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The Quadruple Aboodh Transform and Its Properties with Applications to Integral and Partial Differential Equations

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Abstract: In this article, we introduce definition of quadruple Aboodh transform and its inverse. We also apply quadruple Aboodh transform for several functions of four variables. In addition, we establish and prove some main properties and theorems related to Quadruple Aboodh transform. Furthermore, we discuss the convolution theorem and its proof. To check the efficiency and applicability of quadruple Aboodh transform, we apply this transform to integral and partial including fractional differential equations.

Keywords: Partial differential equation, Aboodh transform, inverse Aboodh transform, convolution

1 Introduction

One of the most important subject in mathematics is the topic of differential equations, both ordinary and partial including fractional. The importance came from that differential equations have a lot of applications in real life science and engineering, therefore many researchers are still working with greet efforts and have been interested in finding the new solutions of these kind of equations.
Among the solution methods, transformation methods are rather, popular, Hence in the literature, there are a lot of different integral transforms like Laplace transform [1,2], Sumudu transform [3,4], Ezaki transform[5], Mellin transform[6], and so on. These kind of solutions have a wide variety of applications in various area in physics, statistics, engineering and in other sciences.
One of them is Aboodh transform[7,8,9,10] which was derived from the Fourier integral, Aboodh transform was introduced by khalid suliman Aboohd in 2013 to solve ordinary ,partial and fractional differential equations in time domain. In 2014 Aboodh introduced the double Aboodh transform which is a higher version of simple Aboodh transform, in 2019 S.Alfaqeih and T.ozis [11]introduced the triple Aboodh transform and used this transform to solve homogeneous and non-homogeneous partial differential equations, Aboodh transform appeared to be a very efficient and powerful method for solving a wide class of partial, integral and fractional differential equations[12,13,14,15,16,17].
The main objective of this article is to introduce a higher version of Aboodh transform and to study its properties with examples, several theorems and properties of quadruple Aboodh transform are discussed in some details, Quadruple Aboodh transform is successfully applied to solve integral ,partial and fractional differential equations.

2 Preliminaries

In this section, we recall the definitions of simple, double and triple Aboodh transforms and there inverses.

Definition 2.1[7] The simple Aboodh transform of a function $f(x)$ is given by:

$$K(p) = A_x[f(x)] = K(p) = \frac{1}{p}\int_0^\infty f(x)e^{-px}dx, \mathrm{x} > 0, \quad (1)$$

and the inverse of simple Aboodh transform is given by:

$$f(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} p \; e^{px}K(p)dp. \quad (2)$$

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Definition 2.2[8] The double Aboodh transform of a function $f(x,y)$ is given by:

$$K(p,q)=A_{xy}\left[f(x,y)\right]=\frac{1}{pq}\int_0^\infty\int_0^\infty f(x,y)\,e^{-(px+qy)}dxdy, \tag{3}$$

and the inverse of double Aboodh transform is given by:

$$f(x,y)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}p\ e^{px}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}q\ e^{qy}K(p,q)dq\right]dp. \tag{4}$$

Definition 2.3[11] The triple Aboodh transform of a function $f(x,y,t)$ is given by:

$$K(p,q,r)=$$
$$A_{xyt}\left(f(x,y,t)\right)=\frac{1}{pqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rt}f(x,y,t)dxdydt, \tag{5}$$

and the inverse of triple Aboodh transform is given by:

3 Quadruple Aboodh transform and Examples

In this section, we present definitions of the quadruple Aboodh transform, its inverse and quadruple Aboodh transform of some functions.

Definition 3.1 Let f be a continuous function of four variables , then the Quadruple Aboodh transform of$f(x,y,z,t)$ is defined by :

$$K(p,q,r,s)=A_{xyzt}\left(f(x,y,z,t)\right)$$
$$=\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}f(x,y,z,t)\,dxdydzdt, \tag{6}$$

Definition 3.2 The inverse of quadruple Aboodh transform is given by:

$$f(x,y,z,t)=$$
$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}pe^{px}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}q\ e^{qy}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}re^{rz}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}se^{st}K(p,q,r,s)\,ds\right]dr\right]dq\right]dp \tag{7}$$

where $K(p,q,r,s)$, for all p,q,r and s must be an analytic function in the region defined by $\mathrm{Re}p\geq\alpha,\mathrm{Re}q\geq\beta,\mathrm{Re}r\geq\gamma$, and $\mathrm{Re}s\geq\mu$.

3.1 Examples

(a) If $f(x,y,z,t)=1$ for $x>0,\ y>0,z>0,t>0$,, then:

$$K(p,q,r,s)=$$
$$A_{xyzt}(1)=\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}dxdydzdt$$
$$=\frac{1}{p}\int_0^\infty e^{-px}dx\frac{1}{q}\int_0^\infty e^{-qy}dy\frac{1}{r}\int_0^\infty e^{-rz}dz\frac{1}{s}\int_0^\infty e^{-st}dt=\frac{1}{(pqrs)^2}.$$

(b) if $f(x,y,z,t)=e^{ax+by+cz+dt}$, then:

$$A_{xyzt}\left(e^{ax+by+cz+dt}\right)=$$
$$\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(p-a)x}e^{-(q-b)y}e^{-(r-c)z}e^{-(s-d)t}dxdydzdt$$
$$=\frac{1}{p}\int_0^\infty e^{-(p-a)x}dx\frac{1}{q}\int_0^\infty e^{-(q-b)y}dy\frac{1}{r}\int_0^\infty e^{-(r-c)z}dz\frac{1}{s}\int_0^\infty e^{-(s-d)t}dt$$
$$=\frac{1}{p(p-a)q(q-b)r(r-c)s(s-d)}.$$

