{R^{\#}}$ is denoted as $\sum_{i\in I} a_i$ in the sequel, if $\sum_{i\in I} a_i \in EX^*$ is not specified. For $\sum_{i\in I} a_i, \sum_{j\in J} b_j \in EX^*$, $\sum_{i\in I} a_i$ and $\sum_{j\in J} b_j$ are equivalent under the equivalence relation $R^{\#}$ means $[\sum_{i\in I} a_i]_{R^{\#}} = [\sum_{j\in J} b_j]_{R^{\#}}$.

Theorem 2.3. *For any* $\sum_{i\in I} a_i$, $\sum_{j\in J} b_j \in E^{\#}X$, *then* $(E^{\#}X, \vee, \wedge)$ *forms a completely distributive lattice under the binary compositions* $\vee$, $\wedge$ *defined as follows,*

$$\sum_{i\in I} a_i \vee \sum_{j\in J} b_j = \sum_{k\in I\sqcup J} c_k, \quad \sum_{i\in I} a_i \wedge \sum_{j\in J} b_j = \sum_{i\in I, j\in J} (a_i \cap b_j). \tag{2.5}$$

$(E^{\#}X, \vee, \wedge)$ is called an $E^{\#}I$ algebra over X. For $\alpha = \sum_{i\in I} u_i$, $\beta = \sum_{j\in J} v_j \in EXM$, $\alpha \leq \beta \Longleftrightarrow \alpha \vee \beta = \beta \Leftrightarrow \forall u_i\ (i \in I), \exists v_h\ (h \in J)$ such that $u_i \subseteq v_h$. $\emptyset$, X are the minimum and maximum element in $E^{\#}X$ respectively.

In Example 2.1, let $\mu_1 = \{x_1, x_2, x_5\} + \{x_2, x_3\}$, $\mu_2 = \{x_4\} + \{x_1, x_2\} \in E^{\#}X$. By (2.5), the algebra operations of them are shown as follows:

$$\mu_1 \vee \mu_2 = \{x_1, x_2, x_5\} + \{x_2, x_3\} + \{x_4\},\ \mu_1 \wedge \mu_2 = \{x_1, x_2\}.$$

In [9,10], authors have been established the homomorphisms relationships between EI, EII and $E^{\#}I$ (i.e., the arrows **1**, **2** and **3** in Figure 2.1). In next section, we will explore that there exist some homomorphism maps (i.e., the arrows **4**, **5** and **6** in Figure 2.1) to reflect the relationship between VTCL and EI, EII and $E^{\#}I$, respectively.

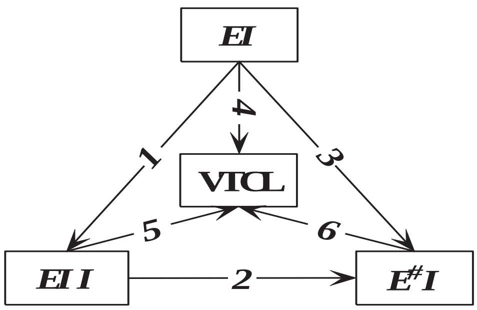

Figure 2.1: The Relationship between AFS Algebra and VTCL

3 The relationship between AFS algebras and VTCL

For given two sets X, M, we can establish the EII algebra $(EXM, \vee, \wedge)$, which is a completely distributive lattice. First, we will discuss the relationship between the lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ and $(EXM, \vee, \wedge)$. To conveniently, we first define a subsets of EXM as following:

$$\mathcal{I}(EXM) = \{\gamma \in EXM | \gamma = \sum_{i\in I} b_i B_i, i \in I, b_i \in X, B_i \in M, b_i = B_i^{'\delta}\} \tag{3.1}$$

where operation $'\delta$ defined by Definition 2.2.

