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Sufficient Condition Starlikeness and Convexity of Integral Operators Related to Multivalent Functions

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Abstract: We define two new general integral operators for certain analytic multivalent functions in the unit disc $\mathcal{U}$ and give some sufficient conditions for these integral operators on some subclasses of analytic multivalent functions.

Keywords: Multivalent functions, Starlike Functions, Convex Functions, Convolution
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1 Introduction

Let $\mathscr{A}_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \,(p,n \in N = \{1,2,3...\}). \quad (1)$$

which is analytic in open unit disc $\mathcal{U} = \{z \in \mathbb{C} | |z| < 1\}$. In particular, we set

$$\mathscr{A}_p(1) = \mathscr{A}_p, \mathscr{A}_1(1) = \mathscr{A}_1 := \mathscr{A}.$$

If $f \in \mathscr{A}_p(n)$ is given by (1) and $g \in \mathscr{A}_p(n)$ is given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \,(p,n \in N = \{1,2,3...\}). \quad (2)$$

then the Hadamard product (or convolution) $f * g$ of f and g is given by

$$(f*g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g*f)(z). \quad (3)$$

We observe that several known operators are deducible from the convolutions. That is, for various choices of g in (3), we obtain some interesting operators. For example, for functions $f \in \mathscr{A}_p(n)$ and the function g is defined by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} \psi_{k,m}(\alpha,\lambda,l,p) z^k \quad (m \in N_0 = N \cup \{0\}) \quad (4)$$

where

$$\psi_{k,m}(\alpha,\lambda,l,p) = \left[\frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} \cdot \frac{p+\lambda(k-p)+l}{p+l}\right]^m.$$

The convolution (3) with the function g is defined by (4) and using operator $D^{m,\alpha}_{\lambda,l,p}$ studied by Bulut ([1]), we introduce an operator $D^{m,\alpha}_{\lambda,l,p}(f*g)(z)$ and introduce new classes $\mathcal{U}\mathscr{S}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b)$ and $\mathcal{U}\mathscr{K}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b)$ as follows

Definition 1. *A function $f \in \mathscr{A}_p(n)$ is in the class $\mathcal{U}\mathscr{S}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b)$ if and only if f satisfies*

$$\mathrm{Re}\left\{p + \frac{1}{b}\left(\frac{zD^{m,\alpha}_{\lambda,l,p}(f*g)'(z)}{D^{m,\alpha}_{\lambda,l,p}(f*g)(z)} - p\right)\right\} > \delta\left|\frac{1}{b}\left(\frac{zD^{m,\alpha}_{\lambda,l,p}(f*g)'(z)}{D^{m,\alpha}_{\lambda,l,p}(f*g)(z)} - p\right)\right| + \beta, \quad (5)$$

where $z \in \mathcal{U}, b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$.

Definition 2. *A function $f \in \mathscr{A}_p(n)$ is in the class $\mathcal{U}\mathscr{S}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b)$ if and only if f satisfies*

$$\mathrm{Re}\left\{p + \frac{1}{b}\left(1 + \frac{zD^{m,\alpha}_{\lambda,l,p}(f*g)''(z)}{D^{m,\alpha}_{\lambda,l,p}(f*g)'(z)} - p\right)\right\} > \delta\left|\frac{1}{b}\left(1 + \frac{zD^{m,\alpha}_{\lambda,l,p}(f*g)''(z)}{D^{m,\alpha}_{\lambda,l,p}(f*g)'(z)} - p\right)\right| + \beta, \quad (6)$$

where $z \in \mathcal{U}, b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$.

Note that $f \in \mathcal{U}\mathscr{K}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{U}\mathscr{S}^{p,\lambda,l,m,\alpha}_g(\delta,\beta,b)$.