(c)

$$A_{xyzt}\left(e^{i(ax+by+cz+dt)}\right)=\frac{1}{p(p-ia)q(q-ib)r(r-ic)s(s-id)}$$
$$=\frac{(p+ia)(q+ib)(r+ic)(s+id)}{p(p^2+a^2)q(q^2+b^2)r(r^2+c^2)s(s^2+d^2)}$$
$$=\frac{[(pq-ab)(rs-cd)-(aq+pb)(sc+rd)]}{p(p^2+a^2)q(q^2+b^2)r(r^2+c^2)s(s^2+d^2)}$$
$$+i\frac{[(rs-cd)(aq+bp)+(pq-ab)(sc+rd)]}{p(p^2+a^2)q(q^2+b^2)r(r^2+c^2)s(s^2+d^2)}$$

Consequently,

$$A_{xyzt}\left(\sin(ax+by+cz+dt)\right)$$
$$=\frac{(rs-cd)(aq+bp)+(pq-ab)(sc+rd)}{p(p^2+a^2)q(q^2+b^2)r(r^2+c^2)s(s^2+d^2)}. \tag{8}$$

$$A_{xyzt}\left(\cos(ax+by+cz+dt)\right)$$
$$=\frac{(pq-ab)(rs-cd)-(aq+pb)(sc+rd)}{p(p^2+a^2)q(q^2+b^2)r(r^2+c^2)s(s^2+d^2)}. \tag{9}$$

$$A_{xyzt}\left(\sinh(ax+by+cz+dt)\right)=$$
$$\frac{1}{2}\left[A_{xyzt}\left(e^{(ax+by+cz+dt)}\right)-A_{xyzt}\left(e^{-(ax+by+cz+dt)}\right)\right]$$
$$=\frac{1}{2}\left[\frac{1}{p(p-a)q(q-b)r(r-c)s(s-d)}\right]$$
$$-\frac{1}{2}\left[\frac{1}{p(p+a)q(q+b)r(r+c)s(s+d)}\right]. \tag{10}$$

Similarly,

$$A_{xyzt}\left(\cosh(ax+by+cz+dt)\right) = \frac{1}{2}\left[\frac{1}{p(p-a)q(q-b)r(r-c)s(s-d)}\right] + \frac{1}{2}\left[\frac{1}{p(p+a)q(q+b)r(r+c)s(s+d)}\right].$$

(d)

$$A_{xyzt}\left((xyzt)^n\right) = \frac{1}{p}\int_0^\infty x^n e^{-px}dx \frac{1}{q}\int_0^\infty y^n e^{-qy}dy \frac{1}{r}\int_0^\infty z^n e^{-rz}dz \frac{1}{s}\int_0^\infty t^n e^{-st}dt = \frac{n!}{p^{n+2}}\frac{n!}{q^{n+2}}\frac{n!}{r^{n+2}}\frac{n!}{s^{n+2}} = \frac{(n!)^4}{(pqrs)^{n+2}}, \text{n} \in N.$$

(e) If $\alpha,\beta,\gamma,\mu > -1$, then:

$$A_{xyzt}\left(x^\alpha y^\beta z^\gamma t^\mu\right) = \frac{1}{p}\int_0^\infty x^\alpha e^{-px}dx \frac{1}{q}\int_0^\infty y^\beta e^{-qy}dy \frac{1}{r}\int_0^\infty z^\gamma e^{-rz}dz \frac{1}{s}\int_0^\infty t^\mu e^{-st}dt = \frac{\Gamma(\alpha+1)}{p^{\alpha+2}}\frac{\Gamma(\beta+1)}{q^{\beta+2}}\frac{\Gamma(\gamma+1)}{r^{\gamma+2}}\frac{\Gamma(\mu+1)}{s^{\mu+2}}.$$

(f) If $f(x,y,z,t) = g(x)h(y)m(z)c(t)$, then:

$$A_{xyzt}\left[f(x,y,z,t)\right] = \frac{1}{p}\int_0^\infty g(x)e^{-px}dx \frac{1}{q}\int_0^\infty h(y)e^{-qy}dy \frac{1}{r}\int_0^\infty m(z)e^{-rz}dz \frac{1}{s}\int_0^\infty c(t)e^{-st}dt = A_x\left[g(x)\right]A_y\left[h(y)\right]A_z\left[m(z)\right]A_t\left[c(t)\right].$$

In particular,

$$A_{xyzt}\left(\sqrt{xyzt}\right) = \frac{\pi^2}{16p^2.5q^2.5r^2.5s^2.5}. \quad (11)$$

$$A_{xyzt}\left(\cos(ax)\ \cos(by)\cos(cz)\cos(dt)\ \right) = \frac{1}{(p^2-a^2)(q^2-b^2)(r^2-c^2)(s^2-d^2)}. \quad (12)$$

$$A_{xyzt}\left(\sin(ax)\ \sin(by)\sin(cz)\sin(dt)\ \right) = \frac{abcd}{p(p^2-a^2)q(q^2-b^2)r(r^2-c^2)s(s^2-d^2)}. \quad (13)$$

4 Existence and Uniqueness of Quadruple Aboodh transform

Definition 4.1 A function $f(x,y,z,t)$ is said to be of exponential order $a > 0, b > 0, c > 0, d > 0$, on $0 \leq x,y,z,t < \infty$, if there exists a positive constant C such that for all $x > X, y > Y, z > Z, t > T$

$$|f(x,y,z,t)| \leq Ce^{(ax+by+cz+dt)} \quad (14)$$

and we write

$$f(x,y,z,t) = O\left(e^{(ax+by+cz+dt)}\right) \text{ as } x,y,z,t \to \infty \quad (15)$$