Theorem 3.1. *Let $(X, M, \mathcal{I})$ be a fuzzy context. Then $\mathcal{I}(EXM)$ is a sub EII algebra of EXM, i.e., $\zeta_k \in \mathcal{I}(EXM), k \in K$, $\vee_{k\in K}\zeta_k \in \mathcal{I}(EXM)$ and $\wedge_{k\in K}\zeta_k \in \mathcal{I}(EXM)$, and $\mathcal{I}(EXM, \vee, \wedge)$ is a completely distribute lattice.*

Proof: It is easy to show that $\vee_{k\in K}\zeta_k \in \mathcal{I}(EXM)$. Since EXM is a completely distributive lattice, so that

$$\wedge_{k\in K}\zeta_k = \sum_{f\in\prod_{k\in K} I_k} \left(\cap_{k\in K} b_{kf(k)}, \cup_{k\in K} B_{kf(k)}\right).$$

where $f \in \prod_{k\in K} I_k$ means that f is a map $f : K \to \cup_{k\in K} I_k$ such that $f(k) = I_k$ for $\forall k \in K$. By Lemma 2.1 and the definition of $\mathcal{I}(EXM)$, we can get that for $\forall k \in K$, $j \in I_k$

$$\left(\cup_{k\in K} B_{kf(k)}\right)^{'\delta} = \cap_{k\in K}\left(B_{kf(k)}\right)^{'\delta} = \cap_{k\in K} b_{kf(k)}.$$

Therefore, $\wedge_{k\in K}\zeta_k \in \mathcal{I}(EXM)$. Moreover, $\mathcal{I}(EXM, \vee, \wedge)$ is a completely distributive lattice because $(EXM, \vee, \wedge)$ is a completely distributive lattice.

Theorem 3.2. *Let $(X, M, \mathcal{I})$ be a fuzzy context, then $p_1^{\mathcal{I}}$ is a homomorphism map from the lattice $(EM, \vee, \wedge)$ to the lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$, provided that for any $\sum_{i\in I} B_i \in EM$, $p_1^{\mathcal{I}}$ is defined by*

$$p_1^{\mathcal{I}}\left(\sum_{i\in I} B_i\right) = \vee_{i\in I}(B_i^{'\delta}, B_i^{'\delta'\delta}) = \left(\left(\cup_{i\in I} B_i^{'\delta}\right)^{'\delta'\delta}, \cap_{i\in I} B_i^{'\delta'\delta}\right). \tag{3.2}$$

where operation $'\delta$ defined by Definition 2.2.

Proof: By Lemma 2.1, one can see that for any $\sum_{i\in I} B_i \in EM$, and for $\forall i \in I$, $(B_i^{'\delta}, B_i^{'\delta'\delta}) \in \mathcal{B}_\delta(X, M, \mathcal{I})$. Since $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ be a complete lattice, so for any $\sum_{i\in I} B_i \in EM$,

$$p_1^{\mathcal{I}}\left(\sum_{i\in I} B_i\right) = \left(\left(\cup_{i\in I} B_i^{'\delta}\right)^{'\delta'\delta}, \cap_{i\in I} B_i^{'\delta'\delta}\right) = \vee_{i\in I}(B_i^{'\delta}, B_i^{'\delta'\delta}) \in \mathcal{B}_\delta(X, M, \mathcal{I}).$$

Now, we will show that $p_1^{\mathcal{I}}$ is a map from EM to $\mathcal{B}_\delta(X, M, \mathcal{I})$. Suppose that $\sum_{i\in I_1} B_i = \sum_{k\in I_2} B_k \in EM$, by definition of EI algebra and Lemma 2.1, which means that $\forall i \in I_1, \exists k \in I_2$ such that $B_i \supseteq B_k$, $B_i^{'\delta} \subseteq B_k^{'\delta}$, and $\forall k \in I_2, \exists i \in I_1$ such that $B_k \supseteq B_i$, $B_k^{'\delta} \subseteq B_i^{'\delta}$. Therefore, $\cup_{i\in I_1} B_i^{'\delta} = \cup_{k\in I_2} B_k^{'\delta}$ and $(\cup_{i\in I_1} B_i^{'\delta})^{'\delta'\delta} = (\cup_{k\in I_2} B_k^{'\delta})^{'\delta'\delta}$ hold.