Remark.(i) For $\delta = 0$, we have

$$\mathcal{U}\mathscr{K}^{p,\lambda,l,m,\alpha}_g(0,\beta,b) = \mathscr{K}^{p,\lambda,l,m,\alpha}_g(\beta,b)$$
$$\mathcal{U}\mathscr{S}^{p,\lambda,l,m,\alpha}_g(0,\beta,b) = \mathscr{S}^{p,\lambda,l,m,\alpha}_g(\beta,b)$$

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(ii) For $\delta = 0$ and $\beta = 0$

$$\mathscr{U}\mathscr{K}_g^{p,\lambda,l,m,\alpha}(0,0,b) = \mathscr{K}_g^{p,\lambda,l,m,\alpha}(b)$$
$$\mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(0,0,b) = \mathscr{S}_g^{p,\lambda,l,m,\alpha}(b)$$

(iii) For $\delta = 0$, $\beta = 0$ and $b = 1$

$$\mathscr{U}\mathscr{K}_g^{p,\lambda,l,m,\alpha}(0,0,b) = \mathscr{K}_g^{p,\lambda,l,m,\alpha}$$
$$\mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(0,0,b) = \mathscr{S}_g^{p,\lambda,l,m,\alpha}$$

(iv) For $g(z) = z^p/(1-z)$, we have two classes $\mathscr{U}\mathscr{K}_{\alpha,\lambda,l}^{m,p,n}(\delta,\beta,b)$ and $\mathscr{U}\mathscr{S}_{\alpha,\lambda,l}^{m,p,n}(\delta,\beta,b)$ which is introduced by Guney and Bulut [1].

Now we define two integral operator

Definition 3. *Let $\eta \in N, m = (m_1,...,m_\eta) \in N_0^\eta$ and $k = (k_1,...,k_\eta) \in R_+^\eta$. One defines the following general integral operators:*

$$\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k} : \mathscr{A}_p(n)^\eta \to \mathscr{A}_p(n)$$
$$\mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k} : \mathscr{A}_p(n)^\eta \to \mathscr{A}_p(n) \tag{7}$$

such that

$$\begin{aligned}\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(t)}{t^p}\right)^{k_j}dt,\\ \mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(t)}{pt^{p-1}}\right)^{k_j}dt,\end{aligned} \tag{8}$$

where $z \in \mathscr{U}$, $f_j, g \in \mathscr{A}_p(n), 1 \le j \le \eta$.

Remark.(i) For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have the new two new integral operators

$$\begin{aligned}\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(t)}{t^p}\right)^{k}dt,\\ \mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(t)}{pt^{p-1}}\right)^{k}dt,\end{aligned} \tag{9}$$

(ii) For $g(z) = z^p/(1-z)$, we have

$$\begin{aligned}\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}f_j(t)}{t^p}\right)^{k_j}dt,\\ \mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}f_j(t)'(t)}{pt^{p-1}}\right)^{k_j}dt,\end{aligned} \tag{10}$$

These operator were introduced by Bulut [1].
(iii) If we take $g(z) = z^p/(1-z)$, the we have

$$\begin{aligned}\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{(f_j)(t)}{t^p}\right)^{k_j}dt,\\ \mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) &= \int_0^z pt^{p-1}\prod_{j=1}^{\eta}\left(\frac{(f_j)'(t)}{pt^{p-1}}\right)^{k_j}dt,\end{aligned} \tag{11}$$

These two operators were introduced by Frasin [3].

2 Sufficient Conditions for $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$

Theorem 1. *Let $\eta \in N, m = (m_1,...,m_\eta) \in N_0^\eta$ and $k = (k_1,...,k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \ge 0, 0 \le \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(\delta_j,\beta_j,b)$ for $1 \le j \le \eta$. If*

$$0 \le p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \tag{1}$$

then the integral operator $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$, defined by (8) , is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Proof. From the definition (8), we observe that $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) \in \mathscr{A}_p(n)$. We can easy to see that

$$\left(\mathscr{I}^{p,\eta,m,k}(z)\right)' = pz^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)}{z^p}\right)^{k_j}. \tag{2}$$

Differentiating (2) logarithmically and multiplying by 'z', we obtain

$$\frac{z\left(\mathscr{I}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{I}^{p,\eta,m,k}(z)\right)'} = p - 1 + \sum_{j=1}^{\eta} k_j\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right) \tag{3}$$

or equivalently

$$1 + \frac{z\left(\mathscr{I}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{I}^{p,\eta,m,k}(z)\right)'} - p = \sum_{j=1}^{\eta} k_j\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right) \tag{4}$$