Theorem 4.1 If $f(x,y,z,t)$ is continuous in every finite intervals $(0,X),(0,Y),(0,Z),(0,T)$, and of exponential order $e^{(ax+by+cz+dt)}$, then the quadruple Aboodh transform of $f(x,y,z,t)$ exists for all p,q,r,s provided $Re\ p > a, Re\ q > b, Re\ r > c, Re\ s > d$.
proof:

$$|K(p,q,r,s)| = \left|\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}f(x,y,z,t)dxdydzdt\right| \leq C\frac{1}{p}\int_0^\infty e^{-(p-a)x}dx\frac{1}{q}\int_0^\infty e^{-(q-b)y}dy\frac{1}{r}\int_0^\infty e^{-(r-c)z}dz\frac{1}{s}\int_0^\infty e^{-(s-d)t}dt = \frac{C}{p(p-a)q(q-b)r(r-c)s(s-d)} \text{for, Re p} > \text{a, Re q} > \text{b, Re r} > \text{c, Re s} > \text{d.} \quad (16)$$

from (16) it follows that

$$\lim_{\substack{p\to\infty,q\to\infty \\ r\to\infty,s\to\infty}} K(p,q,r,s) = 0, \quad (17)$$

(17) can be regarded as the limiting property of quadruple Aboodh transform.
Clearly $K(p,q,r,s) = pqrs$ or $K(p,q,r,s) = p^2+p^2+q^2+r^2$ is not the quadruple Aboodh transform of any function $f(x,y,z,t)$ since $K(p,q,r,s)$ does not goes to zero as $p,q,r,s \to \infty$.

Theorem 4.2 let $h(x,y,z,t)$ and $l(x,y,z,t)$ be continuous functions defined for $x,y,z,t \geq 0$ and having the quadruple Aboodh transform $H(p,q,r,s)$ and $L(p,q,r,s)$ respectively. If $H(p,q,r,s) = L(p,q,r,s)$ then, $h(p,q,r,s) = l(p,q,r,s)$.

Proof: If we assume α,β,γ,μ to be sufficiently large, then since

$$f(x,y,z,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} pe^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} qe^{qy}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} re^{rz}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} se^{st}K(p,q,r,s)ds\right]\right]dr\right)dp$$

we deduce that

$$h(x,y,z,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} p\ e^{px}\Big(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} q\ e^{qy}$$

$$\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} r\ e^{rz}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} s\ e^{st}\, H(p,q,r,s)\,ds\right]dr\right]dq\Big)dp$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} p\ e^{px}\Big(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} q\ e^{qy}$$

$$\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} r\ e^{rz}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} s\ e^{st}\, L(p,q,r,s)\,ds\right]dr\right]dq\Big)$$

$$= l(x,y,z,t)$$

5 Some Properties of Quadruple Aboodh Transform

(1) The triple Aboodh transform is a linear operator, that is:

$$A_{xyzt}\left[(af+bg)(x,y,z,t)\right] = aA_{xyzt}\left[f(x,y,z,t)\right] + bA_{xyzt}\left[g(x,y,z,t)\right]$$

proof: The proof is direct from definition (3.1)

(2) Changing of scale property:

$$A_{xyzt}\left[f(ax,by,cz,dt)\right] = \frac{1}{abcd}K\left(\frac{p}{a},\frac{q}{b},\frac{r}{c},\frac{s}{d}\right)$$

proof:

$$A_{xyzt}\left[f(ax,by,cz,dt)\right] =$$

$$\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rz-st} f(ax,by,cz,dt)\,dxdydzdt$$

$$= \frac{1}{pqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rz}$$

$$\left[\frac{1}{s}\int_0^\infty e^{-st} f(ax,by,cz,dt)\,dt\right] dxdydz$$

$$= \frac{1}{pqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rz}\frac{1}{d}K\left(x,y,z,\frac{s}{d}\right)dxdydz$$

$$= \frac{1}{d}\frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px}e^{-qy}\left[\frac{1}{r}\int_0^\infty e^{-rz}K\left(x,y,z,\frac{s}{d}\right)dz\right]dxdy$$

$$= \frac{1}{d}\frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(a+p)x}e^{-(b+q)y}\frac{1}{c}K\left(x,y,\frac{r}{c},\frac{s}{d}\right)dxdy$$

Continue in the same manner, we get the result.

(3) Shifting property:

$$A_{xyzt}\left[e^{-ax-by-cz-dt} f(x,y,z,t)\right] = K(p+a,q+b,r+c,s+d)$$

proof:

$$A_{xyzt}\left[e^{-ax-by-cz-dt} f(x,y,z,t)\right] = \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz-dt} e^{-px-qy-rz-st} f(x,y,z,t)\,dxdydzdt$$

$$= \frac{1}{pqrs}\times\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(a+p)x-(b+q)y-(c+r)z-(s+d)t} f(x,y,z,t)\,dxdydzdt$$

$$= \frac{1}{pqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(a+p)x}e^{-(b+q)y}e^{-(c+r)z}$$

$$\times\left[\frac{1}{s}\int_0^\infty e^{-(s+d)t} f(x,y,z,t)\,dt\right] dxdydz$$

$$= \frac{1}{pqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(a+p)x}e^{-(b+q)y}e^{-(c+r)z}K(x,y,z,d+s)\,dxdydz$$

$$= \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(a+p)x}e^{-(b+q)y}$$

$$\times\left[\frac{1}{r}\int_0^\infty e^{-(c+r)z}K(x,y,z,d+s)\,dz\right] dxdy$$

$$= \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(a+p)x}e^{-(b+q)y}K(x,y,c+r,d+s)\,dxdy$$

Continue in the same manner, we get the result.