Notice that both $((\cup_{k\in I_2} B_k^{'\delta})^{'\delta'\delta}, \cap_{k\in I_2} B_k^{'\delta'\delta})$ and $((\cup_{i\in I_1} B_i^{'\delta})^{'\delta'\delta}, \cap_{i\in I_1} B_i^{'\delta'\delta})$ are variable threshold concepts in $\mathcal{B}_\delta(X, M, \mathcal{I})$, hence

$$\left((\cup_{i\in I_1} B_i^{'\delta})^{'\delta'\delta}, \cap_{i\in I_1} B_i^{'\delta'\delta}\right) = \left((\cup_{k\in I_2} B_k^{'\delta})^{'\delta'\delta}, \cap_{k\in I_2} B_k^{'\delta'\delta}\right),$$

which implies that $p_1^{\mathcal{I}}(\sum_{i\in I_1} B_i) = p_1^{\mathcal{I}}(\sum_{k\in I_2} B_k)$ hold.

Moreover, for any $\zeta = \sum_{i\in I} A_i$, $\eta = \sum_{j\in J} B_i \in EM$, by Lemma 2.1 and 2.3, we have

$$\begin{aligned}
& p_1^{\mathcal{I}}(\zeta \vee \eta) \\
= \ & \left(\left[\left(\cup_{i\in I} A_i^{'\delta} \right) \cup \left(\cup_{j\in J} B_j^{'\delta} \right) \right]^{'\delta'\delta}, \left[\left(\cap_{i\in I} A_i^{'\delta'\delta} \right) \cap \left(\cap_{j\in J} B_j^{'\delta'\delta} \right) \right] \right). \\
& p_1^{\mathcal{I}}(\zeta) \vee p_1^{\mathcal{I}}(\eta) \\
= \ & \left(\left(\cup_{i\in I} A_i^{'\delta} \right)^{'\delta'\delta}, \cap_{i\in I} A_i^{'\delta'\delta} \right) \vee \left(\left(\cup_{j\in J} B_j^{'\delta} \right)^{'\delta'\delta}, \cap_{j\in J} B_j^{'\delta'\delta} \right) \\
= \ & \left(\left[\left(\cup_{i\in I} A_i^{'\delta} \right)^{'\delta'\delta} \cup \left(\cup_{j\in J} B_j^{'\delta} \right)^{'\delta'\delta} \right]^{'\delta'\delta}, \left[\left(\cap_{i\in I} A_i^{'\delta'\delta} \right) \cap \left(\cap_{j\in J} B_j^{'\delta'\delta} \right) \right] \right).
\end{aligned}$$

Notice that both $p_1^{\mathcal{I}}(\zeta \vee \eta)$ and $p_1^{\mathcal{I}} \vee p_1^{\mathcal{I}}(\eta)$ are variable threshold concepts in $\mathcal{B}_\delta(X, M, \mathcal{I})$, hence $p_1^{\mathcal{I}}(\zeta \vee \eta) = p_1^{\mathcal{I}} \vee p_1^{\mathcal{I}}(\eta)$. From (3.2) and definition of EI algebra, we have

$$p_1^{\mathcal{I}}(\zeta \wedge \eta) = p_1^{\mathcal{I}} \left(\sum_{i\in I, j\in J} A_i \cup B_i \right) = \vee_{i\in I, \in J}((A_i \cup B_j)^{'\delta}, (A_i \cup B_j)^{'\delta'\delta}). \tag{3.3}$$