Then, by multiplying (4) with '1/b', we have

$$\frac{1}{b}\left(1 + \frac{z\left(\mathscr{I}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{I}^{p,\eta,m,k}(z)\right)'} - p\right) = \sum_{j=1}^{\eta} k_j\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right) \tag{5}$$

or

$$p + \frac{1}{b}\left(1 + \frac{z\left(\mathscr{I}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{I}^{p,\eta,m,k}(z)\right)'} - p\right) = p + \sum_{j=1}^{\eta} k_j\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p + p - p\Sigma_{j=1}^{\eta}k_j\right) \tag{6}$$

Since $f_j \in \mathscr{U}\mathscr{S}_g^p(\delta_j,\beta_j,b)$ $(1 \le j \le \eta)$, we get

$$\mathrm{Re}\left\{p + \frac{1}{b}\left(1 + \frac{z\left(\mathscr{I}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{I}^{p,\eta,m,k}(z)\right)'} - p\right)\right\} = p + \sum_{j=1}^{\eta} k_j\mathrm{Re}\left\{\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right)\right\} + p - \sum_{j=1}^{\eta} pk_j \tag{7}$$

$$> \sum_{j=1}^{\eta} k_j\delta_j\left|\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right)\right| + p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Since

$$\sum_{j=1}^{\eta} k_j\delta_j\left|\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)'}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)} - p\right)\right| > 0$$

because the integral operator $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$, defined by (8) , is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

3 Sufficient Conditions for $\mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$

Theorem 2. *Let $\eta \in N, m = (m_1,...,m_\eta) \in N_0^\eta$ and $k = (k_1,...,k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(\delta_j,\beta_j,b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \tag{1}$$

then the integral operator $\mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$, defined by (8) , is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Proof. From the definition (8), we observe that $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z) \in \mathscr{A}_p(n)$. We can easy to see that

$$\left(\mathscr{G}^{p,\eta,m,k}(z)\right)' = pz^{p-1}\prod_{j=1}^{\eta}\left(\frac{D_{\lambda,l,p}^{m,\alpha}(f_j * g)'(z)}{pz^{p-1}}\right)^{k_j}. \tag{2}$$

Differentiating (2) logarithmically and multiplying by 'z', we obtain

$$\frac{z\left(\mathscr{G}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{G}^{p,\eta,m,k}(z)\right)'} = p - 1 + \sum_{j=1}^{\eta} k_j\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p\right) \tag{3}$$

or equivalently

$$1 + \frac{z\left(\mathscr{G}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{G}^{p,\eta,m,k}(z)\right)'} - p = \sum_{j=1}^{\eta} k_j\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z))'} + 1 - p\right) \tag{4}$$

Then, by multiplying (4) with '1/b', we have

$$\frac{1}{b}\left(1 + \frac{z\left(\mathscr{G}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{G}^{p,\eta,m,k}(z)\right)'} - p\right) = \sum_{j=1}^{\eta} k_j\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p\right) \tag{5}$$

or

$$p + \frac{1}{b}\left(\frac{z\left(\mathscr{G}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{G}^{p,\eta,m,k}(z)\right)'} + 1 - p\right) = p + \sum_{j=1}^{\eta} k_j\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p + p - p\Sigma_{j=1}^{\eta}k_j\right) \tag{6}$$