(4) Multiplying by $x^n y^m z^v t^\mu$

$$A_{xyzt}\left[x^m y^n z^v t^\mu f(x,y,z,t)\right] = \frac{(-1)^{m+n+v+\mu}}{pqrs}\frac{\partial^{m+n+v+\mu}}{\partial p^m\partial q^n\partial r^v\partial s^\mu}\left[pqrsK(p,q,r,s)\right]$$

proof:

$$A_{xyzt}\left[x^n y^m z^v t^\mu f(x,y,z,t)\right]$$

$$= \frac{1}{pqrs}\times$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}x^n y^m z^v t^\mu f(x,y,z,t)\,dxdydzdt$$

$$= \frac{1}{p}\int_{0}^{\infty} x^n e^{-px} \times$$

$$\left[\frac{1}{qrs}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{-qy-rt} y^m z^v t^\mu f(x,y,z,t)dydzdt\right]dx$$

The expression in the bracket satisfies the property of the triple Aboodh transform [11], that is

$$A_{yzt}\left[y^m z^v t^\mu f(x,y,z,t)\right] = \frac{(-1)^{m+v+\mu}}{qrs}\frac{\partial^{m+v+\mu}}{q^m r^v s^\mu}\left[qrsK(x,q,r,s)\right]$$

thus,

$$\frac{1}{p}\int_{0}^{\infty} x^n e^{-px}\frac{(-1)^{m+v+\mu}}{qrs}\frac{\partial^{m+v+\mu}}{q^m r^v s^\mu}\left[qrsK(x,q,r,s)\right]dx$$

$$= \frac{(-1)^{n+m+v+\mu}}{pqrs}\frac{\partial^{n+m+v+\mu}}{\partial p^n \partial q^m \partial r^v \partial s^\mu}\left[pqrsK(p,q,r,s)\right]. \quad (5)$$

Theorem 5.1 If $K(p,q,r,s) = A_{xyzt}(f(x,y,z,t))$, then

$$A_{xyzt}\left[f(x-a,y-b,z-c,t-d)H(x-a,y-b,z-c,t-d)\right] = e^{-pa-qb-rc-sd}K(p,q,r,s)$$

Where $H(x,y,z,t)$ is the Heaviside unit step function defined by :

$$H(x-a,y-b,z-c,t-d) = \begin{cases} 1 & x>a, y>b, z>c, t>d \\ 0 & x<a, y<b, z<c, t<d \end{cases}$$

By definition we have:

$$A_{xyzt}\left[f(x-a,y-b,z-c,t-d)H(x-a,y-b,z-c,t-d)\right]$$

$$= \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rz-st}\begin{bmatrix} f(x-a,y-b,z-c,t-d) \\ H(x-a,y-b,z-c,t-d)\end{bmatrix} dxdydzdt$$

$$= \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-qy-rz-st} f(x-a,y-b,z-c,t-d)\, dxdydzdt$$

by letting $x-a=u_1$, $y-b=u_2$, $z-c=u_3$, $t-d=u_4$,we have

$$\frac{1}{pqrs}e^{-pa-qb-rc-sd}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-pu_1-qu_2-ru_3-su_4} f(u_1,u_2,u_3,\, u_4)\, du_1du_2du_3du_4$$

$$= e^{-pa-qb-rc-sd} \quad K(p,q,r,s) \quad (6)$$

Theorem 5.2 If a function $f(x,y,z,t)$ is periodic of periods a,b,c,d, and if $A_{xyzt}[f(x,y,z,t)]$ exists, then

$$A_{xyzt}\left[f(x,y,z,t)\right] = \frac{\left(1-e^{-pa-qb-rc-sd}\right)^{-1}}{pqrs}$$

$$\int_0^a\int_0^b\int_0^c\int_0^d e^{-px-qy-rz-st} f(x,y,z,t)dxdydzdt.$$

By definition, we have

$$A_{xyzt}\left[f(x,y,z,t)\right] =$$

$$\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)} f(x,y,z,t)\, dxdydzdt$$

$$= \frac{1}{pqrs}\int_0^a\int_0^b\int_0^c\int_0^d e^{-(px+qy+rz+st)} f(x,y,z,t)dxdydzdt$$

$$+ \frac{1}{pqrs}\int_a^\infty\int_b^\infty\int_c^\infty\int_d^\infty e^{-(px+qy+rz+st)} f(x,y,z,t)\, dxdydzdt.$$

By putting $u_1+a=x, u_2+b=y, u_3+c=y, u_4+d=z$ in the second quadruple integral, we get:

$$A_{xyzt}\left[f(x,y,z,t)\right]$$

$$= \frac{1}{pqrs}\int_0^a\int_0^b\int_0^c\int_0^d e^{-(px+qy+rz+st)} f(x,y,z,t)dxdydzdt$$

$$+ \frac{1}{pqrs}e^{-pa-qb-rc-sd}$$

$$\times \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(pu_1+qu_2+ru_3+su_4)} \times \left(\begin{matrix} f(u_1+a,u_2+b,u_3+c,u_4+d) \\ du_1du_2du_3du_4 \end{matrix}\right)$$

$$= \frac{1}{pqrs}\int_0^a\int_0^b\int_0^c\int_0^d e^{-(px+qy+rz+st)} f(x,y,z,t)dxdydzdt$$

$$+ \frac{1}{pqrs}e^{-pa-qb-rc-sd}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(pu_1+qu_2+ru_3+su_4)} f(u_1,u_2,u_3,u_4)\, du_1du_2du_3du_4$$

$$= \frac{1}{pqrs}\int_0^a\int_0^b\int_0^c\int_0^d e^{-(px+qy+rz+st)} f(x,y,z,t)dxdydzdt$$

$$+ e^{-pa-qb-rc-sd}A_{xyzt}\left[f(x,y,z,t)\right]$$

Consequently,

$$A_{xyzt}\left[f(x,y,z,t)\right] =$$

$$\frac{\left(1-e^{-pa-qb-rc-sd}\right)^{-1}}{pqrs} \times$$

$$\int_0^a\int_0^b\int_0^c\int_0^d e^{-px-qy-rz-st} f(x,y,z,t)dxdydzdt. \quad (7)$$