In addition, for any $i \in I, j \in J$, it follows by Lemma 2.3 that

$$(A_i^{'\delta}, A_i^{'\delta'\delta}) \wedge (B_j^{'\delta}, B_j^{'\delta'\delta}) = \left((A_i^{'\delta} \cap B_j^{'\delta}), (A_i^{'\delta'\delta} \cup B_j^{'\delta'\delta})^{'\delta'\delta} \right).$$

By Lemma 2.1, we have

$$\begin{aligned}
(A_i^{'\delta'\delta} \cup B_j^{'\delta'\delta})^{'\delta'\delta} &= \left((A_i^{'\delta'\delta} \cup B_j^{'\delta'\delta})^{'\delta} \right)^{'\delta} \\
&= (A_i^{'\delta} \cap B_j^{'\delta})^{'\delta} \\
&= (A_i \cup B_j)^{'\delta'\delta}.
\end{aligned}$$

Therefore, for any $i \in I, j \in J$,

$$((A_i \cup B_j)^{'\delta}, (A_i \cup B_j)^{'\delta'\delta}) = (A_i^{'\delta}, A_i^{'\delta'\delta}) \wedge (B_j^{'\delta}, B_j^{'\delta'\delta}),$$

and

$$\begin{aligned}
p_1^{\mathcal{I}}(\zeta \wedge \eta) &= \vee_{i\in I, j\in J}[(A_i^{'\delta}, A_i^{'\delta'\delta}) \wedge (B_j^{'\delta}, B_j^{'\delta'\delta})] \\
&= [\vee_{i\in I}(A_i^{'\delta}, A_i^{'\delta'\delta})] \wedge [\vee_{j\in J}(B_j^{'\delta}, B_j^{'\delta'\delta})] \\
&= p_1^{\mathcal{I}}(\zeta) \wedge p_1^{\mathcal{I}}(\eta).
\end{aligned}$$

Thus, $p_1^{\mathcal{I}}$ is a homomorphism map from $(EM, \vee, \wedge)$ to $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$.

Theorem 3.3. *Let $(X, M, \mathcal{I})$ be a fuzzy context, then $p_2^{\mathcal{I}}$ is a homomorphism map from the lattice $(\mathcal{I}(EXM), \vee, \wedge)$ to the lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$, provided that for any*

$\sum_{i\in I} b_iB_i \in EM$, $p_2^{\mathcal{I}}$ is defined by

$$p_2^{\mathcal{I}}\left(\sum_{i\in I} b_iB_i\right) = \vee_{i\in I}(b_i, b_i^{'\delta}) = \left((\cup_{i\in I}b_i)^{'\delta'\delta},\ \cap_{i\in I} b_i^{'\delta}\right) \tag{3.4}$$

where operation $'\delta$ defined by Definition 2.2.

Proof: By Lemma 2.1, for any $\sum_{i\in I} b_iB_i \in \mathcal{I}(EXM)$, one can derive that $\forall i \in I$, $(b_i, b_i^{'\delta}) = (B_i^{'\delta}, B_i^{'\delta'\delta}) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, this implies that

$$\left((\cup_{i\in I}b_i)^{'\delta'\delta}, \cap_{i\in I}b_i^{'\delta}\right) = \vee_{i\in I}(b_i, b_i^{'\delta}) \in \mathcal{B}_\delta(X, M, \mathcal{I}).$$