Since $f_j \in \mathscr{U}\mathscr{K}_g^p(\delta_j,\beta_j,b)$ $(1 \leq j \leq \eta)$, we get

$$\mathrm{Re}\left\{p + \frac{1}{b}\left(1 + \frac{z\left(\mathscr{G}^{p,\eta,m,k}(z)\right)''}{\left(\mathscr{G}^{p,\eta,m,k}(z)\right)'} - p\right)\right\}$$
$$= p + \sum_{j=1}^{\eta} k_j \mathrm{Re}\left\{\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p\right)\right\} + p - \sum_{j=1}^{\eta} pk_j + p + \sum_{j=1}^{\eta} k_j(\beta_j - p). \tag{7}$$

$$> \sum_{j=1}^{\eta} k_j\delta_j\left|\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p\right)\right| + p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Since

$$\sum_{j=1}^{\eta} k_j\delta_j\left|\frac{1}{b}\left(\frac{z\left(D_{\lambda,l,p}^{m,\alpha}(f_j*g)(z)\right)''}{D_{\lambda,l,p}^{m,\alpha}(f_j*g)'(z)} + 1 - p\right)\right| > 0$$

because the integral operator $\mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$, defined by (8) , is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

4 Corollaries and Consequences

For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have

Corollary 1. *Let $\eta \in N, m \in N_0^\eta$ and $k \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(\delta_j,\beta_j,b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + k(\beta - p) < p, \tag{1}$$

then the integral operator $\mathscr{I}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$ is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ where

$$\tau = p + k(\beta - p).$$

Corollary 2. *Let $\eta \in N, m \in N_0^\eta$ and $k \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(\delta_j,\beta_j,b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + k(\beta - p) < p, \tag{2}$$

then the integral operator $\mathscr{G}_g^{p,\eta,\lambda,l,m,\alpha,k}(z)$ is in the class $\mathscr{K}_g^{p,\lambda,l,m,\alpha}(\tau,b)$ where

$$\tau = p + k(\beta - p).$$

For $(f_j * g)(z) = D_{\lambda,l,p}^{m,\alpha}f_j(z)$, we have

Corollary 3. *Let $\eta \in N, m = (m_1,...,m_\eta) \in N_0^\eta$ and $k = (k_1,...,k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}_{\alpha,\lambda,l}^{m,j,p,n}(\delta_j,\beta_j,b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \tag{3}$$

then the integral operator $\mathscr{I}_{p,\eta,m,k}(z)$ is in the class $\mathscr{K}^{p,n}(\tau,b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 4. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $\mathscr{U}\mathscr{K}_{\alpha,\lambda,l}^{m,j,p,n}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4)$$

then the integral operator $\mathscr{G}_{p,\eta,m,k}(z)$ is in the class $\mathscr{K}^{p,n}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

which are known results obtained by Guney and Bulut [2]. Further, if put $p = 1$, we have

Corollary 5. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < 1$, and $f_j \in \mathscr{U}\mathscr{S}_g^{p,\lambda,l,m,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (5)$$

then the integral operator $\mathscr{I}_g^{1,\lambda,l,m,\alpha}(z)$ is in the class $\mathscr{K}_g^1(\tau, b)$ where

$$\tau = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1).$$

Corollary 6. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < 1$, and $f_j \in \mathscr{S}_g^{1,\lambda,l,m,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (6)$$

then the integral operator $\mathscr{G}_g^{1,\eta,m,k}(z)$ is in the class $\mathscr{K}_g^{1,\lambda,l,m,\alpha}(\tau, b)$ where

$$\tau = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1).$$

Upon setting $g(z) = z^p/(1-z)$, we have

Corollary 7. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}^{p,\lambda,l,m,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (7)$$

then the integral operator $\mathscr{G}^{p,\eta,m,k}(z)$ is in the class $\mathscr{U}\mathscr{K}^{p,\lambda,l,m,\alpha}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 8. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}^{p,\lambda,l,m,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (8)$$

then the integral operator $\mathscr{G}^{p,\eta,m,k}(z)$ is in the class $\mathscr{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Upon setting $g(z) = z^p/(1-z)$ and $\delta = 0$, we have

Corollary 9. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}^{p,\lambda,l,m,\alpha}(0, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (9)$$

then the integral operator $\mathscr{G}^{p,\eta,m,k}(z)$ is in the class $\mathscr{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 10. *Let $\eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta$ and $k = (k_1, ..., k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathscr{U}\mathscr{S}^{p,\lambda,l,m,\alpha}(0, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (10)$$

then the integral operator $\mathscr{G}^{p,\eta,m,k}(z)$ is in the class $\mathscr{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

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