Theorem 5.3 If $f(x,y,z,t)$ is continuous on $[0,\infty)$ and of exponential order, then f, only can be recovered from $A_{xyzt}[f(x,y,z,t)]$ as

$$f(x,y,z,t) = \lim_{m_1,m_2,m_3,m_4\to\infty} \frac{(-1)^{m_1+m_2+m_3+m_4}}{m_1!m_2!m_3!m_4!} \left(\frac{m_1}{x}\right)^{m_1+1}\left(\frac{m_2}{y}\right)^{m_2+1}\left(\frac{m_3}{z}\right)^{m_3+1}\left(\frac{m_4}{t}\right)^{m_4+1} \frac{\partial^{m_1+m_2+m_3+m_4}}{\partial p^{m_1}\partial q^{m_2}\partial r^{m_3}\partial s^{m_4}}\left[pqrsK\left(\frac{m_1}{x},\frac{m_2}{y},\frac{m_3}{z}\frac{m_4}{t}\right)\right].$$

proof: similar to proof given by[17].
to check the efficiency and applicability of theorem (5.3) we give the following example let $f(x,y,z,t) = e^{-ax-by-cz-dt}$, the quadruple Aboodh transform is

$$K(p,q,r,s) = \frac{1}{p(p+a)q(q+b)r(r+c)s(s+d)}$$

thus, we have

$$\frac{\partial^{m_1+m_2+m_3+m_4}}{\partial p^{m_1}\partial q^{m_2}\partial r^{m_3}\partial s^{m_4}}[pqrsK(p,q,r,s)] = \frac{(-1)^{m_1+m_2+m_3+m_4}m_1!m_2!m_3!m_4!}{(p+a)^{m_1+1}(q+b)^{m_2+1}(r+c)^{m_3+1}(s+d)^{m_3+1}}.$$

By using theorem (5.3), we get

$$f(x,y,z,t) = \lim_{m_1,m_2,m_3,m_4\to\infty}\left(\frac{m_1}{x}\right)^{m_1+1}\left(\frac{m_2}{y}\right)^{m_2+1}\left(\frac{m_3}{z}\right)^{m_3+1}\left(\frac{m_4}{t}\right)^{m_4+1} \left(a+\frac{m_1}{x}\right)^{-m_1-1}\left(b+\frac{m_2}{y}\right)^{-m_2-1}\left(c+\frac{m_3}{z}\right)^{-m_3-1}\left(d+\frac{m_4}{t}\right)^{-m_4-1}$$
$$= \lim_{m_1,m_2,m_3,m_4\to\infty}\left(1+\frac{ax}{m_1}\right)^{-m_1-1}\left(1+\frac{by}{m_2}\right)^{-m_2-1}\left(1+\frac{cz}{m_3}\right)^{-m_3-1}\left(1+\frac{dt}{m_4}\right)^{-m_4-1}.$$

By applying the logarithm and and L?Hopital?s rule we have:

$$\log f(x,y,z,t) = -ax-by-cz-dt,$$
$$f(x,y,z,t) = e^{-ax-by-cz-dt}.$$

6 Convolution Theorem of quadruple Aboodh transform

Theorem 5.1 If at the point (p,q,r,s) the integral

$$G_1(p,q,r,s) = \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}g_1(x,y,z,t)\,dxdydzdt,$$

is converge and in addition if

$$G_2(p,q,r,s) = \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}g_2(x,y,z,t)\,dxdydzdt,$$

is absolutely converge, then the following expression

$$G(p,q,r,s) = pqrsG_1(p,q,r,s)G_2(p,q,r,s) \tag{18}$$

is the quadruple Aboodh transform of the function

$$g(x,y,z,t) = \int_0^t\int_0^z\int_0^y\int_0^x g_1(x-x_1,y-y_1,z-z_1,t-t_1)\times g_2(x_1,y_1,z_1,t_1)dx_1dy_1dz_1dt_1$$

and the integral

$$G(p,q,r,s) = \frac{1}{pqrs}\times \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}g(x,y,z,t)\,dxdydzdt$$

is converge at the point (p,q,r,s).
proof:

$$G(p,q,r,s) = \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}g(x,y,z,t)dxdydzdt$$

$$= \frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)} \left[\begin{array}{l}\int_0^t\int_0^z\int_0^y\int_0^x g_1(x-x_1,y-y_1,z-z_1,t-t_1)\\ g_2(x_1,y_1,z_1,t_1)dx_1dy_1dz_1dt_1\end{array}\right] dxdydzdt$$

By using Heaviside unit step function

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g_2(x_1,y_1,z_1,t_1)dx_1dy_1dz_1dt_1 \left[\begin{array}{l}\frac{1}{pqrs}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+rz+st)}g_1(x-x_1,y-y_1,z-z_1,t-t_1)\\ H(x-x_1,y-y_1,z-z_1,t-t_1)dxdydzdt\end{array}\right]$$

thus,

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^0 g_2(\phi,\varphi,\gamma,\eta)e^{-p\phi-q\varphi-r\gamma-s\eta}G_1(p,q,r,s)\,d\phi d\varphi d\gamma d\eta$$
$$= pqrsG_1(p,q,r,s)G_2(p,q,r,s)$$

7 Quadruple Aboodh transform of some partial and fractional derivatives

1) The Quadruple Aboodh transform of nth derivative of a function of four variables is given by:

$$A_{xyzt}\left(\frac{\partial^n f(x,y,z,t)}{\partial x^n}\right) = p^n A_{xyzt}(f(x,y,z,t)) - \sum_{m=0}^{n-1} p^{n-m-2} A_{yzt}\left(\frac{\partial^m f(0,y,z,t)}{\partial x^m}\right), \quad (19)$$

$$A_{xyzt}\left(\frac{\partial^n f(x,y,z,t)}{\partial y^n}\right) = q^n A_{xyzt}(f(x,y,z,t)) - \sum_{m=0}^{n-1} q^{n-m-2} A_{xzt}\left(\frac{\partial^m f(x,0,z,t)}{\partial y^m}\right) \quad (20)$$

$$A_{xyzt}\left(\frac{\partial^n f(x,y,z,t)}{\partial z^n}\right) = r^n A_{xyzt}(f(x,y,z,t)) - \sum_{m=0}^{n-1} r^{n-m-2} A_{xyt}\left(\frac{\partial^m f(x,y,0,t)}{\partial z^m}\right) \quad (21)$$

$$A_{xyzt}\left(\frac{\partial^n f(x,y,z,t)}{\partial t^n}\right) = s^n A_{xyzt}(f(x,y,z,t)) - \sum_{m=0}^{n-1} s^{n-m-2} A_{xyz}\left(\frac{\partial^m f(x,y,z,0)}{\partial t^m}\right). \quad (22)$$