First, we need to prove that $p_{\mathcal{I}}$ is a map from $\mathcal{I}(EXM)$ to $\mathcal{B}_\delta(X, M, \mathcal{I})$. Suppose that $\sum_{i\in I_1} b_iB_i = \sum_{k\in I_2} b_kB_k \in \mathcal{I}(EXM)$, i.e., $\forall i \in I_1$, $\exists k \in I_2$ such that $B_i \supseteq B_k$, $b_k \supseteq b_i$ and $\forall k \in I_2$, $\exists i \in I_1$ such that $B_k \supseteq B_i$, $b_i \supseteq b_k$, these imply that $b_k^{'\delta} \subseteq b_i^{'\delta}$ and $b_k^{'\delta} \supseteq b_i^{'\delta}$, so $\cup_{i\in I_1}b_i = \cup_{k\in I_2}b_k$, $\cap_{i\in I_1} b_i^{'\delta} = \cap_{k\in I_2}b_k^{'\delta}$. Therefore, $p_2^{\mathcal{I}}(\sum_{i\in I_1} b_iB_i) = p_2^{\mathcal{I}}(\sum_{k\in I_2} b_kB_k)$, i.e., $p_2^{\mathcal{I}}$ is a map from $\mathcal{I}(EXM)$ to $\mathcal{B}_\delta(X, M, \mathcal{I})$. Then for any $\zeta = \sum_{i\in I} a_iA_i$, $\eta = \sum_{j\in J} b_jB_j \in \mathcal{I}(EXM)$, by (3.4) and Lemma 2.3, we have

$$\begin{aligned}
& p_2^{\mathcal{I}}(\zeta \vee \eta) \\
= & \left([(\cup_{i\in I}a_i) \cup (\cup_{j\in J}b_j)]^{'\delta'\delta},\ \left[(\cap_{i\in I}a_i^{'\delta}) \cap (\cap_{j\in J}b_j^{'\delta})\right]\right), \\
& p_2^{\mathcal{I}}(\zeta) \vee p_2^{\mathcal{I}}(\eta) \\
= & \left((\cup_{i\in I}a_i)^{'\delta'\delta},\ \cap_{i\in I} a_i^{'\delta}\right) \vee \left((\cup_{i\in I}b_i)^{'\delta'\delta},\ \cap_{i\in I} b_i^{'\delta}\right) \\
= & \left(\left[(\cup_{i\in I}a_i)^{'\delta'\delta} \cup (\cup_{j\in J}b_j)^{'\delta'\delta}\right]^{'\delta'\delta},\ \left[(\cap_{i\in I}a_i^{'\delta}) \cap (\cap_{j\in J}b_j^{'\delta})\right]\right).
\end{aligned}$$

Recall that both $p_2^{\mathcal{I}}(\zeta \vee \eta)$ and $p_2^{\mathcal{I}}(\zeta) \vee p_2^{\mathcal{I}}(\eta)$ are variable threshold concepts in $\mathcal{B}_\delta(X, M, \mathcal{I})$, hence $p_2^{\mathcal{I}}(\zeta \wedge \eta) = p_2^{\mathcal{I}}(\zeta) \vee p_2^{\mathcal{I}}(\eta)$. By definition of EI algebra, Lemma 2.3 and (3.4), we have

$$p_{\mathcal{I}}(\zeta \wedge \eta) = p_{\mathcal{I}}\left(\sum_{i\in I, j\in J} a_i \cap b_iA_i \cup B_i\right) = \vee_{i\in I,\in J}(a_i \cap b_j, (a_i \cap b_j)^{'\delta}). \tag{3.5}$$

Notice that for any $i \in I$, $j \in J$,

$$(a_i, a_i^{'\delta}) \wedge (b_j, b_j^{'\delta}) = (a_i \cap b_j, (a_i^{'\delta} \cup b_j^{'\delta})^{'\delta'\delta}).$$

By Lemma 2.1, for any $i \in I$, $j \in J$, we have

$$(a_i^{'\delta} \cup b_j^{'\delta})^{'\delta'\delta} = ((a_i^{'\delta} \cup b_j^{'\delta})^{'\delta})^{'\delta} = (a_i^{'\delta'\delta} \cap b_j^{'\delta'\delta})^{'\delta} = a_i^{'\delta} \cup b_j^{'\delta} = (a_i \cap b_j)^{'\delta}.$$

So, $(a_i, a_i^{'\delta}) \wedge (b_j, b_j^{'\delta}) = (a_i \cap b_j, (a_i \cap b_j)^{'\delta})$, and $p_2^{\mathcal{I}}(\zeta \vee \eta) = \vee_{i\in I, j\in J}(a_i \cap b_j, (a_i \cap b_j)^{'\delta}) = \vee_{i\in I, j\in J}[(a_i, a_i^{'\delta}) \wedge (b_j, b_j^{'\delta})] = p_2^{\mathcal{I}}(\zeta) \wedge p_2^{\mathcal{I}}(\eta)$.