2) The quadruple Aboodh transform of mixed derivative of a function of four variables is given by:

$$\begin{aligned} A_{xyzt}\left(\frac{\partial^4 f(x,y,z,t)}{\partial x\partial y\partial z\partial t}\right) &= pqrsA_{xyzt}(f(x,y,z,t)) \\ &- \frac{pqr}{s}A_{xyz}(f(x,y,z,0)) - \frac{pqs}{r}A_{xyt}(f(x,y,0,t)) \\ &- \frac{prs}{q}A_{xzt}(f(x,0,z,t)) - \frac{qrs}{p}A_{yzt}(f(0,y,z,t)) \\ &+ \frac{rs}{pq}A_{zt}(f(0,0,z,t)) + \frac{qs}{pr}A_yA_t(f(0,y,0,t)) \\ &+ \frac{qr}{ps}A_{yz}(f(0,y,z,0))\frac{ps}{qr}A_{xt}(f(x,0,0,t)) \\ &+ \frac{pr}{qs}A_{xz}(f(x,0,z,0)) + \frac{pq}{rs}A_{xy}(f(x,y,0,0)) \\ &- \frac{s}{pqr}A_t(f(0,0,0,t)) - \frac{r}{pqs}A_z(f(0,0,z,0)) \\ &- \frac{q}{prs}A_y(f(0,y,0,0)) - \frac{p}{qrs}A_x(f(x,0,0,0)) \\ &+ \frac{1}{pqrs}f(0,0,0,0). \end{aligned}$$

$$\begin{aligned} A_{xyzt}\left(\frac{\partial^3 f(x,y,z,t)}{\partial y\partial x^2}\right) &= qp^2A_{xyzt}(f(x,y,z,t)) \\ &- qA_{yzt}(f(0,y,z,t)) - \frac{q}{p}A_{yzt}\left(\frac{\partial f(0,y,z,t)}{\partial x}\right) \\ &- \frac{p^2}{q}A_{xzt}(f(x,0,z,t)) + \frac{1}{pq}A_{zt}\left(\frac{\partial f(0,0,z,t)}{\partial x}\right) \\ &+ \frac{1}{q}A_{zt}(f(0,0,z,t)). \end{aligned}$$

3) The quadruple Aboodh transform of the partial fractional Caputo derivatives of a function of four variables is given by:

$$A_{xyzt}\left(\frac{\partial^\alpha f(x,y,z,t)}{\partial x^\alpha}\right) = p^\alpha A_{xyzt}(f(x,y,z,t)) - \sum_{k=0}^{j-1} p^{\alpha-k-2} A_{yzt}\left(\frac{\partial^k f(0,y,z,t)}{\partial x^k}\right).$$

$$A_{xyzt}\left(\frac{\partial^\beta f(x,y,z,t)}{\partial y^\beta}\right) = q^\beta A_{xyzt}(f(x,y,z,t)) - \sum_{k=0}^{n-1} q^{\beta-k-2} A_{xzt}\left(\frac{\partial^k f(x,0,z,t)}{\partial y^k}\right).$$

$$A_{xyzt}\left(\frac{\partial^\delta f(x,y,z,t)}{\partial z^\delta}\right) = s^\delta A_{xyzt}(f(x,y,z,t)) - \sum_{k=0}^{l-1} s^{\delta-k-2} A_{xyt}\left(\frac{\partial^k f(x,y,0,t)}{\partial z^k}\right).$$

$$A_{xyzt}\left(\frac{\partial^\gamma f(x,y,z,t)}{\partial t^\gamma}\right) = r^\gamma A_{xyzt}(f(x,y,z,t)) - \sum_{k=0}^{m-1} r^{\gamma-k-2} A_{xyz}\left(\frac{\partial^k f(x,y,z,0)}{\partial t^k}\right).$$

Where, $\frac{\partial^\alpha f(x,y,z,t)}{\partial x^\alpha}, \frac{\partial^\beta f(x,y,z,t)}{\partial y^\beta}, \frac{\partial^\delta f(x,y,z,t)}{\partial z^\delta}, \frac{\partial^\gamma f(x,y,z,t)}{\partial t^\gamma}$ are the Caputo fractional derivatives of function $f(x,y,z,t)$, see [18].

8 Applications of Quadruple Aboodh transform

In this section, we apply quadruple Aboodh transform to some different examples to show the effectiveness and applicability of this transform.

Application of quadruple Aboodh tranorm to integral equations

Example 8.1 Consider the following quadruple integral equation of the form

$$f(x,y,z,t)=g(x,y,z,t)+\mu\times \int_0^x\int_0^y\int_0^z\int_0^t f(x-\phi,y-\varphi,z-\gamma,t-\eta)h(\phi,\varphi,\gamma,\eta)d\phi d\varphi d\gamma d\eta \quad (23)$$

where $f(x,y,z,t)$ is the unknown function, μ is constant and $g(x,y,z,t),h(x,y,z,t)$ are known functions.
By applying the quadruple Aboodh transform to equation (23) we get:

$$A_{xyzt}(f(x,y,z,t))=A_{xyzt}(g(x,y,z,t)) +\mu A_{xyzt}\left[(f**h)(x,y,z,t)\right].$$

By using Convolution theorems, we get:

$$A_{xyzt}(f(x,y,z,t))=A_{xyzt}(g(x,y,z,t)) +\mu\ pqrsA_{xyzt}(f(x,y,z,t))A_{xyzt}(h(x,y,z,t))$$

Consequently,

$$A_{xyzt}(f(x,y,z,t))=\frac{A_{xyzt}(g(x,y,z,t))}{1-\mu\ pqrsA_{xyzt}(h(x,y,z,t))},$$

by taking the inverse of Quadruple Aboodh transform we get:

$$f(x,y,z,t)=A^{-1}_{xyzt}\left[\frac{A_{xyzt}(g(x,y,z,t))}{1-\mu\ pqrsA_{xyzt}(h(x,y,z,t))}\right]$$
$$=A^{-1}_{xyzt}\left(A_{xyzt}(g(x,y,z,t)).A_{xyzt}(v(x,y,z,t))\right)=$$
$$\int_0^x\int_0^y\int_0^z\int_0^t f(x-\phi,y-\varphi,z-\gamma,t-\eta)v(\phi,\varphi,\gamma,\eta)d\phi d\varphi d\gamma d\eta.$$

where,

$$A_{xyzt}(v(x,y,z,t))=\frac{1}{1-\mu pqrsA_{xyzt}(h(x,y,z,t))}.$$

The above method can be illustrated by the following two simple examples.
Example 8.1a Consider the following integral equation:

$$xyzt=\int_0^x\int_0^y\int_0^z\int_0^t e^{-(x-\phi)-(y-\varphi)-(z-\gamma)-(t-\eta)}f(\phi,\varphi,\gamma,\eta)d\phi d\varphi d\gamma d\eta \quad (24)$$

By applying the Quadruple Aboodh transform to equation (24) we get:

$$A_{xyzt}(f(x,y,z,t))=\left(\frac{1}{p^2}+\frac{1}{p^3}\right)\left(\frac{1}{q^2}+\frac{1}{q^3}\right)\left(\frac{1}{r^2}+\frac{1}{r^3}\right)\left(\frac{1}{s^2}+\frac{1}{s^3}\right)$$

thus,

$$f(x,y,z,t)=A^{-1}_{xyzt}\left[\left(\frac{1}{p^2}+\frac{1}{p^3}\right)\left(\frac{1}{q^2}+\frac{1}{q^3}\right)\left(\frac{1}{r^2}+\frac{1}{r^3}\right)\left(\frac{1}{s^2}+\frac{1}{s^3}\right)\right],$$

by taking the inverse quadruple Aboodh transform, we have

$$f(x,y,z,t)=(1+x)(1+y)(1+z)(1+t).$$

Example 8.1b Consider the following integral equation:

$$c^2=\int_0^x\int_0^y\int_0^z\int_0^t f(x-\phi,y-\varphi,z-\gamma,t-\eta)f(\phi,\varphi,\gamma,\eta)d\phi d\varphi d\gamma d\eta \quad (25)$$

where c is constant.
By applying of quadruple Aboodh transform to (25) we get:

$$\frac{c^2}{(pqrs)^2}=pqrs[A_{xyzt}(f(x,y,z,t))]^2,$$

thus,

$$A_{xyzt}(f(x,y,z,t))=\frac{c}{(pqrs)^{1.5}},$$

by taking inverse of quadruple Aboodh transform we get:

$$f(x,y,z,t)=A^{-1}{}_{xyzt}\left[\frac{c}{(pqrs)^{1.5}}\right]=\frac{c}{\pi^2\sqrt{xyzt}}$$

Application of quadruple Aboodh to partial differential equations
Example 8.2
Consider the three-dimensional diffusion equations

$$\begin{gathered}\frac{\partial^2 f(x,y,z,t)}{\partial x^2}+\frac{\partial^2 f(x,y,z,t)}{\partial y^2}+\frac{\partial^2 f(x,y,z,t)}{\partial z^2}-\frac{\partial f(x,y,z,t)}{\partial t}=4e^{x+y-z-t}\\ f(0,y,z,t)=e^{y-z-t},\quad f(1,y,z,t)=e^{1+y-z-t},\\ f(x,0,z,t)=e^{x-z-t},\quad f(x,1,z,t)=e^{x+1-z-t},\\ f(x,y,0,t)=e^{x+y-t},\quad f(x,y,1,t)=e^{x+1-z-t},\\ f(x,y,z,0)=e^{x+y-z}.\end{gathered} \quad (26)$$

by applying the quadruple Aboodh transform to (26), we get

$$(p^2+q^2+r^2-s)K(p,q,r,s)= U(p,q,r)+\frac{4}{p(p-1)q(q-1)r(r+1)s(s+1)} \quad (27)$$

where,

$$U(p,q,r,s) = A_{yzt}\left(f(0,y,z,t)\right) + \frac{1}{p}A_{yzt}\left(\frac{\partial f(0,y,z,t)}{\partial x}\right)$$
$$+A_{xzt}\left(f(x,0,z,t)\right) + \frac{1}{q}A_{xzt}\left(\frac{\partial f(x,0,z,t)}{\partial y}\right) + A_{xyt}\left(f(x,y,0,t)\right) +$$
$$\frac{1}{r}A_{xyt}\left(\frac{\partial f(x,y,0,t)}{\partial z}\right) + \frac{1}{s}A_{xyz}(f(x,y,z,0))$$
$$= \frac{-4+\left(p^2+q^2+r^2-s\right)}{p(p-1)q(q-1)r(r+1)s(s+1)}.$$

Substituting the value of $U(p,q,r,s)$in equation (27) we get:

$$K(p,q,r,s) = \frac{1}{p(p-1)q(q-1)r(r+1)s(s+1)},$$

bu applying the inverse of quadruple Aboodh transform,we get:

$$f(x,y,z,t) = A^{-1}_{xyzt}\left[K(p,q,r,s)\right] = e^{x+y-z-t}.$$

Example 8.3
Consider the following non-homogeneous Mboctara partial differential equation:

$$\frac{\partial^4 f(x,y,z,t)}{\partial x \partial y \partial z \partial t} + f(x,y,z,t) = 3e^{-x-2y+z+t}, \tag{28}$$

subject to the following initial and boundary conditions:

$$f(0,y,z,t) = e^{-2y+z+t}, f(x,0,z,t) = e^{-x+z+t},$$
$$f(x,y,0,t) = e^{-x-2y+t}, f(x,y,z,0) = e^{-x-2y+z}.$$