Thus, $p_2^{\mathcal{I}}$ is a homomorphism map from $(EXM, \vee, \wedge)$ to $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$.

Theorem 3.4. *Let $(X, M, \mathcal{I})$ be a fuzzy context, then $p_3^{\mathcal{I}}$ is a homomorphism map from the lattice $(E^{\#}X, \vee, \wedge)$ to the lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$, provided that for any $\sum_{i\in I} b_i \in EM$, $p_3^{\mathcal{I}}$ is defined by*

$$p_3^{\mathcal{I}}\left(\sum_{i\in I} b_i\right) = \vee_{i\in I}(b_i, b_i^{\prime\delta}) = \left((\cup_{i\in I} b_i)^{\prime\delta\prime\delta},\ \cap_{i\in I} b_i^{\prime\delta}\right). \tag{3.6}$$

where operation $'\delta$ defined by Definition 2.2.

Proof: The proof is similar to Theorem 3.3.

By Theorems 3.2, 3.3 and 3.4, one can see that $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ has algebraic properties similar to the EI algebra $E^{\#}I$ algebra and $\mathcal{I}(EGM)$, the sub EII algebra of EGM.

Theorem 3.5. *Let $(X, M, \mathcal{I})$ be a context. If $e : \mathcal{B}_\delta(X, M, \mathcal{I}) \to EXM$ is defined as following: for any $(b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $e(b, B) = bB \in EXM$. Then the following conclusions hold:*

T5-1. If $(a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $(a, A) \leq (b, B)$, then *$e(a, A) \leq e(b, B)$;*

T5-2. For any $(a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $e((a, A) \vee (b, B)) \geq e(a, A) \vee e(b, B)$, $e((a, A) \wedge (b, B)) \leq e(a, A) \wedge e(b, B)$.

Proof: T5-1 $(a, A) \leq (b, B) \Rightarrow a \subseteq b$, $A \supseteq B$. From definition of EII algebra, one has

$$e(a, A) \vee e(b, B) = aA + bB = bB = e(b, B).$$

This implies that $e(a, A) \leq e(b, B)$ in lattice EXM.

T5-2 $e((a, A) \vee (b, B)) = e((a \cup b)^{\prime\delta\prime\delta}, A \cap B) = (a \cup b)^{\prime\delta\prime\delta} A \cap B$, moreover, by Lemma 2.1,

$$(a \cup b)^{\prime\delta\prime\delta} \supseteq a \cup b,\ (A \cup B)^{\prime\delta\prime\delta} \supseteq A \cup B.$$

Therefore, by definition of EII algebra, we have

$$e((a, A) \vee (b, B)) = (a \cup b)^{\prime\delta\prime\delta} A \cap B \geq aA + bB = e(a, A) \vee e(b, B),$$

$$e((a, A) \wedge (b, B)) = (a \cap b)(A \cup B)^{\prime\delta\prime\delta} \leq aA \wedge bB = e(a, A) \wedge e(b, B).$$

The proof is complete.

Theorem 3.6. *Let $(X, M, \mathcal{I})$ be a context. If $f : \mathcal{B}_\delta(X, M, \mathcal{I}) \to EXM$ is defined as following: for any $(b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $f(b, B) = b \in E^{\#}X$. Then the following conclusions hold:*

T6-1. If $(a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $(a, A) \leq (b, B)$, then $f(a, A) \leq f(b, B)$;

T6-2. For $(a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $f((a, A) \vee (b, B)) \geq f(a, A) \vee f(b, B)$, $f((a, A) \wedge (b, B)) = f(a, A) \wedge f(b, B)$.