By applying the quadruple Aboodh transform to both side of equation (28) we get:

$$(pqrs+1)K(p,q,r,s) = U(p,q,r,s) + \frac{3}{p(p+1)q(q+2)r(r-1)s(s-1)} \tag{29}$$

where,

$$U(p,q,r,s) = \frac{pqr}{s}A_{xyz}\left(f(x,y,z,0)\right) + \frac{pqs}{r}A_{xyt}\left(f(x,y,0,t)\right)$$
$$+\frac{prs}{q}A_{xzt}\left(f(x,0,z,t)\right) + \frac{qrs}{p}A_{yzt}\left(f(0,y,z,t)\right)_2$$
$$-\frac{rs}{pq}A_{zt}\left(f(0,0,z,t)\right) - \frac{qs}{pr}A_{yt}\left(f(0,y,0,t)\right)$$
$$-\frac{qr}{ps}A_{yz}\left(f(0,y,z,0)\right) - \frac{ps}{qr}A_{xt}\left(f(x,0,0,t)\right)$$
$$-\frac{pr}{qs}A_{xz}\left(f(x,0,z,0)\right) - \frac{pq}{rs}A_{xy}\left(f(x,y,0,0)\right)$$
$$+\frac{s}{pqr}A_{t}\left(f(0,0,0,t)\right) + \frac{r}{pqs}A_{z}\left(f(0,0,z,0)\right)$$
$$+\frac{q}{prs}A_{y}\left(f(0,y,0,0)\right) + \frac{p}{qrs}A_{x}\left(f(x,0,0,0)\right)$$
$$-\frac{1}{pqrs}f(0,0,0,0)$$
$$= \frac{pqrs-2}{p(p+1)q(q+2)r(r-1)s(s-1)}.$$

After substituting the value of $U(p,q,r,s)$, we get:

$$K(p,q,r,s) = \frac{1}{p(p+1)q(q+2)r(r-1)s(s-1)},$$

by taking the inverse of quadruple Aboodh transform,we get:

$$f(x,y,z,t) = A^{-1}_{xyzt}\left[K(p,q,r,s)\right] = e^{-x-2y+z+t}.$$

Example 8.4
Consider the following fractional partial differential equation.

$$D^{\alpha}_{t} f(x,y,z,t) = \frac{\partial^2 f(x,y,z,t)}{\partial y^2}, 0 < \alpha \leq 1, \tag{30}$$

With the initial and boundary conditions:

$$f(x,0,z,t) = 0, f_y(x,0,z,t) = \sin(x)\sin(z)E_{\alpha}(-t^{\alpha}),$$
$$f(x,y,z,0) = \sin(x)\sin(y)\sin(z).$$

By applying the quadruple Aboodh transform to equation (30) and for the initial and boundary conditions we get:

$$s^{\alpha}A_{xyzt}\left(f(x,y,z,t)\right) - s^{\alpha-2}A_{xyz}\left(f(x,y,z,0)\right) = q^2 A_{xyzt}\left(f(x,y,t)\right) - A_{xzt}\left(f(x,0,z,t)\right) - \frac{1}{q}A_{xzt}\left(\frac{\partial f(x,0,z,t)}{\partial y}\right), \tag{31}$$

where,

$$A_{xyz}\left(f(x,y,z,0)\right) = A_{xyz}\left(\sin x \sin y \sin z\right) = \frac{1}{p(p^2+1)}\frac{1}{q(q^2+1)}\frac{1}{r(r^2+1)},$$

$$A_{xzt}\left(f(x,0,z,t)\right) = A_{xzt}\left(0\right) = 0,$$

$$A_{xzt}\left(\frac{\partial f(x,0,z,t)}{\partial y}\right) = \frac{1}{p(p^2+1)}\frac{1}{r(r^2+1)}\frac{s^{\alpha-2}}{1+s^{\alpha}}.$$

thus, (31) becomes

$$\left(s^{\alpha}-q^2\right)A_{xyzt}\left(f(x,y,z,t)\right) = \frac{s^{\alpha-2}}{qp(p^2+1)}\frac{1}{r(r^2+1)}\left[\frac{1}{(q^2+1)} - \frac{1}{1+s^{\alpha}}\right] \tag{32}$$

after simple calculations, (32) becomes:

$$A_{xyzt}\left(f(x,y,z,t)\right) = \frac{1}{p(p^2+1)q(q^2+1)r(r^2+1)}\frac{s^{\alpha-2}}{1+s^{\alpha}} \tag{33}$$

by taking the quadruple inverse Aboodh transform to (33), we get

$$f(x,y,z,t) = \sin(x)\sin(y)\sin(z)E_{\alpha}(-t^{\alpha}). \tag{34}$$

9 Perspective

In this paper, we have introduced the definition of quadruple Aboodh transform and its inverse. First, we applied the quadruple Aboodh transform on some functions, we also, discussed the existence and uniqueness theorems of Quadruple Aboodh transform and there proofs. Next, some properties and theorems related to this transform are presented and proved. To illustrate the efficiency and applicability of our method, we implemented the quadruple Aboodh transform to solve integral and partial including fractional differential equations. Finally it's worthwhile to mention that the quadruple Aboodh transform method can be combined to some other methods to solve nonlinear partial differential equations arising in different field of sciences, which will be discussed in a subsequent papers.

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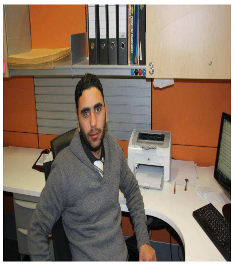

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