Proof: T6-1 $(a, A) \leq (b, B) \Rightarrow a \subseteq b$, $A \supseteq B$. From definition of $E^{\#}I$ algebra, one has

$$f(a, A) \vee f(b, B) = a + b = b = f(b, B).$$

This implies that $f(a, A) \leq f(b, B)$ in lattice EXM.

T6-2 $f((a, A) \vee (b, B)) = f((a \cup b)^{'\delta'\delta}, A \cap B) = (a \cup b)^{'\delta'\delta}$, and by Lemma 2.1,

$$(a \cup b)^{'\delta'\delta} \supseteq a \cup b.$$

So, $(a \cup b)^{'\delta'\delta} \supseteq a$, $(a \cup b)^{'\delta'\delta} \supseteq b$. Therefore, from the definition of $E^{\#}I$ algebra, we have

$$f((a, A) \vee (b, B)) = (a \cup b)^{'\delta'\delta} \geq a + b = f(a, A) \vee f(b, B)$$
$$f((a, A) \wedge (b, B)) = a \cap b = a \wedge b = f(a, A) \wedge f(b, B).$$

The proof is complete.

Theorem 3.7. *Let $(X, M, \mathcal{I})$ be a context. If $g : \mathcal{B}_\delta(X, M, \mathcal{I}) \to EM$ is defined as following: for any $(b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I})$, $g(b, B) = B \in EM$. Then the following conclusions hold:*

$T7-1.$ $\quad If\ (a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I}), (a, A) \leq (b, B), then\ g(a, A) \leq g(b, B);$

$T7-2.$ $\quad For (a, A), (b, B) \in \mathcal{B}_\delta(X, M, \mathcal{I}),$

$$g((a, A) \vee (b, B)) \geq g(a, A) \vee g(b, B),$$

$$g((a, A) \wedge (b, B)) \leq g(a, A) \wedge g(b, B),$$

Proof: T7-1 $(a, A) \leq (b, B) \Rightarrow a \subseteq b$, $A \supseteq B$. From definition of EI algebra, one has

$$g(a, A) \vee g(b, B) = A + B = B = g(b, B).$$

This implies that $g(a, A) \leq g(b, B)$ in EM.

T7-2 $g((a, A) \vee (b, B)) = g((a \cup b)^{'\delta'\delta}, A \cap B) = A \cap B$, we have

$$g((a, A) \vee (b, B)) = A \cap B \geq A + B = g(a, A) \vee g(b, B),$$

by Lemma 2.1, $(A \cup B)^{'\delta'\delta} \supseteq (A \cup B)$, so $(a \cap b)(A \cup B)^{'\delta'\delta} \leq (a \cap b)(A \cup B)$,

$$g((a, A) \wedge (b, B)) = (a \cap b)(A \cup B)^{'\delta'\delta} \leq aA \wedge bB = e(a, A) \wedge (b, B).$$

The proof is complete.

Theorems 3.5, 3.6, 3.7 imply that some properties of the $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ can be studied in the framework of the AFS algebras. Moreover, the AFS algebras are more general algebra structures and can be applied to study fuzzy attributes, such as fuzzy clustering analysis, fuzzy decision trees, etc. About the detail application of AFS algebras, please see [11–14, 20].

4 Conclusion

In this paper, we discuss the homomorphism relationship between VTCL and AFS algebras. Three algebra homomorphism maps (i.e., Theorems 3.2, 3.3, 3.4) between AFS algebras and variable threshold concept lattice are established, by which one can see that the threshold concept lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ has algebraic properties similar to the AFS algebras. Some properties of the complete lattice $(\mathcal{B}_\delta(X, M, \mathcal{I}), \leq)$ can be studied in the framework of the AFS algebras.

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